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THE VORONOI REGION OF E_6^*

R. T. WORLEY

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Abstract

The Voronoi region and covering radius of the lattice E_6^* are determined, and the normalised second moment is calculated, confirming the estimate given by Conway and Sloane.

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0. Introduction

For an n-dimensional lattice $\Lambda \subset \mathbb{R}^n$ the Voronoi region is the convex polytope

$$V_\Lambda = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leqslant |\mathbf{x} - \mathbf{l}| \text{ for all } \mathbf{l} \in \Lambda\}$$

where $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ denotes the length of $\mathbf{x}$. Conway and Sloane [2, 3] have investigated the Voronoi regions of certain special lattices, and calculated their second moments in connection with use of the lattices as quantizers. The lattice E_6 is of interest because it is the best packing lattice for spheres in $\mathbb{R}^6$ [1], and its dual E_6^* is the best known quantizer in 6 dimensions. Conway and Sloane calculated the second moment of E_6^* by Monte Carlo integration as its Voronoi region was not known. In this paper the Voronoi region is determined. Its automorphism group is transitive on the set of vertices. The vertices have distance $(8/9\sqrt{3})^{1/6}$ from the origin when the lattice is normalised to have determinant 1, so for spheres it provides a lattice covering in $\mathbb{R}^6$ of density $J_6 \cdot 8/9\sqrt{3} \simeq .5132\ldots \cdot J_6$, where J_6 is the volume of a sphere in $\mathbb{R}^6$ of radius 1. This is not quite as good as the lattice A_6^* which has covering density $.4936\ldots \cdot J_6$.

The determination of the Voronoi region V for E_6^* makes use of the quantizing algorithm of [3]. A polytope W containing V is defined, and its vertices calculated. The quantizing algorithm is applied to the vertices, showing the vertices lie in V, so $W = V$.

1. The lattice E_6^* and its isometries

Points of $\mathbb{R}^6$ will be represented by row vectors $\mathbf{x} = (x_1, \ldots, x_6)$. E_6^* is generated by the rows $\mathbf{e}_1, \ldots, \mathbf{e}_6$ of the matrix

$$\begin{bmatrix} 0 & 2\sqrt{3} & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -2 & 0 \\ 3 & \sqrt{3} & 0 & 0 & 0 & 0 \\ -1 & \sqrt{3} & 1 & -\sqrt{3} & 0 & 0 \\ -1 & \sqrt{3} & 0 & 0 & 1 & -\sqrt{3} \end{bmatrix}.$$

This is the generating matrix given in [3], except that $\mathbf{e}_4$ has been changed in sign and the lattice has been doubled in scale.

An isometry of the lattice is an automorphism of the lattice that preserves length. This makes it an automorphism of the Voronoi region. There are some obvious isometries of E_6^* that arise from its representation as a complex lattice depending on the cube root of unity. These are:

(i) central symmetry, given by C: $\mathbf{x} \to -\mathbf{x}$,

(ii) reflection in the plane $x_{2i} = 0$ (arising from complex conjugacy) given by M_i: $x_{2i} \leftarrow -x_{2i}$, $x_k \leftarrow x_k$ $(k \neq i)$ for $i = 1, 2, 3$,

(iii) transposition of two complex pairs, given by

$$T_{ij}\colon x_{2i-1} \leftrightarrow x_{2j-1},\ x_{2i} \leftrightarrow x_{2j},\ x_{2k-1} \leftarrow x_{2k-1},\ x_{2k} \leftarrow x_{2k}\ (k \neq i, j) \quad \text{for } 1 \leqslant i < j \leqslant 3,$$

(iv) multiplication of one coordinate pair by a complex cube root of unity, given by R_i: $x_{2i-1} \leftarrow -(x_{2i-1} - x_{2i}\sqrt{3})/2$, $x_{2i} \leftarrow -(x_{2i-1}\sqrt{3} + x_{2i})/2$, other x_k remaining fixed ($i = 1, 2, 3$).

