



A Fractal Function Related to the John–Nirenberg Inequality for $Q_\alpha(\mathbb{R}^n)$

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Abstract. A borderline case function f for $Q_\alpha(\mathbb{R}^n)$ spaces is defined as a Haar wavelet decomposition, with the coefficients depending on a fixed parameter $\beta > 0$. On its support $I_0 = [0, 1]^n$, $f(x)$ can be expressed by the binary expansions of the coordinates of x . In particular, $f = f_\beta \in Q_\alpha(\mathbb{R}^n)$ if and only if $\alpha < \beta < \frac{n}{2}$, while for $\beta = \alpha$, it was shown by Yue and Dafni that f satisfies a John–Nirenberg inequality for $Q_\alpha(\mathbb{R}^n)$. When $\beta \neq 1$, f is a self-affine function. It is continuous almost everywhere and discontinuous at all dyadic points inside I_0 . In addition, it is not monotone along any coordinate direction in any small cube. When the parameter $\beta \in (0, 1)$, f is onto from I_0 to $[-\frac{1}{1-2^{-\beta}}, \frac{1}{1-2^{-\beta}}]$, and the graph of f has a non-integer fractal dimension $n + 1 - \beta$.

1 Introduction

Recently, Essén, Janson, Peng, and Xiao [8] introduced the spaces $Q_\alpha(\mathbb{R}^n)$, corresponding to a parameter $\alpha \in \mathbb{R}$. $Q_\alpha(\mathbb{R}^n)$ are subspaces of the space $BMO(\mathbb{R}^n)$ (see [10, 11, 13] for more information), which are proper and nontrivial when $0 \leq \alpha < 1$ (if $n \geq 2$), or $0 \leq \alpha \leq \frac{1}{2}$ (if $n = 1$) (see [8]). For α in this range, $Q_\alpha(\mathbb{R}^n)$ share some important properties with BMO , such as the relation with Carleson measures (see [8]), duality (see [3]), and decomposition via wavelets or quasi-orthogonal “atoms” (see [4, 8]). Moreover, $Q_\alpha(\mathbb{R}^n)$ are also related to Besov spaces, Sobolev spaces, and Morrey spaces, and hence have important applications to partial differential equations (see [16]).

In particular, analogous to the characterization of functions in $BMO(\mathbb{R}^n)$ in terms of the John–Nirenberg inequality (see [13, 14]), Essén, Janson, Peng, and Xiao [8] conjectured a John–Nirenberg type inequality for $Q_\alpha(\mathbb{R}^n)$. Yue and Dafni [17] proved modified, separate sufficient, and necessary versions of the conjecture. A function constructed in [17], as a counterexample, shows the necessity of the modification.

The function $f = \sum_{l=0}^{\infty} f_l$ is defined as a sum of multiples of Haar functions, with the coefficients depending on the parameter α of the space. When $0 < \alpha < \frac{1}{2}$, $f_l \in Q_\alpha(\mathbb{R}^n)$, $\forall l \geq 0$. In addition, the sum function f is bounded and hence $f \in L^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$. However, $f \notin Q_\alpha(\mathbb{R}^n)$ though it satisfies the John–Nirenberg type inequality for $Q_\alpha(\mathbb{R}^n)$. Therefore, f provides us with a sort of borderline case for $Q_\alpha(\mathbb{R}^n)$. Replacing α in the definition of f by another parameter β and denoting the new function by f_β , we have $f_\beta \in Q_\alpha(\mathbb{R}^n)$ for all $\beta > \alpha$.

Furthermore, f has some fractal properties on its support $I_0 = [0, 1]^n$, such as self-affinity and fine structure. It is interesting to explore the fractal properties and

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dimensions of f in order to measure the complexity of the function. Understanding those fractal properties will help us to grasp the nature of $Q_\alpha(\mathbb{R}^n)$.

In this paper, we discuss the analytic and fractal properties of the function f on its support $I_0 = [0, 1]^n$. The definition, as well as an equivalent binary expression of $f(x)$, are given in Section 2. In particular, when $0 < \beta < 1$, f maps I_0 onto $[-\frac{1}{1-2^{-\beta}}, \frac{1}{1-2^{-\beta}}]$. Next, Section 3 addresses the relation of f to the space $Q_\alpha(\mathbb{R}^n)$. Then, the analytic properties of the function are discussed in Section 4 for $n = 1$, and in Section 5 for $n > 1$. In the case \mathbb{R}^1 , when $\beta \neq 1$, f is discontinuous at every dyadic point in $[0, 1]$, and continuous elsewhere. In addition, f is not monotone in any subinterval of $[0, 1]$. When $\beta = 1$, however, f is a linear function on $[0, 1]$. Most of these properties can be generalized to \mathbb{R}^n for $n > 1$, except that in the case of $\beta = 1$, f is no longer continuous everywhere and its discontinuity set is also dense in $[0, 1]^n$. Finally, following some preliminaries on fractal geometry in Section 6, Section 7 is devoted to the fractal properties and dimensions of the graph of f . It is shown that the closure of the graph of f over $I_0 = [0, 1]^n$ ($n \geq 1$) is a self-affine set in \mathbb{R}^{n+1} . When $0 < \beta < 1$, the graph has a non-integer ‘‘Falconer dimension’’ and Box dimension $n + 1 - \beta$, hence f is a fractal function for β in this range.

