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A Characterization of Minimal Legendrian Submanifolds in $\mathbb{S}^{2n+1}$

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Abstract. Let $x\colon L^n \to \mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2}$ be a minimal submanifold in $\mathbb{S}^{2n+1}$. In this note, we show that L is Legendrian if and only if for any $A \in su(n+1)$ the restriction to L of $\langle Ax, \sqrt{-1}x\rangle$ satisfies $\Delta f = 2(n+1)f$. In this case, $2(n+1)$ is an eigenvalue of the Laplacian with multiplicity at least $\frac{1}{2}(n(n+3))$. Moreover if the multiplicity equals to $\frac{1}{2}(n(n+3))$, then L^n is totally geodesic.

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1. Introduction

Let $x\colon L^n \to \mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ be an embedded submanifold in the standard unit sphere $\mathbb{S}^{2n+1}$. Let $\langle\cdot,\cdot\rangle$ be the standard inner product on $\mathbb{R}^{2n+2}$. For any

$$x = (x_1, y_1, x_2, y_2, \ldots, x_{n+1}, y_{n+1}),$$

we identify it with the constant vector field

$$\sum_{j=1}^{n+1}\left(x_j\frac{\partial}{\partial x_j} + y_j\frac{\partial}{\partial y_j}\right).$$

By this identification, x, the position vector can be seen as the normal vector field of $\mathbb{S}^{2n+1}$. Let J be the standard complex structure, i.e., $J\partial/\partial x_j = \partial/\partial y_j$ and $J\partial/\partial y_j = -(\partial/\partial x_j)$. The standard contact structure ξ on $\mathbb{S}^{2n+1}$ is determined by the distribution

$$\xi = \{Y \in T_x\mathbb{S}^{2n+1} | \langle Y, Jx\rangle = 0\}.$$

Clearly, Jx is the corresponding Reeb vector field. A submanifold $L^n \to \mathbb{S}^{2n+1}$ is

⋆Current Address: Max-Planck Institute for Mathematics in the Sciences, Leipzig 04103, Germany, e-mail: gwang@mis.mpg.de called *Legendrian* if $T_xL \subset \xi$ for any $x \in L$. It is easy to see that L is Legendrian if and only if $JTL \oplus \{Jx\}$ is the normal bundle of the embedding $L \to \mathbb{S}^{2n+1}$.

Let CL be a cone in $\mathbb{C}^{n+1}$ over L. In other words, L is the link of CL with S^{2n+1}. It is easy to check that L is minimal Legendrian submanifold if and only if CL is a special Lagrangian cone w.r.t. some constant calibrated form (see, for instance, [5] and [6]). Such special Lagrangian cones are possible tangent cones of special Lagrangian varieties. In order to study the regularity of special Lagrangian varieties, we have to understand (or classify) all possible special Lagrangian cones, thus minimal Legendrian submanifolds in $\mathbb{S}^{2n+1}$. In this note, we give a characterization of minimal Legendrian submanifolds.

THEOREM 1.1. *Let $x: L^n \to \mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2}$ be a minimal submanifold. L is Legendrian if and only if for any matrix $M \in su(n+1)$, the function $f_M := \langle Mx, Jx \rangle$, as a function on L, satisfies $\Delta f = (n+2)f$. Here Δ is the (positive) Laplacian w.r.t. the induced metric.*

THEOREM 1.2. *If L is minimal Legendrian submanifold in S^{2n+1}, then $2(n+1)$ is an eigenvalue of the Laplacian with multiplicity at least $\frac{1}{2}(n(n+3))$, the dimension of $su(n+1)/so(n+1)$. If the multiplicity equals to $\frac{1}{2}(n(n+3))$, then L is totally geodesic.*

We hope that this characterization can be used to construct minimal Legendrian submanifolds in $\mathbb{S}^{2n+1}$, as in [1] and [7] for minimal submanifolds in the unit sphere. When $n = 2$, generalized Clifford tori are minimal Legendrian submanifolds. More minimal Legendrian tori with invariance under an $\mathbb{S}^1$ action was constructed in [5].

