ON THE TENSOR PRODUCT OF QUATERNION ALGEBRAS OF CHARACTERISTIC TWO

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1. Introduction. The purpose of this note is to generalize to fields of characteristic two the results obtained in [**4**]. We obtain necessary and sufficient conditions involving quadratic forms for certain tensor products of quaternion algebras to be division algebras.

We apply this to show, as in [**4**], that the Albert criterion does not generalize to tensor products of more than two quaternion algebras.

More precisely, let k be a field of characteristic two, $a \in k$ and $b \in k^{\times}$ $(= k - \{0\})$; we denote by $[a, b)_k$ the quaternion k-algebra generated by two elements e_1 and e_2 subject to the relations:

$$\mathcal{P}(e_1) := e_1^2 + e_1 = a,$$
$$e_2^2 = b,$$
$$e_2 e_1 = e_1 e_2 + e_2.$$

Let us also denote by $[a, b]$ the quadratic form $aX^2 + XY + bY^2$. To the tensor product of quaternion algebras

$$T = [a_1, b_1)_k \otimes \ldots \otimes [a_n, b_n)_k,$$

we associate the quadratic form

$$Q_T = [1, a_1 + \ldots + a_n] \perp_{i=1}^{n} \langle b_i \rangle [1, a_i].$$

In fact, for $n = 1$ and 2, it is well known that T has zero divisors if and only if Q_T is isotropic, see [**1**, p. 29 and p. 131]. In § 3, we show that this assertion is false for $n \geq 3$. A similar question was first proposed by D. W. Lewis over fields of characteristic different from two, see [**3**] and [**4**]. Note that Q_T is, as in [**3**] and [**4**], the (alternating) sum of the reduced norms of the quaternion algebras $[a_i, b_i)$ minus the $(n-1)$ obvious hyperbolic planes.

2. Generic extensions of division algebras. Let $X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}, Z$ be independent indeterminates over k (with $n \geq 3$) and let also

$$F = k(X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}, Z).$$

For $f \in k(X_1, \ldots, X_{n-1})$ and $g \in k(X_1, \ldots, X_{n-1}, Z)$, we define

$$T_f := [X_1, Y_1)_F \otimes \ldots \otimes [X_{n-1}, Y_{n-1})_F \otimes [f, Z)_F$$

and

$$T'_g := [X_1, Y_1)_F \otimes \ldots \otimes [X_{n-1}, Y_{n-1})_F \otimes [Z, g)_F.$$

THEOREM A. (i) *T_f is a division algebra if and only if*

$$f \notin \mathcal{P}(k(\mathcal{P}^{-1}(X_1), \ldots, \mathcal{P}^{-1}(X_{n-1}))).$$

(ii) *T'_g is a division algebra if and only if g is not represented by the quadratic form $[1, Z]$ over $k(X_1, \ldots, X_{n-1}, Z)$.*

THEOREM B. (i) *Q_T is anisotropic over F if and only if the quadratic form $[1, X_1 + \ldots + X_{n-1} + f]$ is anisotropic over $k(X_1, \ldots, X_{n-1})$ and $f \notin \mathcal{P}(k(X_1, \ldots, X_{n-1}))$.*

(ii) *$Q_{T'_g}$ is anisotropic over F if and only if the quadratic form*

$$[1, X_1 + \ldots + X_{n-1} + Z] \perp \langle g \rangle [1, Z]$$

is anisotropic over $k(X_1, \ldots, X_{n-1}, Z)$.

The proofs will follow by repeated use of the following results.

LEMMA A. *Let A be a division algebra over k, $c \in k$ and X an indeterminate over k: then we have:*

(i) *$A \otimes_k [c, X)_{k(x)}$ is a division algebra if and only if $A \otimes_k k(\mathcal{P}^{-1}(c))$ is a division algebra;*

(ii) *$A \otimes_k [X, c)_{k(x)}$ is a division algebra if and only if $A \otimes_k k(\sqrt{c})$ is a division algebra.*

Proof. (i) If $A \otimes_k k(\mathcal{P}^{-1}(c))$ is not a division algebra, we easily see that

$$\mathcal{P}^{-1}(c) \otimes 1 + 1 \otimes \mathcal{P}^{-1}(c)$$

is a zero divisor of $A \otimes_k [c, X)_{k(x)}$. Now suppose that $D := A \otimes_k k(\mathcal{P}^{-1}(c))$ is a division algebra. We first observe that the quaternion algebra $[c, X)_{k(x)}$ can be written in the form $k(\mathcal{P}^{-1}(c))(X; \sigma)$, where σ is the non-trivial k-automorphism of $k(\mathcal{P}^{-1}(c))$.

