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WARD'S SOLITONS II: EXACT SOLUTIONS

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ABSTRACT. In a previous paper, we gave a correspondence between certain exact solutions to a (2 + 1)-dimensional integrable Chiral Model and holomorphic bundles on a compact surface. In this paper, we use algebraic geometry to derive a closed-form expression for those solutions and show by way of examples how the algebraic data which parametrise the solution space dictates the behaviour of the solutions.

RÉSUMÉ. Dans un article précédent, nous avons démontré que les solutions d'un modèle chiral intégrable en dimension (2 + 1) correspondent aux fibrés vectoriels holomorphes sur une surface compacte. Ici, nous employons la géométrie algébrique dans une construction explicite des solutions. Nous donnons une formule matricielle et illustrons avec trois exemples la signification des invariants algébriques pour le comportement physique des solutions.

1. **Introduction.** Nonlinear equations admitting soliton solutions in 3-dimensional space-time have been recently studied numerically and analytically. See [Su] and [Wa95] for a discussion of solitons in planar models.

In this paper, we explicitly construct solutions to an integrable model introduced by Ward which is remarkable in that it possesses interacting soliton solutions of finite energy [Su], [Wa95], [Io]. This SU(N) chiral model with torsion term in (2 + 1) dimensions may be obtained by dimensional reduction and gauge fixing from the (2 + 2) Yang-Mills equations [Wa95] or more directly from the (2 + 1) Yang-Mills-Higgs equations. Static solutions of the model correspond to harmonic maps $\mathbb{R}^2 \longrightarrow \mathrm{U}(N)$.

The basic equations of Ward are

$$\frac{\partial}{\partial t}\left(J^{-1}\frac{\partial}{\partial t}J\right) - \frac{\partial}{\partial x}\left(J^{-1}\frac{\partial}{\partial x}J\right) - \frac{\partial}{\partial y}\left(J^{-1}\frac{\partial}{\partial y}J\right) + \left[J^{-1}\frac{\partial}{\partial y}J, J^{-1}\frac{\partial}{\partial t}J\right] = 0, \tag{1}$$

where $J\colon \mathbb{R}^3 \longrightarrow \mathrm{SU}(N)$ is a function of two space variables and time. To this equation Ward added the boundary condition:

$$J(r,\theta,t) = \mathbb{I} + \frac{1}{r}J_1(\theta) + O\left(\frac{1}{r^2}\right) \quad \text{as} \quad r \longrightarrow \infty; \tag{2}$$

(written in terms of polar space coordinates). We will assume $J_1(\theta)$ is continuous. Ward showed that analytic solutions to (1) correspond to doubly-framed holomorphic bundles

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on the open surface $T\mathbb{P}^1$. We showed (in [An97]) that a necessary and sufficient condition for the bundle to extend trivially to the compactification $\widetilde{T\mathbb{P}}^1$, the second Hirzebruch surface, is that J be analytic and that the operator

$$\frac{d}{du} + \tfrac{1}{2}(1+\cos\theta)\iota^*\left(J^{-1}\frac{\partial}{\partial x}J\right) + \tfrac{1}{2}\sin\theta\,\iota^*\left(J^{-1}\left(\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)J\right) \tag{3}$$

have null monodromy around $u \in \mathbb{R}\cup\{\infty\}$, where

$$\iota(u) \stackrel{\text{def}}{=} (u\cos\theta + x_0, u\sin\theta + y_0, 0), \tag{4}$$

for all $x_0, y_0 \in \mathbb{R}$ and $\theta \in \mathbb{S}^1$, *i.e.*, for all lines in $\mathbb{R}^2$.

Construction of solutions. There are currently three methods of solving this system, to which the author adds a fourth. The first method of Ward was to give a twistor correspondence between solutions of (1) and holomorphic bundles on $T\mathbb{P}^1$, the holomorphic tangent space to the complex projective line. This led to the construction of noninteracting soliton solutions. Thereafter, numerical simulations of these solutions by Sutcliffe led to his discovery of interacting soliton solutions. Exact solutions with two interacting solitons were then constructed by Ward using a Zakharov-Shabat procedure. Using this procedure, more general solutions were constructed by Ioannidou concurrently with the present work. In this paper, we present a closed-form expression for all solutions satisfying (1), (2) and having null (3) monodromy, which includes all known exact soliton solutions.

In [An97], we showed that such solutions correspond to abstract holomorphic bundles. A concrete expression for the solutions requires a concrete representation for such bundles, which is provided by monads (explained in Section 3). In the following theorem, the equivalence of the first two sets was proved in [An97]. The equivalence of the second and third is explained in Section 3.

THEOREM 1. *There are bijections between*

1. *the set of analytic solutions J of (1) satisfying (2) and (3);*
2. *holomorphic, rank N bundles $V \longrightarrow T\mathbb{P}^1$ which are real in the sense that they admit a lift*

$$\begin{array}{ccc} V & \xrightarrow{\tilde\sigma} & V \\ \downarrow & & \downarrow \\ T\mathbb{P}^1 & \xrightarrow{\sigma} & T\mathbb{P}^1 \end{array} \qquad \textit{of the antiholomorphic involution} \qquad \begin{array}{c} \sigma^*\lambda = 1/\bar\lambda \\ \sigma^*\eta = -\bar\lambda^{-2}\bar\eta \end{array} \tag{5}$$

(where λ and η are standard base and fibre coordinates of $T\mathbb{C} \subset T\mathbb{P}^1$) and which extend to bundles on the singular quadric cone $T\mathbb{P}^1\cup\{\infty\}$, such that restricted to the compactified tangent planes $T_\lambda\mathbb{P}^1\cup\{\infty\}$ for $|\lambda| = 1$, V is trivial, with a fixed, real framing; and

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3. *the union of a point set and, for all $k > 0$, the set of 4-tuples of matrices*

$$\alpha_1, \alpha_2 \in \mathrm{gl}(k),$$

(monad data) $$\mathrm{spec}(\alpha_2) \subset \{z \in \mathbb{C} \mid \Re z > 0\}$$

$$a \in \mathrm{M}_{N,k}, \quad b \in \mathrm{M}_{k,N}$$

satisfying

(nondegeneracy) $$\mathrm{rank}\begin{pmatrix} \alpha_2 + w \\ \alpha_1 + z \\ a \end{pmatrix} = \mathrm{rank}\,(\,\alpha_2 + w \quad \alpha_1 + z \quad b\,) = k \quad \forall z, w \in \mathbb{C}$$

(monad equation) $$[\alpha_1, \alpha_2] + ba = 0$$

quotiented by the action of $g \in \mathrm{Gl}(k)$

$$\alpha_1 \longmapsto g\alpha_1 g^{-1} \quad \alpha_2 \longmapsto g\alpha_2 g^{-1}$$
$$a \longmapsto ag^{-1} \quad b \longmapsto gb.$$

The eigenvalues of α_2 are restricted because the given monad data encode the bundle above the equator, i.e., for $|\lambda| < 1$. Reality reflects this structure across the equator.

This statement also improves the Theorem of [An97] in removing the condition that V be trivial on real sections, *i.e.*, σ-stable sections. In Section 5 we show that this condition is implicit in the other conditions imposed on V. The integer k is obviously an invariant, it corresponds to the normalised second Chern class of the bundle and the topological charge defined in [An97] (see Conjecture 2.4 in that paper). The point set corresponds to the constant solution and the trivial bundle.