The coordinates of points of E_6^* can be split into pairs (x_{2k-1}, x_{2k}) and the pairs that occur belong to the 2-dimensional lattice generated by $(1, \sqrt{3})$ and $(2, 0)$. R_k cycles $(-2, 0)$ to $(1, \sqrt{3})$ to $(1, -\sqrt{3})$ and back to $(-2, 0)$, while M_k leaves $(-2, 0)$ fixed and swaps $(1, \sqrt{3})$ with $(1, -\sqrt{3})$. By combining M_k and R_k either of $(1, \sqrt{3})$ or $(1, -\sqrt{3})$ may be kept fixed and the other pair interchanged. In all cases only one (x_{2k-1}, x_{2k}) position is altered in the point of E_6^*. Thus these obvious isometries move $(2, 0, 0, 0, 0, 0)$ to $\pm(2, 0, 0, 0, 0, 0)$, $\pm(-1, \pm\sqrt{3}, 0, 0, 0, 0)$, $\pm(0, 0, 2, 0, 0, 0)$, $\pm(0, 0, -1, \pm\sqrt{3}, 0, 0)$, $\pm(0,0,0,0,2,0)$, $\pm(0,0,0,0,-1,\pm\sqrt{3})$.

There is a further isometry of E_6^* which is of importance. The lattice vectors $\mathbf{u}_1 = (-1, \sqrt{3}, 2, 0, 2, 0)$ and $\mathbf{u}_2 = (0,0,0,0,3,-\sqrt{3})$ meet at an angle of $\pi/3$. Let U denote the plane spanned by $\mathbf{u}_1$, $\mathbf{u}_2$ and $U^\perp$ its orthogonal complement. Let S denote the isometry of $\mathbb{R}^6$ which keeps $U^\perp$ fixed and rotates U by an angle of $2\pi/3$. If we write $\mathbf{x}$ as

$$\mathbf{x} = a\mathbf{u}_1 + b\mathbf{u}_2 + \mathbf{v} \quad \text{where } \mathbf{v} \in U^\perp$$

then the image $\mathbf{x}'$ of $\mathbf{x}$ under S is given by

$$\mathbf{x}' = a(\mathbf{u}_2 - \mathbf{u}_1) - b\mathbf{u}_1 + \mathbf{v}.$$

Taking inner products of $\mathbf{x}$ with $\mathbf{u}_1$, $\mathbf{u}_2$, solving to find a and b, we have

$$\mathbf{x}' = \mathbf{x} - \tfrac{1}{6}\{(\mathbf{x}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{x}\cdot\mathbf{u}_2 - \mathbf{x}\cdot\mathbf{u}_1)\mathbf{u}_2\}.$$

For the basis $\mathbf{e}_1, \ldots, \mathbf{e}_6$ of E_6^* we find that $\mathbf{e}_i \cdot \mathbf{u}_1$, $\mathbf{e}_i \cdot \mathbf{u}_2$ are integer multiples of 6, so $\mathbf{e}_i' = \mathbf{e}_i -$ (point in E_6^*). Thus S maps E_6^* into itself. Since it is a rotation, it is an isometry of E_6^*. It moves $(2,0,0,0,0,0)$ into $(5, \sqrt{3}, 2, 0, -1, \sqrt{3})/3$, so in contrast to the obvious isometries it does not keep the complex coordinate pairs separate.

The reader may be prompted to consider rotating U by $\pi/3$: unfortunately the result is not an automorphism of E_6^*. Likewise, although $(2,0,-2,0,0,0)$ and $(2,0,0,0,-2,0)$ meet at an angle of $\pi/3$, rotations of multiples of π are needed to produce automorphisms of E_6^*.

2. The action of S on vectors of E_6^* of length 2

The rotation S described above is of interest because it fixes many vectors of length $2\sqrt{2}$. The table below gives the action of S on various vectors of length $2\sqrt{2}$ for future reference.

Note that in the table the distinction between $\mathbf{x}$ and $-\mathbf{x}$ has been dropped. This is because in applications both $\mathbf{x}$ and $-\mathbf{x}$ determine the same pair of faces. For convenience vectors have been written with the first nonzero component positive.

TABLE 1

no.	vector	orbit under S	
1	$(1,\sqrt{3},-1,\pm\sqrt{3},0,0)$	fixed	
2	$(1,\sqrt{3},0,0,-1,-\sqrt{3})$	fixed	
3	$(1,-\sqrt{3},2,0,0,0)$	fixed	
4	$(2,0,1,\pm\sqrt{3},0,0)$	fixed	
5	$(2,0,0,0,1,\sqrt{3})$	fixed	
6	$(0,0,1,\pm\sqrt{3},-1,-\sqrt{3})$	fixed	
7	$(1,\sqrt{3},2,0,0,0)$	$(2,0,0,0,-2,0)$,	$(2,0,0,0,1,-\sqrt{3})$
8	$(1,\sqrt{3},0,0,-1,\sqrt{3})$	$(2,0,-2,0,0,0)$,	$(1,\sqrt{3},0,0,2,0)$
9	$(1,-\sqrt{3},-1,\sqrt{3},0,0)$	$(0,0,1,\sqrt{3},2,0)$,	$(0,0,1,\sqrt{3},-1,\sqrt{3})$
10	$(1,-\sqrt{3},-1,-\sqrt{3},0,0)$	$(0,0,-1,-\sqrt{3},2,0)$,	$(0,0,1,-\sqrt{3},-1,\sqrt{3})$
11	$(1,-\sqrt{3},0,0,-1,\sqrt{3})$	$(1,-\sqrt{3},0,0,2,0)$,	$(0,0,2,0,1,\sqrt{3})$
12	$(1,-\sqrt{3},0,0,-1,-\sqrt{3})$	$(0,0,2,0,1,-\sqrt{3})$,	$(0,0,2,0,-2,0)$

