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GROUPOID C^*-ALGEBRAS WITH HAUSDORFF SPECTRUM

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Abstract

Suppose that G is a second countable, locally compact Hausdorff groupoid with abelian stabiliser subgroups and a Haar system. We provide necessary and sufficient conditions for the groupoid C^*-algebra to have Hausdorff spectrum. In particular, we show that the spectrum of $C^*(G)$ is Hausdorff if and only if the stabilisers vary continuously with respect to the Fell topology, the orbit space $G^{(0)}/G$ is Hausdorff, and, given convergent sequences $\chi_i \to \chi$ and $\gamma_i \cdot \chi_i \to \omega$ in the dual stabiliser groupoid $\widehat{S}$ where the $\gamma_i \in G$ act via conjugation, if χ and ω are elements of the same fibre then $\chi = \omega$.

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1. Introduction

One of the reasons why C^*-algebras are so well studied is that they have a very deep representation theory. Understanding the spectrum or primitive ideal space of a C^*-algebra, and in particular the topology on these spaces, can reveal a great deal of information about the underlying algebra. For example, if a separable C^*-algebra A has Hausdorff spectrum $\widehat{A}$ then A is naturally isomorphic to the C_0-section algebra of a bundle over $\widehat{A}$ such that each fibre of the bundle is isomorphic to the compact operators. Given a class of C^*-algebras, it is an interesting problem to characterise those algebras which have Hausdorff spectrum. For example, in [14] Williams proves the following result. Suppose that we are given a transformation group (H, X) such that H is abelian and the group action satisfies any of the conditions in the Mackey–Glimm dichotomy [11]. Then the transformation group C^*-algebra will have Hausdorff spectrum if and only if the stabiliser subgroups of the action vary continuously with respect to the Fell topology and the orbit space X/H is Hausdorff.

In this paper we would like to extend the work of [14] from transformation groups to groupoids. The most straightforward generalisation is the conjecture that, given a groupoid G with abelian stabiliser subgroups which satisfies the conditions of the Mackey–Glimm dichotomy, the groupoid C^*-algebra will have Hausdorff spectrum if and only if the stabilisers vary continuously in G and $G^{(0)}/G$

is Hausdorff. Interestingly, we will show that this 'naive' generalisation fails and that characterising the groupoid C^*-algebras with Hausdorff spectrum requires a third condition. Furthermore, the correct generalisation, presented in Section 2 as Theorem 2.11, is in some ways stronger than the results of [14], even for transformation groups. We finish the paper by providing some further examples in Section 3. In addition, we also prove that, unlike the T_0 or T_1 case, in the Hausdorff case working with the dual stabiliser groupoid is necessary.

Before we begin we should review some preliminary material. Throughout the paper we will let G denote a second countable, locally compact Hausdorff groupoid with a Haar system $\{\lambda_u\}$. We will use $G^{(0)}$ to denote the unit space, r to denote the range map, and s to denote the source map. We will let $S = \{\gamma \in G : s(\gamma) = r(\gamma)\}$ be the stabiliser, or isotropy, subgroupoid of G. Observe that on S the range and source maps are equal and that $r = s : S \to G^{(0)}$ gives S a bundle structure over $G^{(0)}$. Given $u \in G^{(0)}$, the fibre $S_u = r|_S^{-1}(u)$ is a group called the stabiliser subgroup at u. Since S is a closed subgroupoid of G, it is always second countable, locally compact and Hausdorff. However, S will have a Haar system if and only if the stabilisers vary continuously. That is, if and only if the map $u \mapsto S_u$ is continuous with respect to the Fell topology on closed subsets of S [13, Lemma 1.3].

One of the primary examples of groupoids consists of those built from transformation groups. If a second countable locally compact Hausdorff group H acts on a second countable locally compact Hausdorff space X then we can form the transformation groupoid $H \ltimes X$ in the usual fashion. The properties of the transformation groupoid are closely tied to those of the group action. For instance, the orbit space of the groupoid $H \ltimes X^{(0)}/H \ltimes X$ is homeomorphic to the orbit space of the action X/H. Furthermore, the stabiliser groups S_x of $H \ltimes X$ can be naturally identified with the stabiliser subgroups H_x of H and the stabilisers will vary continuously in $H \ltimes X$ if and only if they vary continuously in H.

Given a groupoid G, we can construct the groupoid C^*-algebra $C^*(G)$ as a universal completion of the convolution algebra $C_c(G)$ [12]. Of particular interest to us will be the spectrum $C^*(G)^\wedge$ of the groupoid algebra. One special case which will play a key role in our results is the spectrum of the stabiliser subgroupoid. We paraphrase the following results from [8, Section 3].