The main result of this paper is the explicit construction of the solutions corresponding to this matrix data:

THEOREM 2. *The (multi-)soliton associated to the bundle represented by the monad $(\alpha_1, \alpha_2, a, b)$, as above, has the form*

$$J(x, y, t) = \mathbb{I} + (\,ib^* \quad a\,)\begin{pmatrix} -\alpha_2^* & \\ & \alpha_2 \end{pmatrix}^{-1}\begin{pmatrix} -\Omega^* & \phi_1 \\ \phi_2 & \Omega \end{pmatrix}^{-1}\begin{pmatrix} ia^* \\ b \end{pmatrix}; \tag{6}$$

where

$$\Omega = \alpha_1 + 2i(t + y)\gamma(\mathbb{I} + \gamma) + x\alpha_2 + iy, \quad \gamma = \frac{1}{2}(\alpha_2 - \mathbb{I}),$$

and ϕ_1, ϕ_2 are determined by the linear equations

$$\phi_1\alpha_2 + \alpha_2^*\phi_1 + ia^*a = 0, \tag{7}$$

$$-\phi_2\alpha_2^* - \alpha_2\phi_2 + ibb^* = 0. \tag{8}$$

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Verification. We can verify directly that these are (real) solutions. Nonsingularity is not as easy to verify, but follows from the arguments of Section 5.

Ward's equation can be written as

$$4\partial_{y-t}J^{-1}\partial_{t+y}J + \partial_x J^{-1}\partial_x J = 0. \tag{9}$$

For convenience, we define the following block matrices

$$\underline{\Omega} \stackrel{\text{def}}{=} \begin{pmatrix} -\Omega^* & \phi_1 \\ \phi_2 & \Omega \end{pmatrix},$$
$$\underline{\alpha_2} \stackrel{\text{def}}{=} \begin{pmatrix} -\alpha_2^* & \\ & \alpha_2 \end{pmatrix},$$
$$\underline{a} \stackrel{\text{def}}{=} \begin{pmatrix} ib^* & a \end{pmatrix}$$
$$\underline{b} \stackrel{\text{def}}{=} \begin{pmatrix} ia^* \\ b \end{pmatrix}, \text{ and}$$
$$\omega \stackrel{\text{def}}{=} \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix}.$$

The following relations,

$$\underline{\Omega}^* = -\omega\underline{\Omega}\omega,$$
$$\underline{\alpha_2}^* = -\omega\underline{\alpha_2}\omega, \quad \text{and}$$
$$\underline{a}^* = i\underline{b}\omega,$$

which are equivalent to the real structure of the monad and bundle, and may be verified directly, are needed to compute the inverse of the solution:

$$\begin{aligned} J^{-1} = J^* &= \mathbb{I} + \underline{b}^*\underline{\Omega}^{*-1}\underline{\alpha_2}^{*-1}\underline{a}^* \\ &= \mathbb{I} + (i\underline{a}\omega)(-\omega\underline{\Omega}^{-1}\omega)(-\omega\underline{\alpha_2}^{-1}\omega)(i\omega\underline{b}) \\ &= \mathbb{I} - \underline{a}\underline{\Omega}^{-1}\underline{\alpha_2}^{-1}\underline{b}. \end{aligned} \tag{10}$$

We compute

$$\begin{aligned} \partial_{y+t}\Omega &= 2i\gamma(\mathbb{I} + \gamma) + \frac{i}{2}\mathbb{I} \\ &= \frac{i}{2}(\alpha_2^2 - \mathbb{I}) + \frac{i}{2}\mathbb{I} = \frac{i}{2}\alpha_2^2. \end{aligned}$$

Since this involves only even powers of α_2, it holds as well for the block form: $\partial_{y+t}\underline{\Omega} = \frac{i}{2}\underline{\alpha_2}^2$. Hence

$$\partial_{y+t}J = -\frac{i}{2}\underline{a}\underline{\alpha_2}^{-1}\underline{\Omega}^{-1}\underline{\alpha_2}^2\underline{\Omega}^{-1}\underline{b}.$$

With the help of the substitution $\underline{b}\underline{a} = [\underline{\alpha_2}, \underline{\Omega}]$ derived from the monad equation, we find

$$J^{-1}\partial_{y+t}J = -\frac{i}{2}\underline{a}\underline{\Omega}^{-1}\underline{\alpha_2}\underline{\Omega}^{-1}\underline{b},$$

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and

$$\partial_{y-t}J^{-1}\partial_{y+t}J = -\frac{i}{2}\underline{a}\left(-\frac{i}{2}\underline{\Omega}^{-2}\right)\underline{\alpha_2}\underline{\Omega}^{-1}\underline{b} - \frac{i}{2}\underline{a\Omega}^{-1}\underline{\alpha_2}\left(-\frac{i}{2}\underline{\Omega}^{-2}\right)\underline{b}. \tag{11}$$

Using $\partial_x\underline{\Omega} = \underline{\alpha_2}$, we find,

$$\begin{aligned}\partial_x J &= -\underline{a\alpha_2}^{-1}\underline{\Omega}^{-1}\underline{\alpha_2}\underline{\Omega}^{-1}\underline{b},\\ J^{-1}\partial_x J &= -\underline{a\Omega}^{-2}\underline{b},\\ \partial_x J^{-1}\partial_x J &= -\underline{a}(-\underline{\Omega}^{-1}\underline{\alpha_2}\underline{\Omega}^{-1})\underline{\Omega}^{-1}\underline{b} - \underline{a\Omega}^{-1}(-\underline{\Omega}^{-1}\underline{\alpha_2}\underline{\Omega}^{-1})\underline{b}.\end{aligned} \tag{12}$$

Putting (12) and (11) into (9), we see that these are in fact solutions.

To verify that J is real, we use the big monad equation

$$0 = [\underline{\alpha_1}, \underline{\alpha_2}] + \underline{ba} = [\underline{\Omega}, \underline{\alpha_2}] + \underline{ba},$$

which follows from the (monad equation), (7) and (8):

$$\begin{aligned}JJ^* &= (\mathbb{I} + \underline{a\alpha_2}^{-1}\underline{\Omega}^{-1}\underline{b})(\mathbb{I} - \underline{a\Omega}^{-1}\underline{\alpha_2}^{-1}\underline{b})\\ &= \mathbb{I} + \underline{a}\big([\underline{\alpha_2}^{-1}, \underline{\Omega}^{-1}] - \underline{\alpha_2}^{-1}\underline{\Omega}^{-1}\underline{ba\Omega}^{-1}\underline{\alpha_2}^{-1}\big)\underline{b}\\ &= \mathbb{I} + \underline{a}\big([\underline{\alpha_2}^{-1}, \underline{\Omega}^{-1}] - \underline{\alpha_2}^{-1}\underline{\Omega}^{-1}[\underline{\alpha_2}, \underline{\Omega}]\underline{\Omega}^{-1}\underline{\alpha_2}^{-1}\big)\underline{b}\\ &= \mathbb{I}.\end{aligned}$$

1.1. *Structure of the paper.* In Section 2, we recall notation and ideas from [An95], [An97], [An98], in order to make this paper self-contained.

Theorem 1 strengthens the classification of [An97] by adding a monad description, which is explained in Section 3 and by removing the condition that the holomorphic bundles be trivial on real sections, which is done in Section 5. The proof in Section 5 takes as a starting point the result in the special case that the holomorphic bundle has a fixed pair of conjugate fibres on which the holomorphic bundle may be nontrivial. This was established in [An98] using monad methods.

Theorem 2 builds on the construction of the static solutions [An98]. To apply the result in the static case to the general case, we need to introduce, in Section 3, a suitably normalised $\widetilde{T\mathbb{P}^1}$ monad, and a way of going back and forth between this monad and the $\mathbb{P}^2$ monad representation used in [An98]. This leaves to Section 4 the computation of the general closed-form expression.