3. Faces and vertices of the Voronoi region of E_6^*

The points of the Voronoi region satisfy $\mathbf{x}\cdot\mathbf{l} \leqslant \frac{1}{2}\mathbf{l}\cdot\mathbf{l}$ for all points $\mathbf{l} \neq \mathbf{0}$ of the lattice. Since $-\mathbf{l}$ is in the lattice whenever $\mathbf{l}$ is we may assume $\mathbf{l}$ has first nonzero component positive (that is, $\mathbf{l} > \mathbf{0}$) and determine V_Λ by the inequalities

$$|\mathbf{x}\cdot\mathbf{l}| \leqslant \tfrac{1}{2}\mathbf{l}\cdot\mathbf{l} \qquad \text{all } \mathbf{l}\in\Lambda, \mathbf{l} > \mathbf{0}.$$

In reality V_Λ is determined using only a finite number of lattice points $\mathbf{l} > \mathbf{0}$ close to the origin. A pair of parallel faces of V_Λ has equation

$$\mathbf{x}\cdot\mathbf{l} = \pm\tfrac{1}{2}\mathbf{l}\cdot\mathbf{l}.$$

We shall use "the face $\mathbf{l}$" to denote either of these two faces.

E_6^* contains $\mathbf{l} = (2,0,-2,0,0,0)$ close to the origin. Applying isometries to obtain other points at distance $2\sqrt{2}$ from $\mathbf{0}$ we obtain faces characterised by equations

$$\begin{aligned} &\left(x_1 \pm \sqrt{3}x_2\right) - \left(x_3 \pm \sqrt{3}x_4\right) = \pm 4 && \text{(type 1a)}\\ &\left(x_1 \pm \sqrt{3}x_2\right) + 2x_3 = \pm 4 && \text{(type 1b)}\\ &2x_1 - 2x_3 = \pm 4 && \text{(type 1c)}. \end{aligned}$$

Other faces may be obtained from these by applying interchanges T_{ij}—the "type" is meant to distinguish the number of "2" coefficients. Application of M_i involves toggling the sign before a $\sqrt{3}$; application of C toggles the sign of the right side; application of R_i and S may change the type.

Similarly from $(0, 2\sqrt{3}, 0, 0, 0, 0)$ we obtain faces characterised by equations

$$2\sqrt{3}\,x_2 = \pm 6 \qquad \text{(type 2a)}$$
$$\left(3x_1 \pm \sqrt{3}\,x_2\right) = \pm 6 \qquad \text{(type 2b)}$$
$$\left(x_1 \pm \sqrt{3}\,x_2\right) - 2x_3 - 2x_5 = \pm 6 \qquad \text{(type 2c)}$$
$$\left(x_1 \pm \sqrt{3}\,x_2\right) + \left(x_3 \pm \sqrt{3}\,x_4\right) - 2x_5 = \pm 6 \qquad \text{(type 2d)}$$
$$\left(x_1 \pm \sqrt{3}\,x_2\right) + \left(x_3 \pm \sqrt{3}\,x_4\right) + \left(x_5 \pm \sqrt{3}\,x_6\right) = \pm 6 \qquad \text{(type 2e)}$$
$$2x_1 + 2x_3 + 2x_5 = \pm 6 \qquad \text{(type 2f)}.$$

Let W denote the polytope determined by all the faces of the nine types described above. Then W has the isometries described in Section 1. We will show $W = V$, the Voronoi region of E_6^*. Firstly we show that given a vertex of W isometries may be applied to W to make that vertex $(2, 0, 0, 0, 0, 0)$.