Proposition 1.1. *Let G be a second countable, locally compact Hausdorff groupoid with abelian stabiliser subgroups. If the stabilisers vary continuously then S has a Haar system and the groupoid C^*-algebra $C^*(S)$ is abelian. The spectrum of $C^*(S)$, denoted by $\widehat{S}$, is a second countable, locally compact Hausdorff space which is naturally fibred over $G^{(0)}$. Furthermore, the fibre of $\widehat{S}$ over $u \in G^{(0)}$, which we will write as $\widehat{S}_u$, is the Pontryagin dual of the fibre S_u. We refer to $\widehat{S}$ as the dual stabiliser groupoid.*

One of the things that makes $\widehat{S}$ so useful is that its topology is relatively well understood; [8] gives a complete description of the convergent sequences in $\widehat{S}$. Since we will use this characterisation quite a bit we have restated it below.

Proposition 1.2 [8, Proposition 3.3]. *Suppose that the groupoid G has continuously varying abelian stabilisers and that $\{\chi_n\}$ is a sequence in $\widehat{S}$ with $\chi_n \in \widehat{S}_{u_n}$ for all n. Given $\chi \in \widehat{S}_u$, we have $\chi_n \to \chi$ if and only if:*

(a) *$u_n \to u$ in $G^{(0)}$; and*
(b) *given $s_n \in S_{u_n}$ for all n and $s \in S_u$, if $s_n \to s$ then $\chi_n(s_n) \to \chi(s)$.*

The final thing we need to review is the notion of a groupoid action. A groupoid G can only act on spaces X which are fibred over $G^{(0)}$. If there is a surjective function $r_X : X \to G^{(0)}$ then we define a groupoid action via a map $\{(\gamma, x) : s(\gamma) = r_X(x)\} \to X$ such that for composable γ and η we have $\gamma \cdot (\eta \cdot x) = \gamma\eta \cdot x$. Among other things, this implies that $r_X(x) \cdot x = x$ for all $x \in X$ and $r_X(\gamma \cdot x) = r(\gamma)$. We will use the following three actions in this paper.

- Any groupoid G acts on $G^{(0)}$ by $\gamma \cdot u := \gamma u \gamma^{-1} = r(\gamma)$ for $u \in G^{(0)}$.
- Any groupoid G acts on S by $\gamma \cdot s := \gamma s \gamma^{-1}$ for $s \in S$.
- If S has abelian fibres which vary continuously then there is an action of G on $\widehat{S}$. For $\gamma \in G$ and $\chi \in \widehat{S}_{s(\gamma)}$ we define $\gamma \cdot \chi(s) := \chi(\gamma^{-1} s \gamma)$ for $s \in S_{r(\gamma)}$.

Given an action of G on a space X, we will use $G \cdot x$ to denote the orbit of x in X and $[x]$ to denote the corresponding element of X/G. We would also like to recall that the orbit space X/G is locally compact, but not necessarily Hausdorff, and that the quotient map $q : X \to X/G$ is open if G has a Haar system [7, Lemma 2.1].

2. Groupoid C^*-algebras with Hausdorff spectrum

As mentioned in the introduction, we would like to generalise the main result of [14], which has been restated below, from transformation groups to groupoids.

Theorem 2.1 [14, p. 320]. *Suppose that (H, X) is an abelian transformation group and that the maps of H/H_x onto $H \cdot x$ are homeomorphisms for each $x \in X$. Then the spectrum of the transformation group C^*-algebra $C^*(H, X)$ is Hausdorff if and only if the map $x \mapsto H_x$ is continuous with respect to the Fell topology and X/H is Hausdorff.*

Remark 2.2. The condition that the maps of H/H_x onto $H \cdot x$ are homeomorphisms for each $x \in X$ is one of the equivalent conditions in the Mackey–Glimm dichotomy [11]. Following [3], we will refer to groupoids and transformation groups which satisfy one, and hence all, of the conditions of the Mackey–Glimm dichotomy as *regular*.

Remark 2.3. An important question is how to generalise the hypothesis that the group H is abelian. The most natural replacement is to assume that the stabiliser subgroups S_u are abelian for all $u \in G^{(0)}$. Since, as we will see, the regularity hypothesis can be removed completely, we may conjecture that, given a groupoid G with abelian stabilisers, $C^*(G)$ will have Hausdorff spectrum if and only if the stabilisers vary continuously and $G^{(0)}/G$ is Hausdorff. However, we will find that this conjecture fails and the assumption that G has abelian stabilisers is a weaker condition, even for transformation groups.

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We begin by restating the following result.

LEMMA 2.4 [8, Proposition 3.1] . *Suppose that G is a second countable, locally compact Hausdorff groupoid with abelian stabilisers. If the spectrum $C^*(G)^\wedge$ is Hausdorff then the stabilisers vary continuously.*

Next consider the following useful lemma.