In the final section, we show how to go about constructing examples and discuss how the physical behaviour of the solutions relates to the geometry of the bundles as reflected in the algebraic structure of the monad data, *i.e.*, how energy, Chern class, nilpotency of matrix data, position and degree of jumping lines, number and interaction of solitons are related.

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2. **Review.** Static solutions of Ward's equations are harmonic surfaces. The finite-energy maps are called (multi-)unitons. The ellipticity of the harmonic map equations gives static solutions strong regularity properties. A static, finite-energy solution to 1 always extends to the conformal compactification $\mathbb{S}^2$ as a real analytic function, which in turn implies that the boundary and monodromy conditions are satisfied. On the level of bundles, this implies that "uniton bundles" on $T\mathbb{P}^1$ automatically extend to the fibrewise compactification $\widetilde{T\mathbb{P}^1} \stackrel{\text{def}}{=} \mathbb{P}(T\mathbb{P}^1 \oplus O) = T\mathbb{P}^1 \cup (\text{section at infinity})$.

We will use the results of the analysis of the harmonic maps, explained in [An95], [An98], which culminates in the closed-form expression

$$J(x,y) = \mathbb{I} + \underline{a}\underline{\alpha}_2^{-1}(\underline{\alpha}_1 + x\underline{\alpha}_2 + iy)^{-1}\underline{b}$$

for static solutions. In the remainder of this section, we review the derivation of this result in order to fix notation.

2.1. *Three-dimensional geometry.* We will always have in mind a fixed embedding

$$\lambda \mapsto \left(\frac{\lambda + \bar{\lambda}}{1+\lambda\bar{\lambda}}, -i\frac{\lambda - \bar{\lambda}}{1+\lambda\bar{\lambda}}, \frac{1-\lambda\bar{\lambda}}{1+\lambda\bar{\lambda}} \right) \tag{13}$$

of $\{\lambda \in \mathbb{C}\}$ in $\{(x,y,t) \in \mathbb{R}^3\}$ as the unit sphere, which identifies the directions in $\mathbb{R}^3$ with the points on the complex projective line $\mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$, and which identifies tangent planes of $\mathbb{P}^1$ as subplanes of $\mathbb{R}^3$. We will call the t-axis vertical, and perpendicular directions horizontal. With reference to the unit sphere, we will call $\lambda = 0, \infty$ the poles and $\{|\lambda| = 1\} = \mathbb{S}^1$ the equator, and name fibres $T_\lambda\mathbb{P}^1$ as polar or equatorial in this way.

We recall the (Euclidean) twistor fibration

$$\begin{array}{ccc} & \mathbb{R}^3 \times \mathbb{S}^2 \cong \underline{\mathbb{R}} \oplus T\mathbb{P}^1 & \\ & \swarrow \qquad \searrow & \\ \mathbb{C}^3 \supset \mathbb{R}^3 & & T\mathbb{P}^1, \end{array} \tag{14}$$

which identifies $T\mathbb{P}^1$ with the oriented lines in $\mathbb{R}^3$, either directly, or by taking inverse and direct images, giving correspondences

$$\text{oriented line} \stackrel{\text{with}}{\longleftrightarrow} \text{point} \tag{15}$$

$$\text{point} \stackrel{\text{with}}{\longleftrightarrow} \text{real section}. \tag{16}$$

This picture can be complexified,

$$\begin{array}{ccc} & \mathbb{C}^3 \times \mathbb{P}^1 & \\ & \swarrow \qquad \searrow & \\ \mathbb{C}^3 & & \widetilde{T\mathbb{P}^1}, \end{array} \tag{17}$$

in which case the map on the right becomes the "sections map":

$$\{\text{holomorphic vector fields}\} \times \{\text{base}\} \longrightarrow \{\text{holomorphic tangent bundle}\}.$$

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In terms of complexified variables $(z, t, \overline{z}) \in \mathbb{C}^3$, and η, the coordinate associated to the section $d/d\lambda \in \Gamma(T\mathbb{P}^1)$, the sections map is given by

$$\big((z, t, \overline{z}), \lambda\big) \longmapsto z - 2it\lambda - \overline{z}\lambda^2. \tag{18}$$

In the complexification $\mathbb{C}^3$, $\mathbb{R}^{2+1}$ is a real slice which intersects the standard real slice in a space plane, $\{t = 0\}$.

2.2. *Frames from dbar operators.* Ward's equation is a reduction of the Yang-Mills-Higgs system in 2 + 1 dimensions (*i.e.*, in the indefinite, 3-dimensional signature). In analogy with the Euclidean case (the monopole case), there is a zero curvature (Lax) form for these completely integrable equations, which leads naturally to a twistor construction: The zero curvature condition depends on a complex parameter, λ, and can be rewritten as a family of $\overline{\partial}$ operators on the trivial complex bundle $\mathbb{C}^N \times \mathbb{R}^2$ with respect to a varying complex structure on $\mathbb{R}^2$ parametrised by λ. This defines the structure of a holomorphic bundle on an open complex surface, and analytic arguments show that this structure extends to the compactification $\widetilde{T\mathbb{P}^1}$.

When $|\lambda| = 1$, the Lax-form contains an ordinary differential operator with real, horizontal characteristic direction. This operator corresponds to Hitchin's scattering operator in the Euclidean monopole case. These operators can be integrated (from the standard gauge at infinity) to obtain a circle of distinguished gauges, h_λ, of the trivial $\mathbb{C}^N$ bundle, and Ward's equation is satisfied by the change of gauge $J = h_1^{-1} \cdot h_{-1}$.

To reconstruct J, we must identify h_λ intrinsically in the bundle data.

Since the family of gauges h_λ is parametrised holomorphically by $\lambda \in \mathbb{P}^1$, they lift via (17) to define a gauge on a subset of $\mathbb{C}^3 \times \mathbb{P}^1$, which in the analytic case is open. We began with a trivial $\mathbb{C}^N$ bundle, and consequently have a background trivialisation, f, which pulls back to $\mathbb{C}^3 \times \mathbb{P}^1$ and can be pushed forward to real sections of $T\mathbb{P}^1$. (See Table 1.) Assuming that these frames can be constructed intrinsically on $T\mathbb{P}^1$, we can

(19)

	$\mathbb{C}^3$	$\mathbb{C}^3 \times \mathbb{P}^1$	$\widetilde{T\mathbb{P}^1}$
$f_{z,t}$	Frame over a point (z, t)	Frame over $\{(z, t)\} \times \mathbb{P}^1$	Frame over $\{2\eta = z + 2it\lambda - \overline{z}\lambda^2\}$
h	Family of Frames	Frame over open set	Frame over each equatorial fibre

TABLE 1: The frames f and h, which are most naturally defined over $\mathbb{C}^3$ and $\mathbb{C}^3 \times \mathbb{P}^1$ respectively, induce (restricted) framings on the other spaces in the fibration.

now take the "inverse and direct images" of the definition of J:

$$\begin{aligned} J(z,t) &\stackrel{\text{def}}{=} (h_1^{-1}h_{-1})\big|_{(z,t)\in\mathbb{C}^3} \quad \text{on } \mathbb{C}^3 \\ &= (h^{-1}\cdot f_{z,t})\big|_{\substack{((z,t),1)\\ \in\mathbb{C}^3\times\mathbb{P}^1}} (h^{-1}\cdot f_{z,t})^{-1}\big|_{\substack{((z,t),-1)\\ \in\mathbb{C}^3\times\mathbb{P}^1}} \quad \text{on } \mathbb{C}^3\times\mathbb{P}^1 \end{aligned} \tag{20}$$

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$$= \left.(h^{-1}\cdot f_{z,t})\right|_{\substack{2\eta=z-2it-\bar{z}\\ \lambda=1}} \left.(h^{-1}\cdot f_{z,t})^{-1}\right|_{\substack{2\eta=z+2it-\bar{z}\\ \lambda=-1}} \quad \text{on } T\mathbb{P}^1.$$

We have defined the gauge h_λ to agree with the gauge f "at infinity", *i.e.*, on the line at spatial infinity which we add to compactify $\mathbb{R}^3$ and on the rational curve which we add to compactify $T\mathbb{P}^1$.