2 Definition of the Function f

Let I be a cube in \mathbb{R}^n with sides parallel to the coordinate axes (throughout this paper, all cubes are assumed to be like this). We denote the collection of all dyadic subcubes of I by $\mathcal{D}(I) = \bigcup_k \mathcal{D}_k(I)$ with $\mathcal{D}_0(I) = I$ and $\mathcal{D}_k(I)$ ($k \geq 1$) being the k -th generation of the dyadic subcubes of I , obtained by bisection of all the sides of I .

The collection of all dyadic cubes in \mathbb{R}^n is denoted by \mathcal{D} , that is,

$$\mathcal{D} = \left\{ J = \prod_{i=1}^n [m_i 2^{-l}, (m_i + 1) 2^{-l}] \right\}, \quad l, m_1, \dots, m_n \in \mathbb{Z}.$$

We define a function $f(x) = f_\beta(x)$ ($x \in \mathbb{R}^n$) that depends on a fixed parameter $\beta > 0$ by using a system of orthonormal Haar wavelets in \mathbb{R}^n (see [15]).

First, we give the definition of the Haar wavelets. Denote the Haar function in \mathbb{R}^1 and in \mathbb{R}^n by $h(x)$ and $H(x)$, respectively:

$$h(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \leq x < 1, \\ 0 & \text{if } x \notin [0, 1]. \end{cases}$$

and $H(x) = \prod_{i=1}^n h(x_i)$, where x_i is the i -th coordinate of x .

It is clear that $H(x)$ is supported in the unit cube $I_0 = [0, 1]^n$ and takes the value 1 or -1 on each subcube in $\mathcal{D}_1(I_0)$, the first dyadic partition of I_0 , with any two adjacent cubes having opposite values. Also note that

$$\int H(x) dx = \prod_{i=1}^n \int h(x_i) dx_i = 0.$$

The orthonormal Haar wavelets $\{H_{l,J}\}_{J \in \mathcal{D}}$ in $L^2(\mathbb{R}^n)$ are given by

$$H_{l,J}(x) = 2^{\frac{nl}{2}} H(2^l x - m) = 2^{\frac{nl}{2}} \prod_{i=1}^n h(2^l x_i - m_i).$$

Then, we define a sequence of functions $\{f_l\}_{l \geq 0}$ supported in I_0 by

$$(2.1) \quad f_l(x) = \sum_{J \in \mathcal{D}_l(I_0)} 2^{-(\beta + \frac{n}{2})l} H_{l,J}(x).$$

Namely, $f_l = 2^{-\beta l}$ or $-2^{-\beta l}$ on each dyadic subcube of I_0 with length of 2^{-l} , and $f_l = 0$ outside I_0 . Let $k \geq 0$ and J be a dyadic subcube in $\mathcal{D}_k(I_0)$; then we have $\int_J f_l(x) dx = 0$, for all $l \geq k$. Moreover, $\{f_l\}_{l \geq k}$ is a sequence of orthogonal functions on J .

Lastly, we define f as the sum function:

$$(2.2) \quad f(x) = \sum_{l=0}^{\infty} f_l(x) = \sum_{l=0}^{\infty} \sum_{J \in \mathcal{D}_l(I_0)} a_{l,J} H_{l,J}(x),$$

where $a_{l,J} = 2^{-(\beta + \frac{n}{2})l}$.

It converges absolutely since $\sum_l |f_l| \leq \sum_{l=0}^{\infty} 2^{-\beta l} = \frac{1}{1-2^{-\beta}} := C_\beta < \infty$.

In what follows, the notation C_β is reserved for this constant. We will use C for any other constant, which may also depend on β . In the original definition of the function [17], the parameter in (2.1) is α . Here we replace α by β in order to explore the relation of the function with the spaces $Q_\alpha(\mathbb{R}^n)$ in a wider range.

For $x \in I_0 = [0, 1]^n$, we can also express the function $f(x)$ in terms of the binary expansions of x_i ($i \in \{1, \dots, n\}$), where x_i is the i -th coordinate of x . This equivalent expression is more convenient to use in the discussion of the analytic and fractal properties of f .

Consider the simplest case $n = 1$. Let $x = \sum_{l=0}^{\infty} b_l 2^{-(l+1)}$, corresponding to the expansion $0.b_0 b_1 b_2 \dots$, where $b_l = 0$ or 1 . We will use the expansions ending in infinitely many zeroes rather than infinitely many ones. Thus, the function f can be written as

$$(2.3) \quad f(x) = \sum_{l=0}^{\infty} (-1)^{b_l} 2^{-\beta l} = C_\beta - 2 \sum_{l=0}^{\infty} b_l 2^{-\beta l}.$$

In general, let $x = (x_1, x_2, \dots, x_n) \in I_0$ and let $0.b_0^i b_1^i b_2^i \dots$ be the binary expansion of x_i , i.e., $x_i = \sum_{l=0}^{\infty} b_l^i 2^{-(l+1)}$, for $i \in \{1, \dots, n\}$. Then, we can write

$$(2.4) \quad f(x) = \sum_{l=0}^{\infty} (-1)^{b_l^1 + \dots + b_l^n} 2^{-\beta l}.$$

Theorem 2.1 For $0 < \beta < 1$, $f = f_\beta$ defined by (2.4) is onto from I_0 to $(-C_\beta, C_\beta]$.