3.1 Lemma 1. *Let a point of W lie on two faces of type $2a, \ldots, 2f$. Then the point can be transformed into* $(2, 0, 0, 0, 0, 0)$ *by applying isometries of E_6^*.*

Proof. We show first that isometries can be applied so the faces become the faces $(3, \pm\sqrt{3}, 0, 0, 0, 0)$. Select one of the faces. If it is of type 2a apply R_i to make it of type 2b. If it is of type 2c, ..., 2f apply various R_i as necessary to make it of type 2f, then apply S which makes it of type 2b. Hence the first face may be taken to be of type 2b. Applying T_{ij}, M_j as necessary we have the face $(3, \sqrt{3}, 0, 0, 0, 0)$.

Now select the second face. Consider only isometries M_i, R_i $(i = 2, 3)$, the combination $M_1 R_1$, and S, all of which leave the first face unchanged. If the second face is of type 2c, ..., 2f apply suitable R_i to make it one of $(2, 0, 2, 0, 2, 0)$ or $(1, \pm\sqrt{3}, -2, 0, -2, 0)$. We may assume $+\sqrt{3}$ occurs in the latter case by applying $M_1 R_1$ if necessary. Now S sends the first of these to $(3, -\sqrt{3}, 0, 0, 0, 0)$ and sends $(1, \sqrt{3}, -2, 0, -2, 0)$ to $(0, 2\sqrt{3}, 0, 0, 0, 0)$, which becomes $(3, -\sqrt{3}, 0, 0, 0, 0)$ on applying $M_1 R_1$. Thus the second face is either $(3, -\sqrt{3}, 0, 0, 0, 0)$ or of type 2a or 2b. If the face is neither $(3, -\sqrt{3}, 0, 0, 0, 0)$ nor $(0, 2\sqrt{3}, 0, 0, 0, 0)$ then applying suitable R_i, T_{ij} the two faces may be taken to be $(0, 2\sqrt{3}, 0, 0, 0, 0)$ and $(0, 0, 0, 2\sqrt{3}, 0, 0)$. This makes $|x_2| = |x_4| = \sqrt{3}$, and a suitable choice of signs puts $\mathbf{x}$ on the wrong side of a type 1a face. If the second face is $(0, 2\sqrt{3}, 0, 0, 0, 0)$ apply R_1 and we have the two faces $(3, \pm\sqrt{3}, 0, 0, 0, 0)$.

We have transformed our faces to $(3, \pm\sqrt{3}, 0, 0, 0, 0)$. By applying C if necessary the faces may be taken as

$$3x_1 + \sqrt{3}\,x_2 = 6,$$
$$3x_1 - \sqrt{3}\,x_2 = \pm 6.$$

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The right side of the second equation must be $+6$, else $\sqrt{3}\,x_2 = 6$ and $\mathbf{x}$ lies on the wrong side of a type 2a face. Thus $x_1 = 2$ and $x_2 = 0$. The type 1a, 1b requirements are

$$|2x_1 + x_3 \pm \sqrt{3}\,x_4| \leqslant 4,$$
$$|2x_1 - 2x_3| \leqslant 4$$

and can plainly be satisfied only with $x_3 = x_4 = 0$. Similarly $x_5 = x_6 = 0$. Thus the point has been transformed to $(2, 0, 0, 0, 0, 0)$ as required.

3.2 Lemma 2. *Let a point of W lie on five faces of type* 1a, 1b, 1c. *Then the point lies on two faces of type* 2a, ..., 2f.

Proof. The proof is similar in style to the proof of Lemma 1, selecting faces, transforming them, and eliminating impossible combinations. The table in Section 2 will be very useful: if the current set of faces is fixed under S, for the next face chosen we only need to consider one out of each orbit. For example, note that each orbit contains a representative $(1, \pm\sqrt{3}, \ldots)$.

From the five faces, there are five $(0, 0)$ coordinate pairs. Thus there is one position which has a $(0, 0)$ pair for at most one face. A suitable T_{ij} makes this the first position. There are only three possible pairs $(1, \pm\sqrt{3})$, $(2, 0)$, (still assuming the first nonzero component is positive) for the first position, so one pair must occur twice. Applying R_1 as necessary the pair $(1, +\sqrt{3})$ occurs twice. Applying R_2, R_3 as necessary, we may assume two of the faces are either

$$(1, \sqrt{3}, -1, \sqrt{3}, 0, 0) \quad \text{and} \quad (1, \sqrt{3}, -1, -\sqrt{3}, 0, 0), \qquad \text{or}$$
$$(1, \sqrt{3}, -1, \sqrt{3}, 0, 0) \quad \text{and} \quad (1, \sqrt{3}, 0, 0, -1, -\sqrt{3}).$$

We consider these two cases separately.