LEMMA 2.5. *Suppose that G is a second countable, locally compact Hausdorff groupoid with continuously varying abelian stabilisers. Then the following are equivalent:*

(a) *$C^*(G)$ has T_0 spectrum;*
(b) *$C^*(G)$ is GCR;*
(c) *$G^{(0)}/G$ is T_0.*

Furthermore, if any of these conditions hold then the map $[\gamma] \mapsto r(\gamma)$ from G_u/S_u to $G \cdot u$ is a homeomorphism for all $u \in G^{(0)}$ and G is regular.

PROOF. The groupoid algebra is separable since G is second countable. In this case the equivalence of the first two conditions follows from [9, Theorem 6.8.7]. Since the stabilisers are abelian, and therefore amenable and GCR, the equivalence of the second two conditions now follows from the main result of [1]. Finally, if $G^{(0)}/G$ is T_0 then it follows from [11] that the map $[\gamma] \mapsto r(\gamma)$ from G_u/S_u onto $G \cdot u$ is a homeomorphism for all $u \in G^{(0)}$ and hence G is regular in the sense of [3]. □

We may now use Lemmas 2.4 and 2.5, in conjunction with [3, Theorem 3.5], to conclude that if $C^*(G)^\wedge$ is Hausdorff then $C^*(G)^\wedge$ is homeomorphic to $\widehat{S}/G$. A brief argument shows that $G^{(0)}/G$ is homeomorphic to its image in $\widehat{S}/G$ equipped with the relative topology. Thus $G^{(0)}/G$ is Hausdorff if $C^*(G)^\wedge$ is Hausdorff. This proves the following generalisation of the forward direction of Theorem 2.1.

PROPOSITION 2.6. *Suppose that G is a second countable, locally compact Hausdorff groupoid with abelian stabilisers. If $C^*(G)$ has Hausdorff spectrum then the stabilisers vary continuously and $G^{(0)}/G$ is Hausdorff.*

Assuming that G has continuously varying stabilisers, the following proposition shows that a converse statement holds in the T_0 and T_1 case.

PROPOSITION 2.7. *Suppose that G is a second countable, locally compact Hausdorff groupoid with continuously varying abelian stabilisers. Then the spectrum $C^*(G)^\wedge$ is T_1 (respectively T_0) if and only if $G^{(0)}/G$ is T_1 (respectively T_0).*

PROOF. It follows from Lemma 2.5 that $C^*(G)^\wedge$ is T_0 if and only if $G^{(0)}/G$ is. Now suppose that $C^*(G)^\wedge$ is T_1. Then Lemma 2.5 and [3, Theorem 3.5] imply that $C^*(G)^\wedge$ is homeomorphic to $\widehat{S}/G$. As noted above, $G^{(0)}/G$ is homeomorphic to its image in $\widehat{S}/G$, and as such $G^{(0)}/G$ is T_1. Next suppose that $G^{(0)}/G$ is T_1. Again using Lemma 2.5 and [3, Theorem 3.5], we have $C^*(G)^\wedge \cong \widehat{S}/G$. Thus we will be done if we can show that $\widehat{S}/G$ is T_1.

Suppose that we are given elements $[\rho], [\chi] \in \widehat{S}/G$ such that $[\rho] \neq [\chi]$. Let $p : \widehat{S} \to G^{(0)}$ be the bundle map and $\tilde{p} : \widehat{S}/G \to G^{(0)}/G$ its factorisation. Set $[u] = \tilde{p}([\rho])$ and $[v] = \tilde{p}([\chi])$. Suppose that $[u] \neq [v]$. Since $G^{(0)}/G$ is T_1 we can find open sets U and V such that $[u] \in U$, $[v] \in V$ and $[u] \notin V$, $[v] \notin U$. Then $\tilde{p}^{-1}(U)$ is an open set containing $[\rho]$ and not $[\chi]$ and $\tilde{p}^{-1}(V)$ is an open set containing $[\chi]$ and not $[\rho]$. Next suppose that $[u] = [v]$. Since the fibres of S are abelian,

$$s \cdot \chi(t) = \chi(s^{-1}ts) = \chi(t) \quad \text{for all } s \in S. \tag{2.1}$$