Remark $-C_\beta$ is excluded in the range of the function f . This is clear from its binary expression (2.3) when $n = 1$. The value $f(x) = -C_\beta$ corresponds to $b_l = 1$ for all l , that is, the binary expansion of x is $0.111 \dots = 1$. However, $f(1) = 0$ from the definition of the function.

Proof We prove the lemma for the one dimensional case (2.3), and then for (2.4).

By the second equality in (2.3), we just need to show that for all $y \in [0, C_\beta)$, y can be expanded as

$$(2.5) \quad y = \sum_{l=0}^{\infty} b_l(y)2^{-\beta l},$$

where $b_l(y) = 0$ or 1 , for all $l \geq 0$.

If $0 < \beta < 1$, then $1 < q = 2^\beta < 2$. Using the “greedy algorithm” in [7] (see also [5, 6]), the expansion (2.5) is obtained by putting digits $b_l(y)$ inductively as follows: $b_0(y) = 1$ if $1 \leq y$, or $b_0(y) = 0$ if $1 > y$. For $l \geq 1$,

$$(2.6) \quad b_l(y) := \begin{cases} 1 & \text{if } \sum_{j=0}^{l-1} b_j(y)2^{-\beta j} + 2^{-\beta l} \leq y, \\ 0 & \text{if } \sum_{j=0}^{l-1} b_j(y)2^{-\beta j} + 2^{-\beta l} > y. \end{cases}$$

Claim 2.2 With such $b_l(y)$, $\sum_{l=0}^{\infty} b_l(y)2^{-\beta l}$ converges to $y \in [0, C_\beta)$.

While this result may be found in the literature on base q expansion (see [2]), we give our own proof. Note that for any $y \in [0, C_\beta)$, (2.6) guarantees

$$\sum_{j=0}^l b_j(y)2^{-\beta j} \leq y, \quad \forall l.$$

Since $y \neq C_\beta = \sum_{l=1}^{\infty} 2^{-\beta l}$, there exists l such that $b_l(y) = 0$. If there exists l_0 such that $b_l(y) = 0$, for all $l > l_0$, then $y = \sum_{j=0}^{l_0} b_j(y)2^{-\beta j}$, and we are done.

Otherwise, there are infinitely many l 's such that $b_l(y) = 0$ and $b_{l+1}(y) = 1$. Namely,

$$(2.7) \quad \sum_{l=0}^{l-1} b_l(y)2^{-\beta l} + 2^{-\beta(l+1)} < y < \sum_{l=0}^{l-1} b_l(y)2^{-\beta l} + 2^{-\beta l}.$$

To see this, suppose there were only finite many such l , and denote by l_0 the greatest one of them. Then we have $b_l(y) = 1$ for all $l \geq l_0 + 1$, that is,

$$(2.8) \quad \sum_{l=0}^{l_0-1} b_l(y)2^{-\beta l} + \sum_{j=l_0+1}^{\infty} 2^{-\beta j} \leq y.$$

Comparing (2.8) with the right inequality of (2.7) for $l = l_0$, we get

$$2^{-\beta l_0} > \sum_{j=l_0+1}^{\infty} 2^{-\beta j} = \frac{1}{2^\beta - 1} \cdot 2^{-\beta l_0}.$$

This is a contradiction, since $\frac{1}{2^\beta - 1} > 1$ when $0 < \beta < 1$.

So with $b_l(y)$ given by (2.6), $\sum_{l=0}^{\infty} b_l(y)2^{-\beta l}$ converges to y since its partial sums are positive, monotone increasing, and the subsequence given in the left-hand side of (2.7) clearly converges to y . This proves the claim.

Setting $x = \sum_{l=0}^{\infty} b_l(y)2^{-(l+1)}$, we have $f(x) = C_\beta - 2y$. In addition, this expression cannot end with infinitely many 1's. This proves Lemma 2.1 for $n = 1$.

For $n \geq 2$, (2.4) is onto since for each i , $f(0, \dots, x_i, \dots, 0)$ is onto from $[0, 1)$ to $(-C_\beta, C_\beta]$ based on the case $n = 1$. ■

3 Relation of f_β to $Q_\alpha(\mathbb{R}^n)$

It has been shown that for a fixed $\alpha \in \mathbb{R}$, $Q_\alpha(\mathbb{R}^n)$ is a Banach space modulo constants with three equivalent norms [8]. In this paper, we adopt one that has an analogous form with the John–Nirenberg type inequality for $Q_\alpha(\mathbb{R}^n)$.