Throughout this note we will adopt the following ranges of indices:

$$1 \leqslant A, B, C \cdots \leqslant 2n+1,$$
$$1 \leqslant i, j \cdots \leqslant n,$$
$$n+1 \leqslant \alpha \leqslant 2n+1,$$

2. Minimal Legendrian Submanifolds

Let $x: L^n \to \mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ be an embedded submanifold in $\mathbb{S}^{2n+1}$. Let $e_1, e_2, \ldots, e_{2n+1}$ be an orthonormal frame of tangent vectors to $\mathbb{S}^{2n+1}$ at x and $\theta_1, \theta_2, \ldots, \theta_{2n+1}$ be the dual frame. We have (see [1, 2])

$$\mathrm{d}x = \theta_A \otimes e_A \tag{2.1}$$

and

$$\mathrm{d}e_A = \omega_{AB} \otimes e_B - \theta_A \otimes e_A, \tag{2.2}$$

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where

$$\omega_{AB} + \omega_{BA} = 0.$$

Exterior differentiation of (2.2) gives

$$\mathrm{d}\omega_{AB} - \omega_{AB} - \omega_{AC} \wedge \omega_{CB} = -\omega_A \wedge \omega_B. \tag{2.3}$$

Here we have used the summation convention. It is well-known that L is minimal if and only if for any constant vector $a \in \mathbb{R}^{2n+2}$, $\langle x, a\rangle$, as a function on L, satisfies

$$\Delta\langle x, a\rangle = n\langle x, a\rangle, \tag{2.4}$$

where Δ is the (positive) Laplacian operator w.r.t. the induced metric, see [1]. In this note we characterize the Legendrian property of minimal submanifolds in terms of the embedding x and su$(n+1)$. A matrix $M \in \mathrm{su}(n+1)$, as a real one, means that M satisfies

$$M + M^t = 0 \quad \text{and} \quad JM = MJ$$

and

$$MJ \text{ is traceless.} \tag{2.5}$$

We first have

PROPOSITION 2.1. *Let $x: L \to \mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ be a minimal submanifold in $\mathbb{S}^{2n+1}$. If L is Legendrian, then for any matrix $M \in \mathrm{su}(n+1)$ the function $f_M := \langle Mx, Jx\rangle$, as a function on L, satisfies*

$$\Delta f_M = 2(n+1) f_M. \tag{2.6}$$

Proof. By definition, L is Legendrian if and only if in any small neighborhood of $x \in L$, there is a orthonormal frame $e_1, e_2, \ldots, e_n$ of L so that $e_1, e_2, \ldots, e_n, e_{n+1} = Je_1, e_{n+2} = Je_2, \ldots, e_{2n} = Je_n, e_{2n+1} = Jx$ is an orthonormal frame of $\mathbb{S}^{2n+1}$. Let $\theta_1, \theta_2, \ldots, \theta_{2n+1}$ be the dual frame. On L, $\theta_\alpha = 0$ for $\alpha = n+1, \ldots, 2n, 2n+1$.

From (2.1) and (2.2), on L we have

$$\mathrm{d}x = \theta_i \otimes e_i \tag{2.7}$$

and

$$\mathrm{d}e_A = \theta_{AB} \otimes e_B - \theta_A \otimes x. \tag{2.8}$$

Here $\theta_{AB} = x^*\omega_{AB}$. The θ_{ij} are connection forms of the induced metric on L and $\theta_{\alpha i}$ are the second fundamental forms. Let $\theta_{\alpha i} = h_{\alpha ij}\theta_j$. For any matrix $M \in \mathrm{su}(n+1)$, define $f_M = \langle Mx, Jx\rangle$. In view of (2.7) and (2.8), we have

$$\mathrm{d}f_M = (\langle Me_i, Jx\rangle + \langle Mx, Je_i\rangle)\theta_i =: f_i\theta_i \tag{2.9}$$

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and

$$Df_i = -(2\langle Me_i, Je_j\rangle - 2\delta_{ij}\langle Mx, Jx\rangle - (\langle Me_\alpha, Jx\rangle + \langle Mx, Je_\alpha\rangle)h_{\alpha ij})\theta_j.$$

Together with the minimality of M, it follows

$$\Delta f = 2nf - 2\langle Me_i, Je_i\rangle.$$

Since $M \in su(n+1)$, form (2.5) and a simple fact $\langle Me_{n+i}, Je_{n+i}\rangle = \langle Me_i, Je_i\rangle$, we have

$$\langle Me_i, Je_i\rangle + \langle Mx, Jx\rangle = 0, \tag{2.10}$$

for any $M \in su(n+1)$. Hence, we have $\Delta f = 2(n+1)f$. □

Conversely, we have

PROPOSITION 2.2. *Let* $x\colon L \to \mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ *be a minimal submanifold in* $\mathbb{S}^{2n+1}$. *If for any matrix* $M \in \mathrm{su}(n+1)$ *the function* $f_M := \langle Mx, Jx\rangle$ *satisfies (2.6), then L is Legendrian.*

Proof. In order to prove this Proposition, we have to show that there is an orthonormal basis $e_1, e_2, \ldots, e_n$ of L such that $e_1, e_2, \ldots, e_n, Je_1, Je_2, \ldots, Je_n, Jx$ is an orthonormal basis of S^{2n+1}. Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of L.