Case 1. Assume that the point lies on the two faces $(1, \sqrt{3}, -1, \pm\sqrt{3}, 0, 0)$, which are fixed points of S. Select a third face (from the representatives of the orbits listed in Section 2). If the third face is $(1, \sqrt{3}, \ldots)$ it must be $(1, \sqrt{3}, 2, 0, 0, 0)$, and applying C if necessary the three faces are

$$x_1 + \sqrt{3}\,x_2 - x_3 + \sqrt{3}\,x_4 = 4,$$
$$x_1 + \sqrt{3}\,x_2 - x_3 - \sqrt{3}\,x_4 = \pm 4,$$
$$x_1 + \sqrt{3}\,x_2 + 2x_3 = \pm 4.$$

The second face must have right side $+4$, else combining it with the first gives $\sqrt{3}\,x_4 = 4$, on the wrong side of a type 2a face. The third face must have right side 4, else subtracting the first gives $|3x_3 - \sqrt{3}\,x_4| = 8$, on the wrong side of a type 2b face. Solving the equations gives $x_4 = x_3 = 0$, $x_1 + \sqrt{3}\,x_2 = 4$. But $|x_2| \leqslant \sqrt{3}$ to be on the right side of the type 2a faces, so $x_1 \geqslant 1$, and hence $3x_1 + \sqrt{3}\,x_2 \geqslant 2 + 4 = 6$. To stay on the right side of a type 2b face we must have $x_1 = 1$, $x_2 = \sqrt{3}$, and then $\mathbf{x}$ lies on two type 2 faces.

Without loss of generality we can assume none of the remaining faces can be transformed into $(1, \sqrt{3}, 2, 0, 0, 0)$ by isometries fixing the first two faces. This obviously excludes orbit 7 of Table 1, but noting that orbit 5 can be transformed to orbit 12 by R_1M_1, orbit 12 can be transformed to orbit 11 by R_3, and orbit 11 to $(1, -\sqrt{3}, 0, 0, 2, 0)$ by S and then to $(2, 0, 0, 0, -2, 0)$ by R_1M_1, we see that orbits 5, 11 and 12 are also excluded.

If the third face is $(0, 0, 1, -\sqrt{3}, -1, -\sqrt{3})$ the faces are

$$\begin{aligned} x_1 + \sqrt{3}\,x_2 - x_3 + \sqrt{3}\,x_4 &= 4, \\ x_1 + \sqrt{3}\,x_2 - x_3 - \sqrt{3}\,x_4 &= 4, \\ x_3 - \sqrt{3}\,x_4 - x_5 - \sqrt{3}\,x_6 &= \pm 4. \end{aligned}$$

But if the third face has right side $+4$ then adding to the first face shows $\mathbf{x}$ is on the wrong side of a type 1a face, while if the third face has right side -4 then subtracting from the second face shows $\mathbf{x}$ is on the wrong side of a type 2d face. For similar reasons the third face cannot be $(0, 0, 1, \sqrt{3}, -1, -\sqrt{3})$. This eliminates orbit 6 of Table 1, and hence also eliminates orbits 9 and 10 (apply M_3 to the third representative) and orbit 4 (apply M_1R_1 to give orbit 9 or 10).

Thus all three remaining faces must come from orbits 2, 3 or 8. For the third face orbits 2 and 8 are equivalent under M_3, and orbit 3 is equivalent under M_1R_1. We may therefore assume the three faces are $(1, \sqrt{3}, -1, \sqrt{3}, 0, 0)$, $(1, \sqrt{3}, -1, -\sqrt{3}, 0, 0)$ and $(1, -\sqrt{3}, 2, 0, 0, 0)$. Since orbit 3 has now been used up, the fourth face must come from orbits 2 and 8 (which are still equivalent) and may be assumed to be $(1, \sqrt{3}, 0, 0, -1, \sqrt{3})$, leaving the fifth face to come from orbit 8. Thus we have the faces

$$\begin{aligned} x_1 + \sqrt{3}\,x_2 - x_3 + \sqrt{3}\,x_4 &= 4, \\ x_1 + \sqrt{3}\,x_2 - x_3 - \sqrt{3}\,x_4 &= 4, \\ x_1 - \sqrt{3}\,x_2 + 2x_3 &= \pm 4, \\ x_1 + \sqrt{3}\,x_2 - x_5 + \sqrt{3}\,x_6 &= \pm 4, \\ x_1 + \sqrt{3}\,x_2 - x_5 - \sqrt{3}\,x_6 &= \pm 4. \end{aligned}$$