Let f be a measurable function in \mathbb{R}^n and I be a cube in \mathbb{R}^n . Denote by $f(I)$ the mean-value of f over the cube I , i.e., $f(I) = |I|^{-1} \int_I f(x) dx$, and denote by $\Phi_f(I)$ the square mean oscillation of f over the cube I , that is,

$$\Phi_f(I) = |I|^{-1} \int_I |f(x) - f(I)|^2 dx.$$

The norm of the space $Q_\alpha(\mathbb{R}^n)$ is defined by

$$\|f\|_{Q_\alpha} = \sup_I (\Psi_{f,\alpha}(I))^{1/2},$$

where the supremum is taken over all cubes I in \mathbb{R}^n , and $\Psi_{f,\alpha}(I)$ is given by

$$\Psi_{f,\alpha}(I) = \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} \Phi_f(J),$$

where $\mathcal{D}_k(I)$ is as defined in Section 2.

Let $\lambda_I(t) = |\{x \in I : |f(x) - f(I)| > t\}|$, where $|\cdot|$ denotes the Lebesgue measure of a set. It is shown in [17], on the one hand, for $0 \leq p < 2$, that if there exist positive constants B, C , and c , such that, for all cubes $I \subset \mathbb{R}^n$, and any $t > 0$,

$$(3.1) \quad \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \leq B \max\left\{1, \frac{C}{t^p}\right\} \exp(-ct),$$

then f is a function in $Q_\alpha(\mathbb{R}^n)$.

On the other hand, the above inequality with $p = 2$ is necessary for f to be a function in $Q_\alpha(\mathbb{R}^n)$. Namely, for any $f \in Q_\alpha(\mathbb{R}^n)$, there exist two positive constants B and b , such that, for all cubes $I \subset \mathbb{R}^n$,

$$(3.2) \quad \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \leq B \max\left\{1, \frac{\|f\|_{Q_\alpha}^2}{t^2}\right\} \exp\left(\frac{-bt}{\|f\|_{Q_\alpha}}\right).$$

One can see a gap between (3.1) and (3.2). In BMO, the distribution inequality is both necessary and sufficient for a function in the space (see [1, 14]). However, in Q_α , the sufficient and necessary inequalities, (3.1) and (3.2), are not compatible. The sequence $\{f_l\}$ defined in (2.1) can be used as a counterexample to show that the majorization t^{-2} in (3.2) cannot be sharpened to t^{-p} for any $p < 2$. On the other hand, $p = 2$ cannot be included in (3.1) since $f = f_\beta$ defined in (2.2) is not a function in Q_α when $\beta = \alpha$ though it satisfies (3.1) with p replaced by 2 (see [17, Theorem 3]).

The following theorem reveals the relation between f_β and Q_α for all $\beta > 0$:

Theorem 3.1 Let $0 < \alpha < \frac{1}{2}$.

- (i) If $0 < \beta \leq \alpha$, $f_\beta \notin Q_\alpha(\mathbb{R}^n)$, while if $\beta > \alpha$, $f_\beta \in Q_\alpha(\mathbb{R}^n)$.
- (ii) If $\beta \geq \alpha$, then f_β satisfies a John–Nirenberg type inequality that can be obtained by replacing p by 2 in (3.1), in particular, it is (3.2) when $\beta > \alpha$.

Proof Part (ii) is a combination of [17, Theorem 3] for the case $\beta = \alpha$, and a corollary of part (i) for the case $\beta > \alpha$.

For part (i), let J be a subcube in $\mathcal{D}_k(I_0)$ for a fix integer $k \geq 0$. Recall (2.1), thus we have $f_l(J) = 0$ when $k \leq l$ and $f_l \equiv \pm 2^{-\beta l}$ on J when $k > l$. Hence

$$|f(x) - f(J)| = \left| \sum_{l=0}^{\infty} f_l(x) - \sum_{l=0}^{k-1} f_l(J) \right| = \left| \sum_{l=k}^{\infty} f_l(x) \right|.$$

In addition, by the orthogonality of the sequence $\{f_l\}_{l \geq k}$ on J , we have that

$$\Phi_f(J) = \frac{1}{|J|} \int_J \left| \sum_{l=k}^{\infty} f_l(x) \right|^2 dx = \frac{1}{|J|} \sum_{l=k}^{\infty} \int_J |f_l(x)|^2 dx = \sum_{l=k}^{\infty} 2^{-2\beta l} = C_{2\beta} 2^{-2\beta k}.$$

where $C_{2\beta} = \frac{1}{1-2^{-2\beta}}$. Consequently,

$$\begin{aligned} \Psi_{f,\alpha}(I_0) &= \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I_0)} \Phi_f(J) = \sum_{k=0}^{\infty} 2^{(2\alpha-n)k+nk} \cdot C_{2\beta} 2^{-2\beta k} \\ &= C_{2\beta} \sum_{k=0}^{\infty} 2^{2(\alpha-\beta)k}. \end{aligned}$$

It converges if and only if $\beta > \alpha$. This proves the first part of the claim.