Hence the action of G on S is trivial when fixed to a single fibre and we can assume without loss of generality that $\rho, \chi \in \widehat{S}_u$ with $\rho \neq \chi$. Let $q : \widehat{S} \to \widehat{S}/G$ be the quotient map and recall that it is open. Fix a neighbourhood U of ρ. If $\chi \notin G \cdot U$ then $[\chi] \notin q(U)$ and $q(U)$ separates $[\rho]$ from $[\chi]$. Now suppose that $\chi \in G \cdot U$ for all neighbourhoods U of ρ. Then for each U there exists $\gamma_U \in G$ and $\rho_U \in U$ such that $\rho_U = \gamma_U \cdot \chi$. If we direct ρ_U by decreasing U then it is clear that $\rho_U \to \rho$. This implies that $\gamma_U \cdot u = r(\gamma_U) = p(\rho_U) \to u$. Since G is regular, $[\gamma] \mapsto r(\gamma)$ is a homeomorphism and we must have $[\gamma_U] \to [u]$ in G_u/S_u. However, the quotient map on G_u/S_u is open so that we may pass to a subnet, relabel, and choose $r_U \in S_u$ such that $\gamma_U r_U \to u$. Using (2.1),

$$\gamma_U r_U \cdot \chi = \gamma_U \cdot \chi = \rho_U \to u \cdot \chi = \chi.$$

Thus $\rho = \chi$, which is a contradiction. It follows that we must have been able to separate $[\rho]$ from $[\chi]$. This argument is completely symmetric so that we can also find an open set around $[\chi]$ which does not contain $[\rho]$. It follows that $\widehat{S}/G$, and hence $C^*(G)^\wedge$, is T_1. □

Remark 2.8. The essential component of this proof is the argument that $\widehat{S}/G$ is T_1 if $G^{(0)}/G$ is T_1. If we could extend this to the Hausdorff case then we would have proven the converse to Proposition 2.6. Unfortunately there are topological obstructions, as we will see.

We start by recalling Green's famous example of a free group action that is not proper.

Example 2.9 [4]. The space $X \subset \mathbb{R}^3$ will consist of countably many orbits, with the points $x_0 = (0, 0, 0)$ and $x_n = (2^{-2n}, 0, 0)$ for $n \in \mathbb{N}$ as a family of representatives. The action of $\mathbb{R}$ on X is described by defining maps $\phi_n : \mathbb{R} \to X$ such that $\phi_n(s) = s \cdot x_n$. In particular, we let $\phi_0(s) = (0, s, 0)$ and, for $n \geq 1$,

$$\phi_n(s) = \begin{cases} (2^{-2n}, s, 0) & s \leq n \\ (2^{-2n} - (s-n)2^{-2n-1}, n\cos(\pi(s-n)), n\sin(\pi(s-n))) & n < s < n+1 \\ (2^{-2n-1}, s - 1 - 2n, 0) & s \geq n+1. \end{cases}$$

For instance, brief computations show that

$$2n + 1 \cdot (2^{-2n}, 0, 0) = (2^{-2n-1}, 0, 0) \tag{2.2}$$

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for all n. It is straightforward to observe that the orbit space $X/\mathbb{R}$ is homeomorphic to the subset $\{x_n\}_{n=0}^{\infty}$ of $\mathbb{R}^3$.

We may now build an example of a transformation groupoid G with continuously varying abelian stabilisers such that $G^{(0)}/G$ is Hausdorff and $\widehat{S}/G$ is not.

Example 2.10. Let $\mathbb{R}$ act on X as in Example 2.9. Now restrict this action to the action of $\mathbb{Z}$ on the subset $Y = \{\phi_n(m) : n \in \mathbb{N}, m \in \mathbb{Z}\}$. Let $H = \mathbb{Q}_D \rtimes_\phi \mathbb{Z}$ be the semidirect product, where $\mathbb{Q}_D$ denotes the rationals equipped with the discrete topology and where we define

$$\phi(n)(r) = r2^n$$

for all $n \in \mathbb{Z}$ and $r \in \mathbb{Q}$. It is easy to show that ϕ is a homomorphism from $\mathbb{Z}$ into the automorphism group of $\mathbb{Q}_D$. Thus H is a locally compact Hausdorff group which is second countable because it is a countable discrete space. Recall that the group operations are given by

$$(q, n)(p, m) = (q + 2^n p, n + m) \quad (q, n)^{-1} = (-2^{-n}q, -n).$$

Let the second factor of H act on Y as in Example 2.9. In other words, let $(q, n) \cdot x := n \cdot x$. It is straightforward to show that this is a continuous group action. It follows that the transformation groupoid $G = H \ltimes Y$ is a second countable, locally compact Hausdorff groupoid with a Haar system. Furthermore, the stabiliser subgroup of H at x is $H_x = \{(q, 0) : q \in \mathbb{Q}\}$ for all $x \in Y$. Since $(q, 0)(r, 0) = (q + r2^0, 0) = (q + r, 0)$, the stabilisers are abelian, and since the stabilisers are also constant they must vary continuously in both H and G. It will be important for us to observe that S is isomorphic to $\mathbb{Q}_D \times Y$ via the map $((q, 0), x) \mapsto (q, x)$. Finally, $\{x_n\}_{n=0}^{\infty}$ forms a set of representatives for the orbit space and it is not difficult to show that Y/G is actually homeomorphic to $\{x_n\}_{n=0}^{\infty}$ and is therefore Hausdorff.