The right side of the third equation is -4, else adding to the first puts $\mathbf{x}$ on the wrong side of a type 1b face. The right side of the fourth and fifth equations is 4, else subtracting from the first face puts $\mathbf{x}$ on the wrong side of a type 1a face. Solving these equations gives $x_4 = x_6 = 0$, $x_1 = t$, $x_3 = -2t$, $x_5 = 4t$ and $\sqrt{3}\,x_2 = 4 - 3t$. Now the type 1c faces require $|2x_5 - 2x_3| \leqslant 4$, so $|t| \leqslant \frac{1}{3}$, and

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thus $\sqrt{3}\,x_3 \geqslant 3$. For $\mathbf{x}$ to be on the correct side of the type 2a faces we must have equality. Thus $|t| = \frac{1}{3}$, and $\mathbf{x} = \pm(\frac{1}{3}, \sqrt{3}, -\frac{2}{3}, 0, \frac{4}{3}, 0)$ which lies on the two type 2 faces $(0, 2\sqrt{3}, 0, 0, 0, 0)$ and $(1, -\sqrt{3}, 1, -\sqrt{3}, -2, 0)$.

Case 2. Assume the point lies on the two faces $(1, \sqrt{3}, -1, \sqrt{3}, 0, 0)$ and $(1, \sqrt{3}, 0, 0, -1, -\sqrt{3})$. Apply M_3 and S so the second face becomes $(2, 0, -2, 0, 0, 0)$ and then M_1R_1 and M_2R_2 so the second face is $(1, -\sqrt{3}, 2, 0, 0, 0)$. Assume also that Case 1 cannot be applied, so we can exclude all orbits in Table 1 that can be transformed to $(1, \sqrt{3}, -1, -\sqrt{3}, 0, 0)$ under isometries that preserve the first face. This excludes orbits 7, 5, 11 and 12.

Now consider the remaining faces. None can lie on an orbit 9 or 10 face, else forget the first face, and apply M_1 to make Case 1 apply. A similar argument eliminates the face $(2, 0, 1, -\sqrt{3}, 0, 0)$. Orbit 6 faces become orbit 9 or 10 using M_3, so these can be eliminated. The remaining orbit 4 face yields the equations

$$\begin{aligned} x_1 + \sqrt{3}\,x_2 - x_3 + \sqrt{3}\,x_4 &= 4, \\ x_1 - \sqrt{3}\,x_2 + 2x_3 &= \pm 4, \\ 2x_1 + x_3 + \sqrt{3}\,x_4 &= \pm 4. \end{aligned}$$

The right side of the second face must be -4, else adding to the first puts $\mathbf{x}$ on the wrong side of a type 1b face. The right side of the third face must be -4 else subtracting from the second puts $\mathbf{x}$ on the wrong side of a type 1b face. But now subtracting the third face from the first puts $\mathbf{x}$ on the wrong side of a type 1b face.

This leaves orbits 2 and 8 for the remaining three faces. These are equivalent under M_3, so we can take $(1, \sqrt{3}, 0, 0, -1, -\sqrt{3})$ for the third face and $(1, \sqrt{3}, 0, 0, -1, \sqrt{3})$ for the fourth. But now we have Case 1 on forgetting the first two faces and applying T_{23} to the third and fourth. This completes the proof of Lemma 2.

Since a vertex of W is determined by at least six faces, the two lemmas above show that the vertices of W are obtained from $(2, 0, 0, 0, 0, 0)$ by applying isometries of E_6^*. Plainly $V \subset W$. To show $W \subset V$ and deduce $V = W$ it is only necessary to show that all vertices of W lie in V, and appeal to convexity. Indeed, because the isometry group acts transitively on the vertices, it is only necessary to show $(2, 0, 0, 0, 0, 0)$ is no closer to any other point of E_6^* than it is to the origin.