To prove the second part, we still need to show that, for all cubes $I \subset \mathbb{R}^n$, $\Psi_{f,\alpha}(I)$ is bounded provided $\alpha < \beta \leq \frac{n}{2}$.

We estimate $\Psi_{f,\alpha}(I)$ for a dyadic cube.

Case 1: $I \in \mathcal{D}_j(I_0)$, $j > 0$.

By the orthogonality of the sequence $\{f_l\}_{l \geq k+j}$ on $J \in \mathcal{D}_k(I)$, we have that

$$\begin{aligned} \Phi_f(J) &= \frac{1}{|J|} \int_J \left| \sum_{l=k+j}^{\infty} f_l(x) \right|^2 dx = \frac{1}{|J|} \sum_{l=k+j}^{\infty} \int_J |f_l(x)|^2 dx \\ &= \sum_{l=k+j}^{\infty} 2^{-2\beta l} = C_{2\beta} 2^{-2\beta(k+j)}. \end{aligned}$$

So,

$$\begin{aligned} \Psi_{f,\alpha}(I) &= \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I_0)} \Phi_f(J) \\ &= \sum_{k=0}^{\infty} 2^{(2\alpha-n)k+nk} \cdot C_{2\beta} 2^{-2\beta(k+j)} \\ &= C_{2\beta} 2^{-2\beta j} \sum_{k=0}^{\infty} 2^{2(\alpha-\beta)k} \leq \frac{C_{2\beta}}{1 - 2^{\alpha-\beta}}. \end{aligned}$$

Case 2: $I = [0, 2^j]^n$, $j > 0$.

Consider $J \in \mathcal{D}_k(I)$. If $k \leq j$, then for any l , either J is disjoint from I_0 , in which case $f_l(x) \equiv 0$, or it contains I_0 , which occurs only in one case, namely $J = [0, 2^{j-k}]^n$, in which case $f_l(J) = 0$. Hence, $f_l(x) - f_l(J) = f_l(x)$ if $x \in I_0$ and $f_l(x) = 0$ if $x \in J \setminus I_0$. Thus,

$$|f(x) - f(J)| = \begin{cases} \left| \sum_{l=0}^{\infty} f_l(x) \right| & \text{if } x \in I_0 \\ 0 & \text{if } x \in J \setminus I_0. \end{cases}$$

and by the orthogonality of $\{f_l\}_{l \geq 0}$ on I_0 ,

$$\begin{aligned} \Phi_f(J) &= \frac{1}{|J|} \int_{I_0} \left| \sum_{l=0}^{\infty} f_l(x) \right|^2 dx = \frac{1}{|J|} \sum_{l=0}^{\infty} \int_{I_0} |f_l(x)|^2 dx \\ &= \frac{|I_0|}{|J|} \sum_{l=0}^{\infty} 2^{-2\beta l} = C_{2\beta} 2^{-n(j-k)}. \end{aligned}$$

If $k > j$, then for $0 < k - j \leq l$, again either J is disjoint from I_0 , in which case $f_l(J) = 0$, or it is contained in I_0 , i.e., $J \in \mathcal{D}_{k-j}(I_0)$, and again, $f_l(J) = 0$ and $f_l(x) - f_l(J) = f_l(x)$.

As for $k - j > l$, we also have that either J is disjoint from I_0 and $|f_l(x) - f_l(J)| = 0$, or $J \in \mathcal{D}_{k-j}(I_0)$, but now f_l is constant on J , resulting in $f_l(x) - f_l(J) = 0$.

It follows that

$$|f(x) - f(J)| = \left| \sum_{l=0}^{\infty} [f_l(x) - f_l(J)] \right| = \left| \sum_{l=k-j}^{\infty} f_l(x) \right|.$$

Once more, by the orthogonality of the sequence $\{f_l\}_{l \geq k-j}$ on J , we have that

$$\begin{aligned} \Phi_f(J) &= \frac{1}{|J|} \int_J \left| \sum_{l=k-j}^{\infty} f_l(x) \right|^2 dx = \frac{1}{|J|} \sum_{l=k-j}^{\infty} \int_J |f_l(x)|^2 dx \\ &= \sum_{l=k-j}^{\infty} 2^{-2\beta l} = C_{2\beta} 2^{-2\beta(k-j)}. \end{aligned}$$

Consequently, we have, for $\alpha < \beta$,

$$\begin{aligned} \Psi_{f,\alpha}(I) &= C_{2\beta} \left(\sum_{k=0}^j 2^{(2\alpha-n)k} \cdot 2^{-n(j-k)} + \sum_{k=j+1}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_{k-j}(I_0)} 2^{-2\beta(k-j)} \right) \\ &= C_{2\beta} \left(\sum_{k=0}^j 2^{2(\alpha-\beta)k+2\beta(k-j)+(2\beta-n)j} + \sum_{k=j+1}^{\infty} 2^{(2\alpha-n)k+(k-j)n-2\beta(k-j)} \right) \\ &\leq C_{2\beta} \sum_{k=0}^{\infty} 2^{2(\alpha-\beta)k+(2\beta-n)j} = C_{2\beta} 2^{(2\beta-n)j} \sum_{k=0}^{\infty} 2^{2(\alpha-\beta)k} = \frac{C_{2\beta} \ell(I)^{2\beta-n}}{1 - 2^{\alpha-\beta}}. \end{aligned}$$