2.3. *Intrinsic construction.* On a compact variety, evaluation of a trivial bundle at a point is an isomorphism. Since the bundle is trivial over the section at infinity, a framing at a point induces a framing along the whole section. Similarly, triviality over equatorial fibres and the framing at the infinite points induces a frame, h on their union. Restricted to a real section, we have a frame over a circle of points (the points lying over the equator) and since the bundle is trivial over real sections, we can use any trivialisation, $f_{z,t}$ to compare these frames, as in (20).

We have not said anything about the real structure. The real structure arises from the unitarity of the solutions, much as it does in other twistor constructions. Unlike constructions adapted to a definite signature, however, our real structure, $\sigma\colon \lambda \longmapsto 1/\bar{\lambda}$, $\eta \longmapsto -\bar{\lambda}^{-2}\bar{\lambda}$, on $T\mathbb{P}^1$ has fixed points which occur on equatorial fibres. This is one reason why equatorial fibres have a special role. Fixed points make it harder to treat the triviality over real sections.

2.4. *A note on compactifications.* We have said that solutions satisfying the monodromy condition correspond to bundles on the compactification of $T\mathbb{P}^1$, but have used both the one-point compactification and the fibrewise compactification obtained by adding a whole section at infinity, G_∞. The second compactification is a smooth surface, and is more convenient at present. We can use both compactifications because G_∞ is a rational curve of negative self-intersection, hence bundles on $\widetilde{T\mathbb{P}^1}$ which are trivial on G_∞ are in one-to-one correspondence with bundles on $T\mathbb{P}^1 \cup \{\infty\}$ which is the surface with G_∞ contracted.

3**. Ruled surfaces and holomorphic jumps.** In this section, we prepare for the construction of the solitons by giving monad representations of our bundles.

The surface $\widetilde{T\mathbb{P}^1}$ is a rational ruled surface, meaning that it is birational to $\mathbb{P}^2$ and is a fibration with rational curves as fibres. Since our bundles are framed and are trivial on generic fibres, they are determined by neighbourhoods of their jumping (holomorphically nontrivial) fibres, plus a choice of framing along a transverse section [Hu]. The natural section along which to frame is the section at infinity, and the bundle data includes a unitary framing along this section.

The real structure implies that jumping fibres come in pairs, reflected across the equator, with conjugate framings. This property implies that real bundles which are trivial on equatorial fibres and the section at infinity are automatically trivial on real sections (Section 5).

Birationality implies that by blowing up and down the surface, and taking inverse and direct images of bundles, we can study our bundles on any rational surface. In [An98]

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we used the birational map $\rho\colon \widetilde{T\mathbb{P}^1} \longrightarrow \{(X, Y, W) \in \mathbb{P}^2\}$ whose graph is

$$\operatorname{graph}(\rho) \stackrel{\text{def}}{=} \left\{X = \lambda Y, (X + Y)W = \eta Y^2\right\} \subset \widetilde{T\mathbb{P}^1} \times \mathbb{P}^2,$$

and studied the bundles on $\mathbb{P}^2$. This equivalence results from blowing up $(\lambda = -1, \eta = 0) \in \widetilde{T\mathbb{P}^1}$, blowing down the proper transform of $\{\lambda = -1\}$ and then blowing down the image of $\{\eta = \infty\}$, the section at infinity.

On $\mathbb{P}^2$ we have the representation:

THEOREM 3 ([DO], [OSS]). *Framed holomorphic vector bundles, V, on $\mathbb{P}^2$ with $c_2(V) = k'$ and rank N which are trivialised on $\{X + Y = 0\}$ can be represented as $V = \ker K / \operatorname{im} J$, where*

$$O(-1)^{k'} \xrightarrow{J} O^{2k'+N} \xrightarrow{K} O(1)^{k'}, \tag{21}$$

$$J \stackrel{\text{def}}{=} \begin{pmatrix} \mathbb{I} \\ 0 \\ 0 \end{pmatrix} W + \begin{pmatrix} 0 \\ \mathbb{I} \\ 0 \end{pmatrix} (X - Y) + \begin{pmatrix} \underline{\alpha}_1 \\ \underline{\alpha}_2 \\ \underline{a} \end{pmatrix} (X + Y), \text{ and}$$

$$K \stackrel{\text{def}}{=} (0 \quad \mathbb{I} \quad 0)\, W + (-\mathbb{I} \quad 0 \quad 0)\, (X - Y) + (-\underline{\alpha}_2 \quad \underline{\alpha}_1 \quad \underline{b})\, (X + Y),$$

are linear maps built up from $k' \times k'$, $k' \times k'$, $N \times k'$, $k' \times N$ matrices, $\underline{\alpha}_1, \underline{\alpha}_2, \underline{a}, \underline{b}$. (This representation is unique up to the natural action of $\mathrm{Gl}(k')$ *on these matrices.)*

Because bundle maps are represented by monad maps, and the induced action of σ on $\mathbb{P}^2$ is linear, monads representing real bundles have a real form [An98, Section 3]:

$$\begin{aligned} \underline{\alpha}_2 &= \begin{pmatrix} -\mathbb{I} - 2\gamma^* & \\ & \mathbb{I} + 2\gamma \end{pmatrix} \\ \underline{\alpha}_1 &= \begin{pmatrix} -\alpha_1^* & \phi_1 \\ \phi_2 & \alpha_1 \end{pmatrix} \\ \underline{b} &= \begin{pmatrix} ia^* \\ b \end{pmatrix} \\ \underline{a} &= (ib^* \quad a), \end{aligned} \tag{22}$$

where ϕ_1 and ϕ_2 are determined by linear equations (7) and (8), which are nondegenerate precisely when the eigenvalues of α_2 have strictly positive real part, which can be arranged iff the bundle has no jumping lines on the equator.

In this representation, it is easy to identify projective subspaces on which the bundle is trivial, and to identify natural bases of global sections in terms of the monad data. Since the frames f, h are determined by trivialisations above rational curves, this property facilitates the reconstruction of some of the frames. Unfortunately, not all rational curves in $\mathbb{P}^2$ are hyperplanes. The sections of $T\mathbb{P}^1$, for example, have self-intersection two, not one, so generically they will be transformed into quadric curves. This is why the formula we derive in [An98] is only valid on the hyperplane $\{y + t = 0\} \subset \mathbb{R}^3$.

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Since $\mathbb{R}^3$ is foliated by the translates of $\{y + t = 0\}$, we can compute J globally by computing $J(x, y, t = -y + t_0)$, for any t_0, using the bundle $V_t = (\rho^{-1} \circ \delta_{t_0})^* V$ in place of $\rho^{-1*} V$. Concretely, we need to compute the effect of $\delta_{t_0}^*$ on the monad (21). Since $\rho^{-1} \circ \delta_t \circ \rho$ acts discontinuously on $\mathbb{P}^2$, we need to introduce monads on $\widetilde{T\mathbb{P}^1}$, which we do via the equivalence between trivialised $\mathbb{P}^2$ bundles and trivialised $\widetilde{T\mathbb{P}^1}$ bundles. We thus avoid existence and uniqueness questions for such representations, at the expense of not having a more general theory of such monads.