To show that $\widehat{S}/G$ is not Hausdorff we must first compute the dual. Since S is isomorphic to $\mathbb{Q}_D \times Y$ we can identify $\widehat{S}$ with $\widehat{\mathbb{Q}_D} \times Y$. While $\widehat{\mathbb{Q}_D}$ is fairly mysterious we do know that since $\hat{r}(s) = e^{irs}$ is a character on $\mathbb{R}$ for all $r \in \mathbb{R}$ it must also be a character on $\mathbb{Q}_D$. Now suppose that $((q, n), x) \in G$ and $(\hat{r}, -n \cdot x) \in \widehat{\mathbb{Q}_D} \times Y$. We have

$$\begin{aligned} ((q, n), x) \cdot (\hat{r}, -n \cdot x)(p, x) &= (\hat{r}, -n \cdot x)(((q, n), x)^{-1}((p, 0), x)((q, n), x)) \\ &= (\hat{r}, -n \cdot x)((-2^{-n}q, -n)(p, 0)(q, n), -n \cdot x) \\ &= (\hat{r}, -n \cdot x)((2^{-n}p, 0), -n \cdot x) \\ &= e^{irp2^{-n}} = (\widehat{2^{-n}r}, x)(p, x). \end{aligned}$$

Or, more succinctly,

$$((q, n), x) \cdot (\hat{r}, -n \cdot x) = (\widehat{2^{-n}r}, x). \tag{2.3}$$

Next let $\gamma_n = ((0, 2n + 1), (2^{-2n-1}, 0, 0))$ for all n. Using the inverse of (2.2),

$$r(\gamma_n) = (2^{-2n-1}, 0, 0) \quad \text{and} \quad s(\gamma_n) = (2^{-2n}, 0, 0).$$

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If we set $\chi_n = (\hat{1}, (2^{-2n}, 0, 0))$ then clearly $\chi_n \to \chi = (\hat{1}, (0, 0, 0))$. Using (2.3), we compute $\gamma_n \cdot \chi_n = (\widehat{2^{-(2n+1)}}, (2^{-2n-1}, 0, 0))$. A quick calculation shows that $\gamma_n \cdot \chi_n \to \omega = (\hat{0}, (0, 0, 0))$. Hence $[\chi_n] \to [\chi]$ and $[\chi_n] \to [\omega]$. Since the action of G is trivial on fixed fibres this implies that $\widehat{S}/G$, and hence $C^*(G)^\wedge$, is not Hausdorff. Thus the converse to Proposition 2.6 is false.

The previous example shows that we will need to introduce an additional hypothesis to generalise Theorem 2.1. The appropriate condition is given below and forms the main result of the paper.

Theorem 2.11. *Suppose that G is a second countable, locally compact Hausdorff groupoid with a Haar system and abelian stabilisers. Then $C^*(G)$ has Hausdorff spectrum if and only if the following conditions hold:*

(a) *the stabilisers vary continuously, that is, $u \mapsto S_u$ is continuous with respect to the Fell topology;*
(b) *the orbit space $G^{(0)}/G$ is Hausdorff; and*
(c) *given sequences $\{\chi_i\} \subset \widehat{S}$ and $\{\gamma_i\} \subset G$ with $\chi_i \in \widehat{S}_{s(\gamma_i)}$, if $\chi_i \to \chi$ and $\gamma_i \cdot \chi_i \to \omega$ such that χ and ω are in the same fibre then $\chi = \omega$.*

Remark 2.12. Even in the case of transformation groups Theorem 2.11 is in some ways stronger than Theorem 2.1. The main advantage is that we only require the stabiliser groups to be abelian, and not the whole group. Furthermore, we also removed the regularity hypothesis. The price is that we have added a slightly technical condition that, while not easy to say, is simple enough to check in practice.

Proof. It follows from Proposition 2.6, and its proof, that if $C^*(G)^\wedge$ is Hausdorff then conditions (a) and (b) hold and $\widehat{S}/G$ is Hausdorff. Now suppose that we have $\chi_i \to \chi$ and $\gamma_i \cdot \chi_i \to \omega$ as in condition (c). Then $[\chi_i] \to [\chi]$ and $[\chi_i] \to [\omega]$. Since $\widehat{S}/G$ is Hausdorff this implies $[\omega] = [\chi]$. However, χ and ω live in the same fibre and the action of G on a fixed fibre is free by (2.1) so that $\chi = \omega$.