Case 1. n is odd. Set $p = (n+1)/2$ and $e_{n+1} = x$. Applying Lemma 6.13 in [4] to the simple $2p$-vector $\eta = e_1 \wedge e_2 \wedge \cdots e_n \wedge e_{n+1}$, we have a unitary basis $\bar e_1, J\bar e_1, \bar e_2, J\bar e_2, \ldots, \bar e_{n+1}, J\bar e_{n+1}$ for $\mathbb{C}^{n+1}$ and angles

$$0 \leqslant \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_{p-1} \leqslant \pi/2, \quad \theta_{p-1} \leqslant \theta_p \leqslant \pi$$

such that

$$\begin{aligned}\eta = {}& \bar e_1 \wedge (J\bar e_1 \cos\theta_1 + \bar e_2 \sin\theta_1) \wedge \bar e_3 \wedge (J\bar e_3 \cos\theta_2 + \bar e_4 \sin\theta_2) \wedge \cdots \\ &\wedge \bar e_{2p-1} \wedge (J\bar e_{2p_1} \cos\theta_p + \bar e_{2p} \sin\theta_p).\end{aligned} \tag{2.11}$$

We choose a new basis $\tilde e_1, \tilde e_2, \ldots, \tilde e_n$ of L such that

$$\tilde e_{2i-1} = \bar e_{2i-1} \text{ and } \tilde e_{2i} = J\bar e_{2i-1}\cos\theta_i + \bar e_{2i}\sin\theta_i, \quad \text{for } i = 1, \ldots, p.$$

In view of (2.10), we have

$$\sum_{j=1}^{n+1} \langle M\tilde e_j, J\tilde e_j\rangle = 0 \quad \text{for } M \in \mathrm{su}(n+1). \tag{2.12}$$

We claim that

$$\theta_1 = \theta_2 = \cdots = \theta_p = \pi/2. \tag{2.13}$$

For any fixed i, we first choose a matrix M_i satisfying

$$M_i\bar{e}_{2i-1} = \bar{e}_{2i},$$
$$M_i\bar{e}_{2i} = -\bar{e}_{2i-1},$$
$$M_i\bar{e}_k = 0, \quad \text{for } k \in \{1, 2, \ldots, n+1\}/\{2i-1, 2i\}.$$

It is clear that $M_i \in \mathrm{su}(n+1)$. Inserting such M_i into (2.12), we get

$$\sin 2\theta_i = 0. \tag{2.14}$$

Then we choose another $M \in \mathrm{su}(n+1)$ such that

$$M_i'\bar{e}_{2i-1} = J\bar{e}_{2i-1},$$
$$M_i'\bar{e}_{2i} = -J\bar{e}_{2i},$$
$$M_i'\bar{e}_k = 0, \quad \text{for } k \in \{1, 2, \ldots, n+1\}/\{2i-1, 2i\}.$$

Inserting it into (2.12), we get

$$1 + \cos^2\theta_i - \sin^2\theta_i = 0. \tag{2.15}$$

(2.14) and (2.15) imply that $\theta_i = \pi/2$. The claim follows. Thus $\tilde{e}_1 = \bar{e}_1$, $\tilde{e}_2 = \bar{e}_2, \cdots, \tilde{e}_n = \bar{e}_n$. This implies that L is Legendrian.

Case 2. n is even. Let $e_{n+1} = x$. Choose a new basis $e_1', e_2', \ldots, e_{n+1}'$ such that

$$\langle e_{n+1}', e_k\rangle = \langle e_{n+1}', Je_k\rangle = 0, \quad \text{fork} = 1, 2, \ldots, \text{n}.$$

The existence of such e_{n+1}' follows simply from that $n+1$ is odd. Set $2p = n$. Decompose $\mathbb{C}^{n+1} = \mathbb{C}^n \oplus \{e_{n+1}', Je_{n+1}'\}$. Now, we can apply Lemma 6.13 in [4] again to the simple $2p$-vector

$$\eta_1 = e_1' \wedge e_2' \wedge \cdots \wedge e_n'$$

to get a normal form as in Case 1. The similar argument shows that L is Legendrian in this case. □

Proof of Theorem 1.1. It follows from Propositions 2.1 and 2.2. □

Proof of Theorem 1.2. The proof is inspired by [9], see also [6]. Fix a point x_0 in L and let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of L in a neighborhood U of x_0. Since L is Legendrian, $e_1, Je_1, \ldots, e_n, Je_n, x, Jx$ is a unitary basis of $\mathbb{C}^{n+1}$ for any point $x \in U$. Denote $e_{n+i} = Je_i$ and $e_{2n+1} = Jx$.