The quantizing algorithm for E_6^* given in [3] is as follows. Given a point $\mathbf{x}$, form $\mathbf{x} - \mathbf{a}^{(k)}$ where $\mathbf{a}^{(k)}$ is one of $(0, 0, 0, 0, 0, 0)$, $\pm(2, 0, -2, 0, 0, 0)$, $\pm(2, 0, 0, 0, -2, 0)$, $\pm(0, 0, 2, 0, -2, 0)$, $\pm(2, 0, 2, 0, 2, 0)$. Write

$$\mathbf{y}^{(k)} = \mathbf{x} - \mathbf{a}^{(k)} = \left(y_1^{(1)}, y_2^{(1)}, y_1^{(2)}, y_2^{(2)}, y_1^{(3)}, y_2^{(3)}\right),$$

and for each $(y_1^{(i)}, y_2^{(i)})$ let $(z_1^{(i)}, z_2^{(i)})$ be a point of the hexagonal lattice in $\mathbb{R}^2$ spanned by $\{(0, 2\sqrt{3}), (3, \sqrt{3})\}$ nearest $(y_1^{(i)}, y_2^{(i)})$. Then a point of E_6^* nearest to $\mathbf{x}$ is the point $\mathbf{z}^{(k)} + \mathbf{a}^{(k)}$ where

$$\mathbf{z}^{(k)} = \left(z_1^{(1)}, z_2^{(1)}, z_1^{(2)}, z_2^{(2)}, z_1^{(3)}, z_2^{(3)}\right)$$

and k is chosen so that $|\mathbf{y}^{(k)} - \mathbf{z}^{(k)}|$ is minimal. Applying this to $(2,0,0,0,0,0)$ we find the nearest points of E_6^* are $(0,0,0,0,0,0)$, $(3, \pm\sqrt{3}, 0,0,0,0)$, $(2,0,-2,0,0,0)$, $(2,0,1,\pm\sqrt{3},0,0)$, $(2,0,0,0,-2,0)$ and $(2,0,0,0,1,\pm\sqrt{3})$. As we required, $(0,0,0,0,0,0)$ is among the nearest points—in passing we observe that the other quantizers yield the eight faces of W on which $(2,0,0,0,0,0)$ lies.

The lemmas above give information on the vertices, faces and edges of V. Plainly an edge of V must be determined by exactly four type 1 faces and one type 2 face. A vertex pair $\pm\mathbf{x}$ is uniquely determined by two type 2 faces $\mathbf{l}_1$, $\mathbf{l}_2$ with $\mathbf{l}_1 \cdot \mathbf{l}_2 = 6$. For any given $\mathbf{l}_1$ there are 20 different possible $\mathbf{l}_2$, so on each type 2 face there are 20 vertices. There are 36 $\mathbf{l}$ giving a pair of opposing type 2 faces, so there are 720 vertices of V.

The type 1 faces have 80 vertices on each face. For example, on the face $2x_1 - 2x_3 = 4$ there are the vertices $(2,0,0,0,0,0)$, $(\frac{5}{3}, \pm 1/\sqrt{3}, -\frac{1}{3}, \pm 1/\sqrt{3}, -\theta_1, -\theta_2)$, $(\frac{4}{3}, \varphi_1, -\frac{2}{3}, \varphi_2, \theta_1, \theta_2)$, $(\frac{4}{3}, 0, -\frac{2}{3}, 0, -2\theta_1, -2\theta_2)$, $(1, \pm 1/\sqrt{3}, -1, \pm 1/\sqrt{3}, \pm 1, \pm 1/\sqrt{3})$, $(1, \pm 1/\sqrt{3}, -1, \pm 1/\sqrt{3}, 0, \pm 2/\sqrt{3})$ with $(\theta_1, \theta_2) \in \{(-\frac{2}{3}, 0), (\frac{1}{3}, \pm 1/\sqrt{3})\}$ and $(\varphi_1, \varphi_2) \in \{(\pm 2/\sqrt{3}, 0), (0, \pm 2/\sqrt{3})\}$, together with vertices obtained from these by the transformation $\mathbf{x} \to (-x_3, -x_4, -x_1, -x_2, x_5, x_6)$. Projecting the face into $\mathbb{R}^5$ by the transformation $x_1 \to x_1'/\sqrt{2} + 1$, $x_3 \to x_1'/\sqrt{2} - 1$ and dropping the x_3 coordinate we observe one vertex at $x_1 = \pm\sqrt{2}$, twelve vertices at $x_1 = \pm \frac{2}{3}\sqrt{2}$, fifteen vertices at $x_1 = \pm\frac{1}{3}\sqrt{2}$ and twenty-four vertices at $x_1 = 0$. The automorphism group in $\mathbb{R}^5$ of the face includes the reflections $x_2' = -x_2$, $x_3' = -x_3$ and rotation through $2\pi/3$ on (x_4, x_5). There is also the automorphism $x' = xT$ where T is the symmetric matrix