As a result, when $\beta \leq \frac{n}{2}$, for all dyadic cubes I in \mathbb{R}^n ,

$$\Psi_{f,\alpha}(I) \leq \frac{C_{2\beta} \max\{1, |I|\}^{\frac{2\beta}{n}-1}}{1 - 2^{\alpha-\beta}} := C_{\alpha,\beta} < \infty.$$

Lastly, we use [17, Lemma 1] to estimate $\Psi_{f,\alpha}(I)$ for any cube $I \subset \mathbb{R}^n$. Here we have to restrict to $\alpha < 1/2$.

Lemma 3.2 ([17]) *Assume $\alpha < \frac{1}{2}$. Let I^1, \dots, I^l be l cubes of the same size, that is, $|I^1| = \dots = |I^l| = V$, for some $V > 0$. If a cube $I \subset I^1 \cup \dots \cup I^l$, with $V \leq |I| < 2^n V$, then,*

$$\Psi_{f,\alpha}(I) \leq C_l \left(\sum_{j=1}^l \Psi_{f,\alpha}(I^j) + \sum_{1 \leq i < j \leq l} |f(I^i) - f(I^j)|^2 \right).$$

Given $I \subset \mathbb{R}^n$, there exists an integer j such that the side length of I satisfies $2^j \leq \ell(I) < 2^{j+1}$. Moreover, there exist 2^n adjacent dyadic cubes I^1, \dots, I^{2^n} , with

side length 2^j , such that $I \subset I^1 \cup \dots \cup I^{2^n}$. Since the mean of f_i on each of these dyadic cubes, $f_i(I^i)$, is either zero or $\pm 2^{-\beta l}$, we have $|f_i(I^i) - f_i(I^j)| \leq 2 \cdot 2^{-\beta l}$.

$$\begin{aligned} \Psi_{f,\alpha}(I) &\leq C_n \left(\sum_{j=1}^{2^n} \Psi_{f,\alpha}(I^j) + \sum_{1 \leq i < j \leq 2^n} |f(I^i) - f(I^j)|^2 \right) \\ &\leq C_n \left(2^n C_{\alpha,\beta} + \frac{2^{2n} - 2^n}{2} \cdot 4C_{\beta}^2 \right) = C_{n,\alpha,\beta}. \end{aligned}$$

We are done. ■

From Theorem 3.1 we see that a borderline case occurs when $\beta = \alpha$. Moreover, by [8, Theorem 2.3], $Q_{\alpha'} \subset Q_{\alpha}$ for $\alpha < \alpha'$. f_{β} provides another example showing that this inclusion is strict.

4 Analytic Properties of f_{β} for $n = 1$

In what follows $f = f_{\beta}$ defined on I_0 unless stated otherwise. We mainly discuss the case \mathbb{R}^1 , then we generalize those results to higher dimensions.

Theorem 4.1 *Let $n = 1$. When $\beta \neq 1$, f defined by (2.2) is a right continuous function on $I_0 = [0, 1]$. It is discontinuous at all dyadic points in I_0 , and continuous elsewhere. In addition, f is not monotone in any subinterval of I_0 .*

When $\beta = 1$, f is a linear function: $f(x) = 2 - 4x$, for $x \in I_0$.

Proof Recall that the dyadic points are the end points of the dyadic intervals in $[0, 1]$, or the points with finite binary expansion. Denote by E_k the set of the end points of all dyadic intervals in $\mathcal{D}_k(I_0)$, and let $E = \bigcup_{k=0}^{\infty} E_k$, the set of dyadic points. The proof is in four parts.

Part 1 f is continuous on $I_0 \setminus E$:

By the convergence of $\sum_{l \geq 0} |f_l(y) - f_l(x)|$, we have

$$f(y) - f(x) = \sum_{l \geq 0} (f_l(y) - f_l(x)).$$

Let $x \in I_0 \setminus E$. Since $x \notin E, \forall k \geq 0, x \notin E_k$, there exists a unique subinterval in $\mathcal{D}_k(I_0)$, denote it by $J_k(x)$, such that x is an interior point of $J_k(x)$. Moreover, we have that

$$J_0(x) \supset J_1(x) \supset \dots \supset J_k(x) \supset \dots$$

and $f_l (l < k)$, as well as the partial sum $\sum_{l=0}^{k-1} f_l$, are constant on $J_k(x)$.

Let $\epsilon > 0$. For any $y \in J_k(x)$,

$$\begin{aligned} (4.1) \quad |f(y) - f(x)| &\leq \sum_{l=k}^{\infty} |f_l(y) - f_l(x)| \leq 2 \sum_{l=k}^{\infty} 2^{-\beta l} \\ &= 2 \cdot 2^{-\beta k} \sum_{l=0}^{\infty} 2^{-\beta l} = 2C_{\beta} 2^{-\beta k} < \epsilon, \end{aligned}$$

for k sufficiently large. So, f is continuous at x .