LEMMA 1. *The equivalence of spaces of bundles on $\widetilde{T\mathbb{P}^1}$ and $\mathbb{P}^2$, trivial on $\{\eta = \infty\} \cup \{\lambda = -1\}$ and $\{X + Y = 0\}$ respectively, induced by ρ, is realised explicitly on the level of monad representatives by the map*

$$\begin{array}{ccccc} O(-G_0)^k & \xrightarrow[\omega_2]{\omega_1} & \begin{array}{c} O(-G_0 + P_\infty)^k \\ \oplus \\ O^{k+N} \end{array} & \xrightarrow[\omega_4]{\omega_3} & O(P_\infty)^k \\ {\scriptstyle D}\downarrow & & {\scriptstyle A}\downarrow{\scriptstyle B} & & {\scriptstyle C}\downarrow \\ O(-1)^k & \xrightarrow{J} & O^{2k+N} & \xrightarrow{K} & O(1)^k \end{array} \tag{23}$$

where $P_\infty = \{\lambda = \infty\}$ and $G_0 = \{\eta = 0\}$,

$$\begin{gathered} \omega_1 = (\underline{\alpha}_2 + \mathbb{I})\,\lambda + (\underline{\alpha}_2 - \mathbb{I}) \\ \omega_2 = \begin{pmatrix} -I \\ 0 \end{pmatrix} \eta + \begin{pmatrix} -2\underline{\alpha}_1 - \underline{\alpha}_1\underline{\alpha}_2 \\ \underline{a} \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 \\ 2\underline{a} \end{pmatrix} \lambda + \begin{pmatrix} -2\underline{\alpha}_1 + \underline{\alpha}_1\underline{\alpha}_2 \\ \underline{a} \end{pmatrix} \\ \omega_3 = \eta + (2\underline{\alpha}_1 + \underline{\alpha}_2\underline{\alpha}_1)\lambda^2 + (2\underline{\alpha}_1 - \underline{\alpha}_2\underline{\alpha}_1) \\ \omega_4 = (\underline{\alpha}_2 + \mathbb{I} \quad b)\,\lambda + (\underline{\alpha}_2 - \mathbb{I} \quad b) \end{gathered} \tag{24}$$

and

$$\begin{gathered} A = \begin{pmatrix} -\underline{\alpha}_1 \\ 0 \\ 0 \end{pmatrix} (X - Y) + \begin{pmatrix} 0 \\ \mathbb{I} \\ 0 \end{pmatrix} (X + Y) \\ B = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix} \\ C = \mathbb{I} \\ D = \mathbb{I}\,(X + Y). \end{gathered}$$

PROOF. One verifies that the first row is a monad, *i.e.*,

- that the composition is zero, and
- that the first map is injective and the second surjective at every point.

Since we assume that the second row is a monad, and we can verify that the diagram commutes, we have an induced bundle map lifting ρ.

It is a bundle isomorphism over the smooth points of ρ since $D, (A, B), C$ are full-rank there. (The smooth points form the open subset on which ρ is a bijection, which is isomorphic to $\{X + Y \neq 0\} \subset \mathbb{P}^2$.)

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By assumption, the bundle is trivial on the exceptional curve in $\mathbb{P}^2$. It remains to check that the bundle defined by the first row is trivial on its exceptional curves, $\{\lambda = -1\}$ and $\{\eta = \infty\} \stackrel{\text{def}}{=} G_\infty$.

Let V be the bundle represented by the first row of (23), and let

$$\begin{aligned} \omega_1 &= \omega_1^0 + \omega_1^1\lambda, \\ \omega_2 &= \omega_2^0\eta + \omega_2^1 + \omega_2^2\lambda + \omega_2^3\lambda^2, \\ \omega_3 &= \omega_3^0\eta + \omega_3^1 + \omega_3^2\lambda + \omega_3^3\lambda^2, \\ \omega_4 &= \omega_4^0 + \omega_4^1\lambda. \end{aligned} \tag{25}$$

Then

$$V|_{G_\infty} \text{ is trivial} \quad \text{iff} \quad \mathrm{H}^0\big(G_\infty, V(-1)\big) = 0 = \mathrm{H}^1\big(G_\infty, V(-1)\big). \tag{26}$$

Restricting the monad to G_∞ and twisting by $O(-1)$ gives the monad

$$0 \longrightarrow O(-1)^k \xrightarrow[\omega_2^0]{\omega_1^0+\omega_1^1\lambda} \begin{matrix} O^k \\ \oplus \\ O(-1)^{k+N} \end{matrix} \xrightarrow[\omega_4^0+\omega_4^1\lambda]{\omega_3^0} O^k \longrightarrow 0.$$

To any monad, we can associate a diagram of short exact sequences of bundles called the *display*:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & O(-1)^k & \xrightarrow{\binom{\omega_1}{\omega_2}} & \ker\begin{pmatrix}\omega_3\\ \omega_4\end{pmatrix}^t & \longrightarrow & V|_{G_\infty}(-1) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & O(-1)^k & \xrightarrow{\binom{\omega_1}{\omega_2}} & O^k \oplus O(-1)^{k+N} & \longrightarrow & \operatorname{coker}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle (\omega_3\ \omega_4)^t} & & \downarrow{\scriptstyle (\omega_3\ \omega_4)^t} & & \\ & & & & O^k & = & O^k & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{27}$$

Because the rows and columns are exact, they induce long exact sequences in cohomology. From the first row and the fact $H^i\big(\mathbb{P}^1, O(-1)\big) = 0$ for $i = 0, 1, 2$, we see that

$$\mathrm{H}^i\big(G_\infty, V(-1)\big) \cong \mathrm{H}^i\left(G_\infty, \ker\begin{pmatrix}\omega_3\\ \omega_4\end{pmatrix}^t\right), \tag{28}$$

for $i = 1, 2$. From the second column, it follows that the RHSs are zero iff

$$\mathrm{H}^0\begin{pmatrix} O^k \\ \oplus \\ O(-1)^{k+N} \end{pmatrix} \xrightarrow[\omega_4]{\omega_3^0} \mathrm{H}^0(O^k)$$

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is bijective, which is true since $\omega_3^0 = \mathbb{I}$.

The restriction of V to a fibre $\{\lambda = -1\}$ is given by the restricted monad:

$$0 \longrightarrow O(-1)^k \xrightarrow[\omega_2(-1)]{\omega_1^0-\omega_1^1} \begin{matrix} O(-1)^k \\ \oplus \\ O^{k+N} \end{matrix} \xrightarrow[\omega_4^0-\omega_4^1]{\omega_3(-1)} O^k \longrightarrow 0.$$

We will show that the represented bundle has N nonvanishing sections.

Using the same long exact sequences as above and $H^1\big(O(-1)\big) = 0$, we see that these sections are isomorphic to sections of $\ker \begin{pmatrix} \omega_3 \\ \omega_4 \end{pmatrix}^t \Big|_{\lambda=-1}$, which are naturally contained in the sections of $O(-1)^k \oplus O^{k+N}$. Since the first summand has no sections, $H^0\left(\ker \begin{pmatrix} \omega_3 \\ \omega_4 \end{pmatrix}^t \Big|_{\lambda=-1}\right) \cong \ker(\omega_4^0 - \omega_4^1)$. Since $\omega_4^0 - \omega_4^1 = \begin{pmatrix} -2\mathbb{I} \\ 0 \end{pmatrix}^t$ is surjective V has N sections.