Now suppose that conditions (a)–(c) are satisfied. Then the first two conditions, together with Lemma 2.5 and [3, Theorem 3.5], imply that $C^*(G)^\wedge$ is homeomorphic to $\widehat{S}/G$. Now suppose that $[\chi_i] \to [\chi]$ and $[\chi_i] \to [\omega]$ in $\widehat{S}/G$. Using the fact that the quotient map is open we can pass to a subsequence, relabel, and choose new representatives χ_i so that $\chi_i \to \chi$. As before, let $p : \widehat{S} \to G^{(0)}$ be the bundle map and let $\tilde{p} : \widehat{S}/G \to G^{(0)}/G$ be the natural factorisation. Define $u_i = p(\chi_i)$ and $u = p(\chi)$ and observe that $[u_i] \to [u]$. Furthermore, if $p(\omega) = v$ then $[u_i] \to [v]$ as well. Since $G^{(0)}/G$ is Hausdorff we have $[u] = [v]$ and we may assume, without loss of generality, that $u = v$. Now pass to a subsequence again, relabel, and find $\gamma_i \in G$ such that $\gamma_i \cdot \chi_i \to \omega$. These sequences satisfy the hypothesis of (c), so $\omega = \chi$. It follows that $[\omega] = [\chi]$ and that $\widehat{S}/G$, and hence $C^*(G)^\wedge$, is Hausdorff. □

It should be noted that there are a variety of situations in which condition (c) is guaranteed to hold.

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PROPOSITION 2.13. *Let G be a second countable, locally compact Hausdorff groupoid with continuously varying abelian stabilisers. Then condition* (c) *of Theorem 2.11 automatically holds if G satisfies any of the following:*

(a) *$G = H \ltimes X$ is an abelian transformation groupoid;*
(b) *G is principal;*
(c) *G is proper;*
(d) *G is Cartan; or*
(e) *G is transitive.*

PROOF. Let $\chi_i \to \chi$ and $\gamma_i \cdot \chi_i \to \omega$ be as in condition (c) of Theorem 2.11. Set $u_i = s(\gamma_i)$, $v_i = r(\gamma_i)$, $u = p(\chi) = p(\omega)$ and observe that $u_i \to u$ and $v_i \to u$. Now suppose that $G = H \ltimes X$ where H is abelian. Then we must have $\gamma_i = (t_i, v_i)$ with $u_i = t_i^{-1} \cdot v_i$. Given s in the stabiliser subgroup H_u, we can use the fact that the stabilisers vary continuously to pass to a subsequence, relabel, and find $s_i \in H_{u_i}$ such that $s_i \to s$ in H. Consequently $(s_i, u_i) \to (s, u)$ and, by Proposition 1.2, $\chi_i(s_i, u_i) \to \chi(s, u)$. On the other hand, since the group is abelian, we also have $s_i \in H_{v_i} = H_{t_i \cdot u_i}$ for all i. It follows that $(s_i, v_i) \to (s, u)$ in S and therefore

$$(t_i, v_i) \cdot \chi_i(s_i, v_i) = \chi_i(t_i^{-1} s_i t_i, u_i) = \chi_i(s_i, u_i) \to \omega(s, u).$$

Hence $\chi = \omega$ and condition (c) automatically holds for abelian transformation groups.

Moving on, condition (c) trivially holds if G is principal. For the next two conditions observe the following. Suppose that we can pass to a subsequence, relabel, and find $\gamma \in G$ such that $\gamma_i \to \gamma$. It follows that $\gamma_i \cdot \chi_i \to \gamma \cdot \chi$ and therefore $\gamma \cdot \chi = \omega$. However, the range and source maps are continuous so we must have $r(\gamma) = s(\gamma) = u$ and hence $\gamma \in S_u$. The fibres of S are abelian so that, by (2.1), $\omega = \gamma \cdot \chi = \chi$. Thus it will suffice to show that we can prove γ_i has a convergent subsequence. However, if G is either proper or Cartan then this follows almost by definition.

Finally, suppose that G is transitive. Since G is also second countable, [6, Theorem 2.2] implies that the map $\gamma \mapsto (r(\gamma), s(\gamma))$ is open. Thus we can pass to a subsequence, relabel, and find $\eta_i \in G$ such that $r(\eta_i) = v_i$, $s(\eta_i) = u_i$ and $\eta_i \to u$. Observe that $\eta_i^{-1}\gamma_i \in S_{u_i}$ for all i so that $\gamma_i \cdot \chi_i = \eta_i \cdot (\eta_i^{-1}\gamma_i \cdot \chi_i) = \eta_i \cdot \chi_i$. Thus $\gamma_i \cdot \chi_i = \eta_i \cdot \chi_i \to u \cdot \chi = \chi$. It follows that $\chi = \omega$ and condition (c) holds in this case as well. □

3. Examples and duality

In this section we would like to begin by applying Theorem 2.11 to several examples.