Define a linear map $F\colon \mathrm{su}(n+1) \to C^\infty(L)$ by

$$F(M) = f_M = \langle Mx, Jx\rangle_{|L},$$

where $C^\infty(L)$ is the space of smooth functions. We want to show that the image

$F(su(n+1))$ of F is of dimension not less than $\frac{1}{2}(n(n+3))$. Considering the point $x_0 \in L$ as a vector as before, we define

$$K(x_0) = \left\{M \in \mathrm{su}(n+1)\,|\,Mx_0 \in JT_{x_0}L,\, M(T_{x_0}L) \subset JT_{x_0}L\right\}$$

and

$$P(x_0) = \left\{M \in \mathrm{su}(n+1)\,|\, \begin{array}{l} Mx_0 \in T_{x_0}L \oplus \{x_0\} \oplus \{Jx_0\}, \\ M(T_{x_0}L) \in T_{x_0}L \oplus \{x_0\} \oplus \{Jx_0\}. \end{array}\right\}.$$

It is easy to check that

$$\mathrm{su}(n+1) = P(x_0) \oplus K(x_0).$$

Now we claim that if $M \in K(x_0)$ satisfies $F(M) \equiv 0$, then $M = 0$. Assume that there is a matrix $M \in K(x_0)$ such that $F(M) = 0$, i.e., $f_M = 0$. By (2.9) we know

$$\langle Me_i, Jx\rangle + \langle Mx, Je_i\rangle = 0, \quad \text{on } U,$$

for any $i = 1, 2, \ldots, n$. Since $M \in \mathrm{su}(n+1)$, it follows that

$$\langle Mx, Je_i\rangle = \langle Me_i, Jx\rangle = 0, \quad \text{on } U \tag{2.16}$$

for $i = 1, 2, \ldots, n$. Thus $Mx_0 = 0$, for $Mx_0 \in JT_{x_0}L$. Exterior differentiation of (2.16) gives

$$\langle Me_i, Je_j\rangle + h_{\alpha ij}\langle Me_a, Jx\rangle = 0, \quad \text{on } U. \tag{2.17}$$

Since $M \in K(x_0)$, we have $\langle Me_{n+i}, Jx_0\rangle = \langle Me_i, x_0\rangle = 0$. We also have $h_{(2n+1)ij} = 0$ by Lemma 2.3 below. Hence, (2.17) implies that

$$\langle Me_i, Je_j\rangle = 0 \text{ at } x_0, \text{ for } i, j = 1, 2, \ldots, n.$$

which, in turn, implies that $M = 0$. This proves the claim. It is clear that the dimension of K is $\frac{1}{2}(n(n+3))$. Now the first statement of the Proposition follows from the claim.

If the multiplicity equals to $\frac{1}{2}(n(n+3))$, then from the argument above we know that, for any point $x_0 \in L$ and any $M \in P(x_0)$, $F(M) \equiv 0$ and (2.17) holds in any small neighborhood of x_0. In this case, we claim that

$$h_{\alpha ij} = 0. \tag{2.18}$$

By definition, for any $M \in P(x_0)$, $\langle Me_i, Je_j\rangle = 0$. It follows from (2.17) that

$$h_{\alpha ij}\langle Me_\alpha, Jx\rangle = 0. \tag{2.19}$$

For any k, choosing $M \in P$ such that $Me_k = x$ and inserting it into (2.19) we get that $h_{(n+k)ij} = 0$. This together with Lemma 2.3 below, implies the claim, i.e., L is totally geodesic. □

Clearly if L^n is totally geodesic in $\mathbb{S}^{2n+1}$ then L is a great sphere.

LEMMA 2.3. *Let L be a minimal Legendrian submanifold in* $\mathbb{S}^{2n+1}$. *We have*

$$h_{(2n+1)ij} = 0, \qquad \text{for any } i, j = 1, 2, \ldots, n.$$

Proof. It is a known fact. For convenience, we give a proof. From (2.6), we have

$$\mathrm{d}e_{2n+1} = J\mathrm{d}x = \theta_i \otimes Je_i,$$

which implies that $\theta_{(2n+1)i} = 0$, i.e., $h_{(2n+1)ij} = 0$. □

Remark. The dimension of $\mathrm{su}(n+1)/\mathrm{so}(n+1)$ is $\frac{1}{2}(n(n+3))$.

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