None of these sections acquires a zero in the quotient $\ker \begin{pmatrix} \omega_3 \\ \omega_4 \end{pmatrix}^t / \operatorname{im} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$, because $\omega_1^0 - \omega_1^1 = -2\mathbb{I}$ implies

$$\operatorname{im} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \cap \begin{pmatrix} \{0\} \\ \oplus \\ O^{k+N} \end{pmatrix} = \{0\}. \qquad \blacksquare$$

3.1. *Monad automorphism.* Monads on $\widetilde{T\mathbb{P}^1}$ consist of more matrix data than the standard $\mathbb{P}^2$ monads, and since they represent the same space, have a correspondingly larger group of automorphisms. It is precisely the extra group action which we require to calculate the effect of time translation.

A monad automorphism has the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & O(-G)^k & \xrightarrow[\omega_2]{\omega_1} & \begin{matrix} O(-G+P)^k \\ \oplus \\ O^{k+N} \end{matrix} & \xrightarrow[\omega_4]{\omega_3} & O(P)^k & \longrightarrow & 0 \\ & & \downarrow A & & \downarrow & & \downarrow E & & \\ 0 & \longrightarrow & O(-G)^k & \xrightarrow[\omega_2']{\omega_1'} & \begin{matrix} O(-G+P)^k \\ \oplus \\ O^{k+N} \end{matrix} & \xrightarrow[\omega_4']{\omega_3'} & O(P)^k & \longrightarrow & 0, \end{array}$$

where $A, E \in \operatorname{Aut}(O^k) \cong \operatorname{Gl}(k)$, since they are automorphisms of a sum of identical line bundles. The middle bundle of the monad is not homogeneous, and the component acting on it has the block form:

$$\begin{pmatrix} B & \\ C_0 + C_1\lambda & D \end{pmatrix} : O\,(-G+P)^k \oplus O^{k+N} \longrightarrow O\,(-G+P)^k \oplus O^{k+N}, \tag{29}$$

for $B \in \operatorname{Gl}(k)$, $C_i \in M_{k+N}^k$, $D \in \operatorname{Gl}(N)$. The group of monad automorphisms acts on the monad data by

$$\omega_1 \longmapsto B\omega_1 A^{-1}$$

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$$
\begin{aligned}
\omega_2^0 &\longmapsto D\omega_2^0 A^{-1}\\
\omega_2^1 &\longmapsto D\omega_2^1 A^{-1} + C_0\omega_1^0 A^{-1}\\
\omega_2^2 &\longmapsto D\omega_2^2 A^{-1} + C_0\omega_1^1 A^{-1} + C_1\omega_1^0 A^{-1}\\
\omega_2^3 &\longmapsto D\omega_2^3 A^{-1} + C_1\omega_1^1 A^{-1}\\
\omega_3^0 &\longmapsto E\omega_3^0 B^{-1}\\
\omega_3^1 &\longmapsto E\omega_3^1 B^{-1} - E\omega_4^0 D^{-1} C_0 B^{-1}\\
\omega_3^2 &\longmapsto E\omega_3^2 B^{-1} - E\omega_4^1 D^{-1} C_0 B^{-1} - E\omega_4^0 D^{-1} C_1 B^{-1}\\
\omega_3^3 &\longmapsto E\omega_3^3 B^{-1} - E\omega_4^1 D^{-1} C_1 B^{-1}\\
\omega_4 &\longmapsto E\omega_4 D^{-1}.
\end{aligned}
\tag{30}
$$

4. **Closed form.** Under ρ, the sections $G_{z,t} \stackrel{\text{def}}{=} \{\eta = z - 2it\lambda - \bar{z}\lambda^2\}$ are mapped to quadrics on $\mathbb{P}^2$. Generically they are smooth, but if the section contains $(\lambda = -1, \eta = 0)$, *i.e.*, if $t + y = 0$, then the quadric decomposes into two lines: $(X + Y)(\bar{z}X - zY + 2W)$. The bundle is always trivial on the first line, because this is the exceptional divisor of the birational map. The sections over the quadric are thus the same as the sections over $(\bar{z}X - zY + 2W)$. Knowing these sections sufficed to compute the adapted frames and, using the relationship (20) between the adapted frames and J, to compute

$$
J(x, y, t = -y) = \mathbb{I} + \underline{a}\underline{\alpha}_2^{-1}(\underline{\alpha}_1 + x\underline{\alpha}_2 + iy)^{-1}\underline{b}. \tag{31}
$$

Since unitons are static solutions of Ward's equations, this was sufficient for unitons. We now extend this formula to all values of t by computing its time-translates.

The group $\delta_t : t_0 \longmapsto t_0 + t$ acting on $\mathbb{C}^3 = \{\text{sections of } T\mathbb{P}^1\}$, induces an action on $T\mathbb{P}^1$ which extends (trivially) to a continuous action on $\widetilde{T\mathbb{P}^1}$. The action $\delta_t : \eta \longmapsto \eta - 2it\lambda$ stabilises fibres. The effect of pull-back by $\delta_{t+y} : \eta \longmapsto \eta - 2i(t+y)\lambda$ on the $\widetilde{T\mathbb{P}^1}$ monad is to introduce a λ term into both ω_2 and ω_3 where there was none before.

To restore the normalization, we look for an action of the "extra" parabolic subgroup (see (30)):

$$
\begin{array}{ccc}
\cdots \longrightarrow & \begin{array}{c} O(-G_0 + P_\infty)^k \\ \oplus \\ O^{k+N} \end{array} & \begin{array}{c} \xrightarrow{\omega_3 + \mathbb{I}\lambda} \\ \xrightarrow[\omega_4]{} \end{array} O(P_\infty)^k \\
& \Big(\begin{smallmatrix} \mathbb{I} \\ C_0 + C_1\lambda & \mathbb{I} \end{smallmatrix}\Big) \Big\downarrow & \Big\downarrow \mathbb{I} \\
\cdots \longrightarrow & \begin{array}{c} O(-G_0 + P_\infty)^k \\ \oplus \\ O^{k+N} \end{array} & \begin{array}{c} \xrightarrow{\delta_{(t+y)}(\omega_3)} \\ \xrightarrow[\delta_{(t+y)}(\omega_4)]{} \end{array} O(P_\infty)^k.
\end{array}
\tag{32}
$$

Assuming that the monad data has been put in the real form (22), we calculate the group element to be

$$
C_1 = \begin{pmatrix} i(t+y)(\mathbb{I} - 3\gamma^{*2} - 2\gamma^{*3}) & 0 \\ 0 & -i(t+y)\gamma(3 + 2\gamma) \\ 0 & 0 \end{pmatrix}
$$

$$
C_0 = \begin{pmatrix} i(t+y)\gamma^*(3 + 2\gamma^*) & 0 \\ 0 & -i(t+y)(\mathbb{I} + \gamma)^{-1}(\mathbb{I} - 3\gamma^2 - 2\gamma^3) \\ 0 & 0 \end{pmatrix}
$$

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and the resulting action on the normalised data to be

$$\tilde{\delta}_{(t+y)}(\alpha_1) = \alpha_1 + 2i(t+y)\gamma(\mathbb{I}+\gamma).$$

Putting this back into (31) we obtain the required closed form (6).

5**. Real triviality.** In [An98, Section 6], we used the $\mathbb{P}^2$ monad representation to show that soliton bundles whose only jumping lines among the fibres of $\widetilde{T\mathbb{P}^1}$ are $\{\lambda = 0\}$ and $\{\lambda = \infty\}$ are necessarily trivial on real sections:

LEMMA 2 ([An98, 6.1]). *Any real bundle (in the sense of this paper) which is trivial on nonpolar fibres and the section at infinity is also trivial on the zero section, and hence on all real sections.*

In general, soliton bundles can have a finite number of conjugate pairs of jumping fibres. We first prove the intermediate case of one pair, not necessarily $\{\lambda = 0\}$ and $\{\lambda = \infty\}$ and from there go on to the general case.