Part 2 f is discontinuous on E :

First, we have that $f(x)$ is discontinuous at $x = 0$ and $x = 1$, since

$$f(0^+) = \sum_{l=0}^{\infty} 2^{-\beta l} = C_\beta \neq 0 = f(0^-)$$

$$f(1^-) = -\sum_{l=0}^{\infty} 2^{-\beta l} = -C_\beta \neq 0 = f(1^+).$$

Then consider the unique point $x = \frac{1}{2} \in E_1 \setminus E_0$.

Claim $f(x)$ is right continuous at $\frac{1}{2}$ with

$$f\left(\frac{1}{2}\right) = f\left(\frac{1}{2}^+\right) = -1 + \sum_{l=1}^{\infty} 2^{-\beta l} = C_\beta - 2,$$

while the left limit

$$f\left(\frac{1}{2}^-\right) = 1 - \sum_{l=1}^{\infty} 2^{-\beta l} = -(C_\beta - 2).$$

Since $C_\beta = 2$ when $\beta = 1$, if $\beta \neq 1$, $f(x)$ is discontinuous at $\frac{1}{2}$ with a jump $2(C_\beta - 2)$.

Proof of the claim Let $k \geq 1$, $a_k = \frac{1}{2} - 2^{-k}$, and $d_k = \frac{1}{2} + 2^{-k}$. Let $J_k = [a_k, \frac{1}{2}]$ and $J'_k = [\frac{1}{2}, d_k]$, which are two adjacent subintervals in $\mathcal{D}_k(I_0)$ touching at $\frac{1}{2}$, such that,

$$J_k \supset J_{k+1}, \quad \text{and} \quad J'_k \supset J'_{k+1}, \quad \text{for all } k \geq 0.$$

We have

$$(i) \quad f_0(x) = \begin{cases} 1 & x \in [a_k, \frac{1}{2}); \\ -1 & x \in [\frac{1}{2}, d_k). \end{cases} \quad \forall k \geq 1;$$

$$(ii) \quad f_l(x) = \begin{cases} -2^{-\beta l} & x \in [a_k, \frac{1}{2}); \\ 2^{-\beta l} & x \in [\frac{1}{2}, d_k). \end{cases} \quad \forall l \geq 1 \quad \text{and} \quad k \geq l + 1.$$

Let $A = -1 + \sum_{l=1}^{\infty} 2^{-\beta l} = C_\beta - 2$.

If $x \in [a_k, \frac{1}{2})$, $f(x) = 1 - 2^{-\beta} - \dots - 2^{-\beta(k-1)} + \sum_{l=k}^{\infty} f_l(x)$, then,

$$|f(x) - (-A)| \leq \sum_{l=k}^{\infty} |f_l(x) + 2^{-\beta l}| \leq 2 \sum_{l=k}^{\infty} 2^{-\beta l} = 2C_\beta 2^{-\beta k} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

If $x \in [\frac{1}{2}, d_k)$, $f(x) = -1 + 2^{-\beta} + \dots + 2^{-\beta(k-1)} + \sum_{l=k}^{\infty} f_l(x)$, then

$$|f(x) - A| \leq \sum_{l=k}^{\infty} |f_l(x) - 2^{-\beta l}| \leq 2 \sum_{l=k}^{\infty} 2^{-\beta l} = 2C_\beta 2^{-\beta k} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

So, $f(\frac{1}{2}^+) - f(\frac{1}{2}^-) = 2A$. In addition, $f(\frac{1}{2}) = A = C_\beta - 2$ since $\frac{1}{2} \in [\frac{1}{2}, d_k)$, for all k .

This proves the claim. ■

Furthermore, recall $f(0) = f(0^+) = C_\beta$, $f(1^-) = -C_\beta$. From this and (4.1), we have that $\forall J \in \mathcal{D}_1(I_0)$,

$$(4.2) \quad \sup_{x \in J} f(x) - \inf_{x \in J} f(x) = 2C_\beta 2^{-\beta}.$$

Now, for any $x \in E \setminus \{0, \frac{1}{2}, 1\}$, there exists a $k \geq 1$ such that $x \in E_{k+1} \setminus E_k$. Then, there exists a subinterval $J_k(x) \in \mathcal{D}_k(I_0)$ such that x is the middle point of $J_k(x)$. Furthermore, f_0, \dots, f_{k-1} , as well as the partial sum $\sum_{l=0}^{k-1} f_l$, are constants on $J_k(x)$. Denote by $a = \sum_{l=0}^{k-1} f_l(x)$ for $x \in J_k(x)$. Similarly to the case $x = \frac{1}{2}$, we have the left limit,

$$f(x^-) = a + 2^{-\beta k} - \sum_{l=k+1}^{\infty} 2^{-\beta l} = a + 2^{-\beta k}(2 - C_\beta),$$

and the right limit,

$$f(x^+) = a - 2^{-\beta k} + \sum_{l=k+1}^{\infty} 2^{-\beta l} = a + 2^{-\beta k}(C_\beta - 2).$$

Thus, $f(x^+) - f(x^-) = 2(C_\beta - 2)2^{-\beta k}$. Therefore, there is a jump: $2(C_\beta - 2)2^{-\beta k}$ at x for $\beta \neq 1$.