EXAMPLE 3.1. Let $H = \mathrm{SO}(3, \mathbb{R})$, $X = \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ and let H act on X by rotation. It is clear that H is not abelian, and therefore we cannot apply Theorem 2.1. However, it does have abelian stabiliser subgroups. Given a vector $v \in X$, it is easy to see that S_v is the set of rotations about the line described by v. In particular, this is isomorphic to the circle group and is therefore abelian. What is more, some computations show that the

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stabilisers vary continuously and that the stabiliser subgroupoid S is homeomorphic to $X \times \mathbb{T}$. This in turn implies that the dual groupoid is homeomorphic to $X \times \mathbb{Z}$. Now suppose that $(U_i^{-1}v_i, \chi_i) \to (v, \chi)$ and $(U_i, v_i) \cdot (U_i^{-1}v_i, \chi_i) \to (v, \omega)$ as in condition (c). Given $\theta_i \to \theta$ in $\mathbb{T}$, we have from Proposition 1.2 that

$$(U_i^{-1}v_i, \chi_i)(U_i^{-1}v_i, \theta_i) = \chi_i(\theta_i) \to (v, \chi)(v, \theta) = \chi(\theta).$$

Using the fact that conjugating rotation about an axis w by $V \in H$ gives us the corresponding rotation about Vw, we also have

$$(U_i, v_i) \cdot (U_i^{-1}v_i, \chi_i)(v_i, \theta_i) = (U_i^{-1}v_i, \chi_i)(U_i^{-1}v_i, \theta_i) = \chi_i(\theta_i) \to (v, \omega)(v, \theta) = \omega(\theta).$$

It follows that $\chi = \omega$ and condition (c) of Theorem 2.11 holds. Finally, the orbit space X/H is homeomorphic to the open half-line and is therefore Hausdorff. Thus we can conclude that $C^*(H \ltimes X)$ has Hausdorff spectrum. In fact [3, Theorem 3.5] shows that $C^*(H \ltimes X)$ is homeomorphic to $\widehat{S}/(H \ltimes X) = (0, \infty) \times \mathbb{Z}$.

Example 3.2. Let E be a row finite directed graph with no sources. Recall that we can build the graph groupoid G as in [5]. Elements of G are triples (x, n, y) where x and y are infinite paths which are shift equivalent with lag n, and elements of $G^{(0)}$ are infinite paths. (We will be using the Raeburn convention [10] for path composition.) It is known that the groupoid C^*-algebra $C^*(G)$ is isomorphic to the graph C^*-algebra. Let us consider the conditions of Theorem 2.11. First, the stabilisers are all subgroups of $\mathbb{Z}$ and hence abelian. Furthermore, the groupoid G will have nontrivial stabilisers if and only if there exists an infinite path which is shift equivalent to itself. In other words, if and only if there is a cycle. Suppose that a cycle on the graph has an entry. Let x be the path created by following the cycle an infinite number of times. For each $i \in \mathbb{N}$, let x_i be the path which, at its head, follows the cycle i times and then has a noncyclic tail leading off from the entry. Because x_i eventually agrees with x on any finite segment we have $x_i \to x$. However, none of the x_i are cycles so that S_{x_i} is trivial for all i. On the other hand, $S_x \cong n\mathbb{Z}$ where n is the length of the cycle. Thus the stabilisers do not vary continuously. This shows that in order for the stabilisers to vary continuously no cycles in the graph can have entries. A similar argument shows that the converse holds as well.

For the second condition we require that the orbit space $G^{(0)}/G$ be Hausdorff. In this case the orbit space is the space of shift equivalence classes. Recall that the basic open sets in $G^{(0)}$ are the cylinder sets V_a. More specifically, a is a finite path and V_a is the set of all infinite paths which are initially equal to a. Given $[x] \in G^{(0)}/G$, we will have $x \in G \cdot V_a$ if and only if x is shift equivalent to a path with initial segment a. This is equivalent to there being a path from any vertex on x to the source of a. Conversely, $y \notin G \cdot V_a$ if and only if there is no path from any vertex on y to the source of a. Using these facts, it follows from a brief argument that $G^{(0)}/G$ will be Hausdorff if and only if, given nonshift equivalent paths x and y, there exist vertices u and v such that there is a path from a vertex on x to u, a path from a vertex on y to v, and no vertex w which has a path to both u and v.