LEMMA 3. *Any real bundle (in the sense of this paper) which is trivial on all but one pair of (nonequatorial) conjugate fibres and the section at infinity is also trivial on real sections.*

PROOF. Let V be an arbitrary real bundle with only one pair of conjugate jumping fibres, at $\lambda = 1 + 2\mu$ and $\lambda = 1/(1+2\overline{\mu})$, *i.e.*, μ is the only eigenvalue of γ. Let

$$f(\lambda) = \frac{1}{|1+2\mu|^2}\frac{\lambda - 1 - 2\mu}{\lambda + \frac{1}{1+2\overline{\mu}}}, \tag{33}$$

a map $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$. Then $(f_*)^*V$, where f_* is the tangent map to f, is real with polar jumping lines, so it is trivial on real sections by the previous lemma .

Since $\sigma|_{\mathbb{P}^1} \circ f = f \circ \sigma|_{\mathbb{P}^1}$, it follows from the chain rule that $\sigma|_{\mathbb{P}^1{}_*} \circ f_* = f_* \circ \sigma|_{\mathbb{P}^1{}_*}$, so the map f_* is compatible with the real structure $\sigma = -\sigma|_{\mathbb{P}^1{}_*}$ and sends real sections to real sections. It follows that V is trivial on real sections, as well. ∎

PROPOSITION 1. *Any real bundle which is trivial on the section at infinity and equatorial fibres is trivial on real sections.*

DEFINITION. Given a meromorphic section of a vector bundle, s, let $\zeta_\Sigma(s)$ be the "net zeros" of s on Σ, *i.e.*, only singular points in Σ contribute, $\zeta_{\Sigma_1 \cup \Sigma_2} = \zeta_{\Sigma_1} + \zeta_{\Sigma_2}$, and $\zeta_{\{\lambda=\lambda_0\}}(s)$ is the largest i such that $(\lambda - \lambda_0)^{-i}s$ is holomorphic at $\lambda = \lambda_0$.

Let $P \subset G$ be a discrete set, and for each $p \in P$ let T_p be a meromorphic matrix function defined on a neighbourhood of p which is nonsingular away from p. Let V be a bundle over G and V' another bundle which is isomorphic to V on the complement of P, and defined by transition functions, T_p, near P. Then s can also be interpreted as a meromorphic section of V', and as such $\zeta(s) = \zeta_{G\setminus P}(s) + \sum_{p\in P}\zeta_{\{p\}}(T_p s)$.

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PROOF. Let V be such a bundle. Let J_i, J_i' be its paired jumping fibres. The bundle V is determined by the restriction of V to neighbourhoods of J_i, J_i' together with a framing along G_∞, the section at infinity. We can think of this as constructing V out of a trivial bundle on the complement of $\bigcup\{J_i, J_i'\}$ and transition functions near J_i, J_i'.

Consider a fixed real section, G. A bundle over G with vanishing first Chern class is trivial iff it has no meromorphic sections with more zeros than poles (counted with multiplicity).

Let $\{p_i = J_i \cap G,\ q_i = J_i' \cap G\}$, and let $V|_G$ be defined by transition functions T_i and T_i'. Let s be a meromorphic section of V, expressed in terms of the trivialised bundle. Then the number of zeros of s as a section of $V|_G$ is

$$\zeta_{G\setminus\cup\{p_i,q_i:i=1\ldots l\}}(s) + \sum_{i=1}^{l}\Big(\zeta_{\{p_i\}}(T_i s) + \zeta_{\{q_i\}}(T_i' s)\Big). \tag{34}$$

Since $\zeta_G(s) \leq 0$, the result follows from the fact that, for each pair $\{p_i, q_i\}$,

$$\zeta_{\{p_i\}}(T_i s) + \zeta_{\{q_i\}}(T_i' s) \leq \zeta_{\{p_i,q_i\}}(s). \tag{35}$$

Now fix i. The inequality (35) is satisfied for all choices of s iff the bundle V_i with jumping fibres J_i, J_i', given by T_i, T_i', is trivial when restricted to G. This follows from the previous lemma. ∎

6. **Generating solutions.** To generate solutions, one must find complex matrices satisfying the monad equation and the nondegeneracy condition. If solutions are sought for a unitary group of arbitrarily large rank, then the nondegeneracy condition poses no obstruction. By this we mean that for arbitrary matrices γ, α_1 we can, in a routine way, find a, b to satisfy the monad equation $[\alpha_1, \gamma] + ba = 0$. If the nondegeneracy condition is not satisfied, we can increase N by $2k$ and construct new a', b' in block form

$$a' = \begin{pmatrix} a \\ \mathbb{I} \\ 0 \end{pmatrix}, \qquad b' = \begin{pmatrix} b & 0 & \mathbb{I} \end{pmatrix}$$

which will satisfy the nondegeneracy conditions. For fixed rank, however, nondegeneracy does pose an obstruction.

Since the matrix data is unique up to the action of Gl(k), it makes sense to assume either α_1 or α_2 is in Jordan normal form. Since the eigenvalues of α_2 correspond to the location of jumping lines, α_2 is the logical candidate. The simplest solutions are those with α_2 diagonal with distinct eigenvalues. These solutions consist of k noninteracting solitons whose velocities are determined by the real components of the eigenvalues of γ. For example

$$\begin{aligned} \alpha_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \gamma = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.1 \end{pmatrix} \\ a &= \begin{pmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 5 \end{pmatrix} \end{aligned} \tag{36}$$

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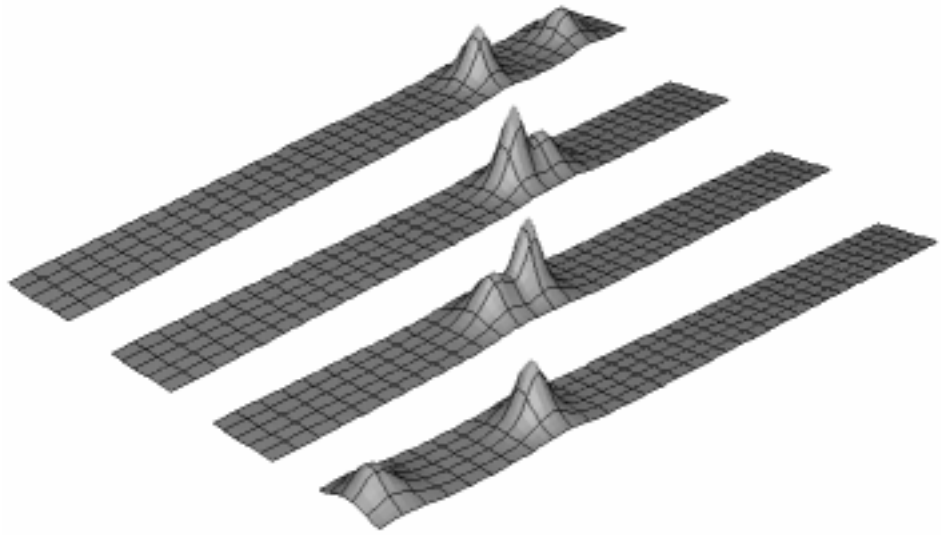

FIGURE 1: Evolution of noninteracting (1-uniton) solitons. The energy density is plotted on the vertical axis and space directions on the horizontal. The scale is arbitrary, but fixed from one picture to the next.

is a solution with two solitons, whose energy densities are plotted in Figure 1. Solutions corresponding to α_2 diagonal with $n < k$ distinct eigenvalues are observed to possess n noninteracting multi-solitons.