Since D is a division algebra, we can extend σ to D in such a way that $\sigma_{|A} = 1_A$. But then we remark that $A \otimes_k [c, X)_{k(x)}$ is nothing else than $D(X; \sigma)$.

(ii) If $A \otimes_k k(\sqrt{c})$ is not a division algebra, we easily see that $\sqrt{c} \otimes 1 + 1 \otimes \sqrt{c}$ is a zero divisor of $A \otimes_k [X, c)_{k(x)}$. Suppose now that $D' := A \otimes_k k(\sqrt{c})$ is a division algebra. Let e be the basis element of $[X, c)_{k(x)}$ such that $e^2 + e = x$ then, if we put $t = c^{-1}\sqrt{c}\, e$, we can verify the following relations: $t^2 = c^{-1}X \in k(x)$ and $t\sqrt{c} = \sqrt{c}\, t + 1$. This shows that we can write the quaternion algebra $[X, c)_{k(x)}$ in the form $k(\sqrt{c})(t; \delta)$, where δ is the derivation defined by $\delta(\sqrt{c}) = 1$. Since D' is a division algebra, we can extend δ to D' so that $\delta_{|A} = 0$. But then, as in the previous case, $A \otimes_k [X, c)_{k(x)}$ is nothing else than $D(t; \delta)$.

LEMMA B. *Let Q_1 and Q_2 be two quadratic forms over k, and X an indeterminate over k. Then $Q_1 \perp \langle x \rangle Q_2$ is anisotropic over $k(x)$ if and only if Q_1 and Q_2 are anisotropic over k.*

Proof. We take first a representation of Q_1 and Q_2 with respect to the symplectic basis and then proceed, as for the case of characteristic different from two (see [**2**, p. 273]), by a degree argument. Details are left to the reader.

Proof of Theorem A. (i) By induction and Lemma A(i), we see that T_f is a division algebra if and only if the quaternion algebra $[f, Z)$ is a division algebra over $k(\mathcal{P}^{-1}(X_1), \ldots, \mathcal{P}^{-1}(X_{n-1}), Z)$. But this condition is equivalent to the following (see the introduction): $[1, f] \perp \langle Z\rangle[1, f]$ is an anisotropic quadratic form over $k(\mathcal{P}^{-1}(X_1), \ldots, \mathcal{P}^{-1}(X_{n-1}), Z)$. Applying now Lemma B for $X = Z$, we see that this condition holds if and only if $[1, f]$ is an anisotropic quadratic form over $k(\mathcal{P}^{-1}(X_1), \ldots, \mathcal{P}^{-1}(X_{n-1}))$, i.e. $f \notin \mathcal{P}(k(\mathcal{P}^{-1}(X_1), \ldots, \mathcal{P}^{-1}(X_{n-1})))$.

(ii) By induction and Lemma A(ii), we see that T'_g is a division algebra if and only if the quaternion algebra $[Z, g)$ is a division algebra over $k(X_1, \ldots, X_{n-1}, Z, \sqrt{Y_1}, \ldots, \sqrt{Y_{n-1}})$. This last condition is clearly equivalent to the following: g is not represented by the quadratic form $[1, Z]$ over

$$k(X_1, \ldots, X_{n-1}, Z, \sqrt{Y_1}, \ldots, \sqrt{Y_{n-1}}),$$

and so, if and only if g is not represented by $[1, Z]$ over $k(X_1, \ldots, X_{n-1}, Z)$.

Proof of Theorem B. Use induction and Lemma B.

Remark. The quadratic form $[1, X_1 + \ldots + X_{n-1} + f]$ is isotropic over $k(X_1, \ldots, X_{n-1})$ if and only if $X_1 + \ldots + X_{n-1} + f \in \mathcal{P}(k(X_1, \ldots, X_{n-1}))$. Since $X_1 + \ldots + X_{n-1} \in \mathcal{P}(k(\mathcal{P}^{-1}(X_1), \ldots, \mathcal{P}^{-1}(X_{n-1}))$, this last condition implies that $f \in \mathcal{P}(k(\mathcal{P}^{-1}(X_1), \ldots, \mathcal{P}^{-1}(X_{n-1}))$. This shows that if T_f is a division algebra then Q_{T_f} is anisotropic.

3. The counterexamples. In the introduction we said that the equivalence, T is a division algebra if and only if Q_T is anisotropic, holds if T is a quaternion algebra or a tensor product of two quaternion algebras. We now provide counterexamples to both implications for $n \geq 3$. Applying Theorems A and B, we can see that

(1) for $f = X_1 + \ldots + X_{n-1}$, T_f is not a division algebra and Q_{T_f} is anisotropic;

(2) for $g = X_1 + \ldots + X_n + Z$, T'_g is a division algebra and $Q_{T'_g}$ is isotropic.

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