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Finally, for the third condition we observe that, given $(y, n, x) \in G$, $(y, m, y) \in S$ and $\chi \in \widehat{S}_x$,

$$(y, n, x) \cdot \chi(y, m, y) = \chi((x, -n, y)(y, m, y)(y, n, x)) = \chi(x, m, x). \tag{3.1}$$

Now suppose that $\chi_i \to \chi$ and $(y_i, n_i, x_i) \cdot \chi_i \to \omega$ in $\widehat{S}$ with $\chi, \omega \in \widehat{S}_x$. Notice that this implies that we must have $x_i \to x$ and $y_i \to x$ in $G^{(0)}$. Let $(x, n, x) \in S_x$. Then $(x_i, n, x_i) \to (x, n, x)$ and, by Proposition 1.2, $\chi_i(x_i, n, x_i) \to \chi(x, n, x)$. On the other hand, we also know that $(y_i, n, y_i) \to (x, n, x)$ so that, using (3.1) and Proposition 1.2,

$$(y_i, n_i, x_i) \cdot \chi_i(y_i, n, y_i) = \chi_i(x_i, n, x_i) \to \omega(x, n, x).$$

This implies that $\chi(x, n, x) = \omega(x, n, x)$. Hence $\chi = \omega$ and condition (c) is automatically satisfied. Put together this shows that the graph groupoid algebra, and therefore the graph algebra, will have Hausdorff spectrum if and only if:

- no cycle has an entry; and
- given nonshift equivalent paths x and y, we can find vertices u and v such that there is a path from a vertex on x to u, a path from a vertex on y to v, and no vertex w which has a path to both u and v.

Remark 3.3. One annoyance of Theorem 2.11 is that condition (c) requires us to deal with the dual stabiliser groupoid. Using the same technique as the proof of Theorem 2.11, one can show that if $G^{(0)}/G$ is Hausdorff and if condition (c) holds for sequences in S (not $\widehat{S}$) then S/G is Hausdorff. If $\widehat{S}/G$ were Hausdorff whenever S/G is Hausdorff then one could verify (c) on S instead of $\widehat{S}$. This would allow us to avoid the use of $\widehat{S}$ altogether.

As in the previous section, let us first consider the T_0 and T_1 case. Using the fact that $\widehat{\widehat{S}} = S$ [2], as well as the topological argument given in Proposition 2.7, one can prove the following result.

Proposition 3.4. *Let G be a second countable, locally compact Hausdorff groupoid with continuously varying abelian stabilisers. Then either $G^{(0)}/G$, S/G, and $\widehat{S}/G$ are all T_1 (respectively T_0) or none of them is T_1 (respectively T_0).*

Unfortunately, again similar to the previous section, this proposition does not extend to the Hausdorff case, as we demonstrate below.

Example 3.5. Let H, Y and G be as in Example 2.10. Recall that we have already shown that in this case $\widehat{S}/G$ is not Hausdorff. The computations from Example 2.10 also show that condition (c) does not hold on $\widehat{S}$. Now we will show that S/G is Hausdorff and that S does satisfy condition (c). First, given $((q, n), y) \in G$ and $(r, x) \in S$, a computation similar to the one preceding (2.3) shows that

$$((q, n), y) \cdot (r, x) = (r2^n, y). \tag{3.2}$$

Suppose that $[s_i] \to [s]$ and $[s_i] \to [t]$ in S/G. Since Y/G is Hausdorff we can follow the same argument given in Theorem 2.11 to pass to subsequences, choose

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new representatives, and find $\gamma_i \in G$ so that $s_i \to s$ and $\gamma_i \cdot s_i \to t$ where $s, t \in S_u$. In particular, this implies that $s = (r, u)$ and $t = (q, u)$ for $r, q \in \mathbb{Q}$. Suppose that $s_i = (r_i, x_i)$ and $\gamma_i = ((p_i, n_i), y_i)$. Then it follows from (3.2) that $\gamma_i \cdot s_i = (r_i 2^{n_i}, y_i)$. Hence $r_i \to r$ and $r_i 2^{n_i} \to q$. However, we gave $\mathbb{Q}_D$ the discrete topology so that, eventually, $q = 2^{n_i} r_i = 2^{n_i} r$. Now, if either $r = 0$ or $q = 0$ then $s = t$. If $r, q \neq 0$ we know that eventually $n_i = n = \log_2(q/r)$. We may as well pass to a subnet and assume that this is always true. Then $n_i \cdot x_i \to n \cdot x$ and, since $n_i \cdot x_i = \gamma_i \cdot x_i = y_i \to x$, we have $n \cdot x = x$. The action of $\mathbb{Z}$ is free so that $n = 0$. Thus $\log_2(q/r) = 0$ and $q = r$. It follows that $s = t$ and that S/G is Hausdorff. This demonstrates that $\widehat{S}/G$ is not necessarily Hausdorff if S/G is Hausdorff. Using the fact that $\widehat{\widehat{S}} = S$ [2], the reverse implication fails as well. What is more, the above argument also shows that condition (c) holds for sequences in S. Hence it is not enough to verify (c) on S and working with the dual is necessary.

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