For interacting solutions, one may as well assume γ to be nilpotent (with high nilpotency degree). It is not hard to find solutions exhibiting k separated solitons (where k is the second Chern class of the bundle which can be interpreted as a topological charge [An97, 2.4]). For example

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad b = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{37}$$

exhibits three solitons which interact pairwise by forming ring-shaped bound states. (See Figure 2.)

6.1. *Static solutions.* To construct static solutions, one must find monad data which represent bundles admitting a lift of the one-parameter group of deformations of $\widetilde{T\mathbb{P}^1}, \delta_t$, induced by time translation in $\mathbb{R}^{2+1}$. If such a lift exists, the generic triviality and bundle framing ensure that it is unique. In terms of the monad data, the lift has a representative $g \in \mathrm{gl}(k)$ such that

$$[g, \alpha_1] = \gamma, \quad [g, \gamma] = 0, \quad gb = 0, \quad ag = 0. \tag{38}$$

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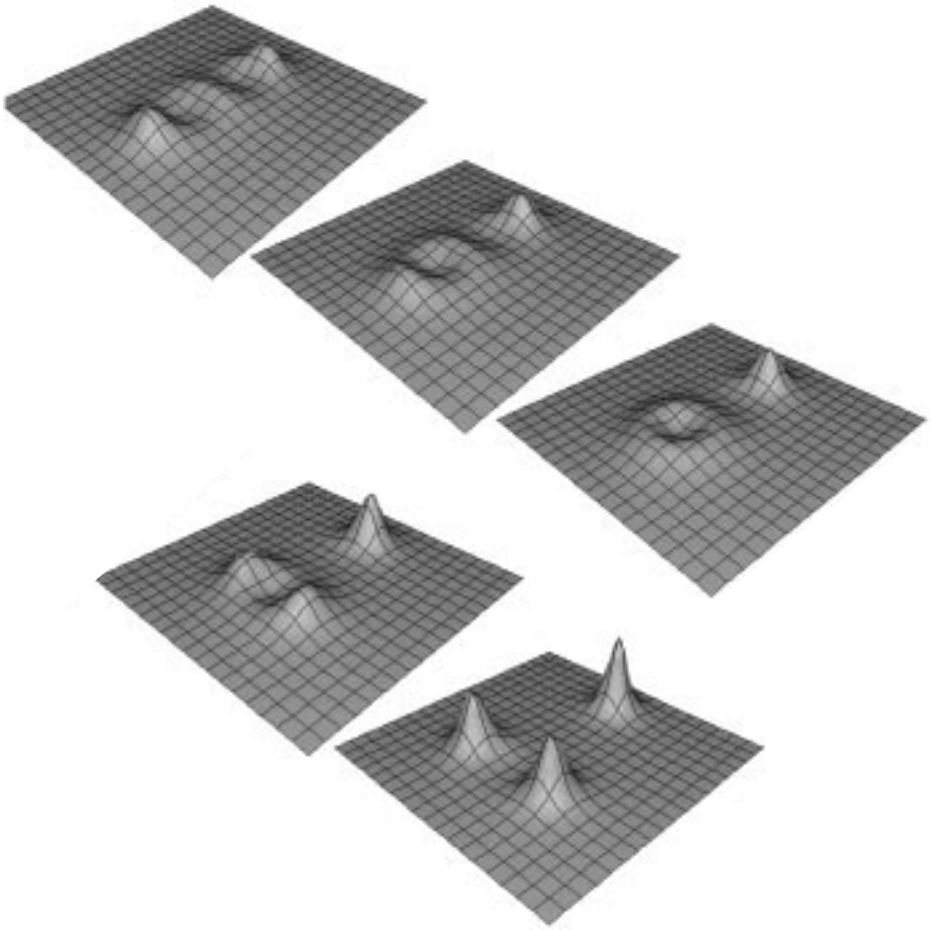

FIGURE 2: With γ maximally nilpotent, we observe three separated solitons. The energy densities are plotted for times $t = 0, 0.25, 0.5, 1, 2$. The 'interaction' is symmetric about $t = 0$ with two solitons approaching from the right and forming a ring which devolves into two solitons one returning to the right and the other meeting a soliton from the left to form a ring (at $t = .5$) which itself devolves into two solitons scattering to the left as shown.

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In the static case, the topological charge k is also the energy level, and the energy seems to be bounded below by the square of the uniton number (the nilpotency degree of γ). An example of an energy four solution is

$$\alpha_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \gamma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{39}$$

whose energy density is plotted in Figure 3.

REMARK. The illustrations for this paper were all generated using Richard Palais's 3D-filmstrip program, available from
http://rsp.math.brandeis.edu/3D-Filmstrip_html/3D-FilmstripHomePage.html
These data are all included in the program (although other data may be entered). Because the calculation of the energy densities has been optimised, new examples with $k \leq 4$ can be rendered on any Power Macintosh in seconds to minutes. Since these are dynamic phenomena, the animations provide much better illustrations. Prepared animations of these and other examples are available from the author's web page:
http://gauss.univ-brest.fr/~anand
Readers with access to Maple may also download a Maple program to generate solutions in the restricted case that γ is nilpotent.

ACKNOWLEDGMENTS. I am grateful to Richard Palais for creating the 3D-Filmstrip mathematical visualisation software and helping me incorporate these solutions into it. Research supported by an NSERC postdoctoral fellowship.

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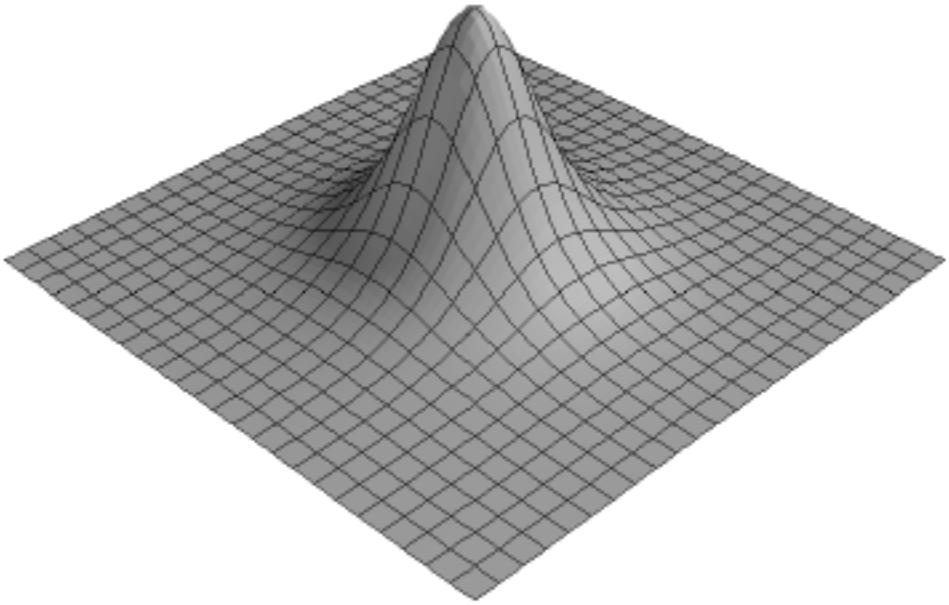

FIGURE 3: This static solution is a bound state of 'energy' four. As a harmonic map, it is a 2-uniton of minimal energy.

[Wa95] ———, *Nontrivial scattering of localised solitons in a* (2 + 1)*-dimensional integrable system*. Phys. Lett. A **208**(1995), 203–208.

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