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{3} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{\sqrt{2}}{3} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Each vertex has twelve 4-dimensional edges from it. For example the edges from $(\sqrt{2}, 0, 0, 0, 0)$ are to $(\frac{2}{3}\sqrt{2}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3}, -\theta_1, -\theta_2)$. The eight faces of V defining the vertex $(2, 0, 0, 0, 0, 0)$ project down to the seven subfaces

$$\frac{3}{\sqrt{2}}x_1 \pm \sqrt{3}x_3 = 3,$$
$$\frac{3}{\sqrt{2}}x_1 \pm \sqrt{3}x_2 = 3,$$
$$\sqrt{2}x_1 - 2x_4 = 2,$$
$$\sqrt{2}x_1 + x_4 \pm \sqrt{3}x_5 = 2$$

of the 5-dimensional face. These have distances $\sqrt{6}/\sqrt{5}$ and $2/\sqrt{6}$ from the origin in $\mathbf{R}^5$, and the closer ones are orthogonal to each other. Details of these faces are given in Table 2 below.

The type 2 faces (all isometric) have a simple structure. Select the face $(0, 2\sqrt{3}, 0, 0, 0, 0)$: the 20 vertices on this face all have $x_2 = \sqrt{3}$. Projecting into $\mathbb{R}^5$ by dropping the x_2 coordinate, the face has vertices $\pm(1, 0, 0, 0, 0)$, $\pm(\frac{1}{3}, \theta_1, \theta_2, \theta_1', \theta_2')$ where (θ_1, θ_2), $(\theta_1', \theta_2') \in \{(-\frac{2}{3}, 0), (\frac{1}{3}, \pm 1/\sqrt{3})\}$. The structure of these faces is described in the table below.

By using the formulae in Theorem 3 of [2], the volume and unnormalised second moment U for the faces can be calculated, eventually giving $U = 50476\sqrt{3}/315$ for V. Since V has volume $72\sqrt{3}$ (the determinant of the generating matrix), we get the normalised second moment $G(E_6^*) = 12619 \cdot 3^{1/6}/204120 = 0.0742437^{(-)}$, which confirms the estimate for $G(E_6^*)$ given in [3].

TABLE 2. Faces of V

dim	type	volume	U	subfaces at distances
2	triangle	$1/\sqrt{3}$	$1/9\sqrt{3}$	3 lines at $h = 1/3$
3	octahedron	$6\sqrt{6}/27$	$8\sqrt{6}/315$	6 triangles at $h = \sqrt{2}/3$
3	tetrahedron	$2\sqrt{6}/27$	$\sqrt{6}/315$	4 triangles at $h = \sqrt{2}/6$
4	24-cell	$32/9$	$832/405$	24 octahedra at $h = \sqrt{6}/3$
4	10-cell	$11\sqrt{5}/54$	$58\sqrt{5}/1215$	5 octahedra at $h = \sqrt{2/15}$
				5 tetrahedra at $h = \sqrt{3/10}$
5	type-1	$496\sqrt{6}/315$	$1952\sqrt{6}/567$	10 24-cells at $h = \sqrt{2/3}$
				32 10-cells at $h = \sqrt{6/5}$
5	type-2	$22/45$	$86/567$	12 10-cells at $h = 1/\sqrt{5}$
6	V	$72\sqrt{3}$	$50476\sqrt{3}/315$	54 type-1 at $h = \sqrt{2}$
				72 type 2 at $h = \sqrt{3}$

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4. Methods

The determination of V and its isometries was carried out as follows. The quantization algorithm for E_6^* was programmed and applied to points $\mathbf{x} = \sum \lambda_i \mathbf{e}_i$ where $|\lambda_i| < \frac{1}{2}$. The nearest lattice points arising were stored and printed, and the general pattern was noticed. However even with many fairly uniformly spaced points $\mathbf{x}$ not all the points nearest $\mathbf{0}$ were found (the λ_i used were determined using multiples of the point $k(1, b, b^2, b^3, b^4, b^5)/N$ with $0 \leqslant k < N$, and N, b were selected from Haber's integration tables [4]). Having obtained a set of points near $\mathbf{0}$ which gave the faces of W, a program to produce the vertices was written. This was a crude program and ran rather slowly, but produced enough vertices to indicate that every vertex lay on exactly six type 1 faces and two type 2 faces, and suggested looking for another isometry. Two vertices that were not obviously related were selected, the faces on which they lay were compared and the angles between the normals calculated. Possible isometries were produced in terms of their effect on the face normals, their matrix representations were calculated using the interactive MATRIX program from the University of Sydney, and the eigenvalues and eigenvectors displayed. The isometry S of Section 1 was discovered in this way.

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Department of Mathematics
Monash University
Clayton, Victoria 3168
Australia