The following corollary will be useful later in the estimate of the Box dimension of the graph of f .

Corollary 4.2 *Given a $k \geq 0$, and a dyadic subinterval $J \in \mathcal{D}_k(I_0)$, we have,*

$$\sup_{x \in J} f(x) - \inf_{x \in J} f(x) = 2C_\beta 2^{-\beta k}.$$

Proof The proof is similar to that of (4.2). The lower bound is obtained by comparing the value of the left end point which is right continuous and the left limit to the right end point. ■

Part 3 f is not monotone in any subinterval of I_0 :

We just give a proof for the case $0 < \beta < 1$. The proof for the case $\beta > 1$ is similar.

First, we show that $f(x)$ is not monotone in $I_0 = [0, 1]$.

On one hand, $f(x)$ is not monotone increasing in I_0 since for $\frac{3}{4} > \frac{1}{2}$, we have

$$f\left(\frac{3}{4}\right) = -1 - 2^{-\beta} + \sum_{l=2}^{\infty} 2^{-\beta l} < A = f\left(\frac{1}{2}\right).$$

On the other hand, we know from the proof of part 2 that $\lim_{k \rightarrow \infty} a_k = -A$. So there exists a $k_0 > 0$, such that, $f(a_{k_0}) < -A/2 < f\left(\frac{1}{2}\right)$. Note that $a_{k_0} = \frac{1}{2} - 2^{-k_0} < \frac{1}{2}$, so $f(x)$ is not monotone decreasing in I_0 .

Next, let $J \in \mathcal{D}_k(I_0)$. Similarly to the case I_0 , we look at x_J , the middle point of J . Also from the proof of part 2, there exist two points x' and x'' in J such that $x' < x_J < x''$, while $f(x') < f(x_J)$ and $f(x'') < f(x_J)$. So, f is not monotone in any dyadic subinterval of I_0 .

Finally, we conclude that f is not monotone on any subinterval in I_0 since any interval contains a dyadic interval.

Part 4 $f_{\beta=1}(x) = 2 - 4x, (x \in I_0)$:

Recall the dyadic expression (2.3) and set $\beta = 1$; then

$$f(x) = 2 - 4 \sum_{l=0}^{\infty} b_l 2^{-(l+1)} = 2 - 4x.$$

So $f_{\beta=1}$ is linear on $[0, 1)$, and 0 and 1 are the only two discontinuous points of $f_{\beta=1}(x)$ for $x \in \mathbb{R}$, since

$$f_{\beta=1}(0) = 2 \neq 0 = f_{\beta=1}(0^-) \text{ and } f_{\beta=1}(1^-) = -2 \neq 0 = f_{\beta=1}(1^+). \quad \blacksquare$$

5 Analytic Properties of f_β for $n > 1$

For the case $n > 1$, we get a parallel theorem to Theorem 4.1.

Theorem 5.1 *Let $n > 1$. For $\beta \neq 1$, f defined by (2.2) is continuous at every point that is not on the surface of any dyadic cube in I_0 , and discontinuous at all dyadic points inside I_0 , i.e., points whose coordinates are dyadic points in $(0, 1)$. Moreover, f is not monotone along any coordinate direction in any subcube of I_0 .*

For $\beta = 1$, f is discontinuous at some dyadic points and the set of those points is still dense in I_0 .

Proof Most of the proof of Theorem 5.1 is analogous to that of Theorem 4.1 and involves more complicated details. Here, we will only explain the difference when $\beta = 1$.

Rewrite the binary expression (2.4):

$$f(x) = \sum_{l=0}^{\infty} (-1)^{b_l} (-1)^{\sum_{i=2}^n b_i} 2^{-\beta l}.$$

Let x be a dyadic point inside I_0 , i.e., $x_i \neq 1$ for $i = 1, \dots, n$. There exists an integer $k \geq 0$ such that for all $l > k$, $b_l^i = 0$ for all $i \in \{1, \dots, n\}$, and $b_k^i = 1$ for some $i \in \{1, \dots, n\}$. Without loss of generality, assume $b_k^1 = 1$. Then we have the following:

- (i) If $\sum_{i=2}^n b_k^i$ is even, $f(x_1^+, x_2, \dots, x_n) - f(x_1^-, x_2, \dots, x_n) = 2(C_\beta - 2)2^{-\beta k}$.
- (ii) If $\sum_{i=2}^n b_k^i$ is odd, $f(x_1^+, x_2, \dots, x_n) - f(x_1^-, x_2, \dots, x_n) = 2C_\beta 2^{-\beta k}$.

So when $\beta \neq 1$, f is discontinuous at all dyadic points inside I_0 . Moreover, on every subcube $J \in \mathcal{D}_k(I_0)$, there exists at least one point satisfying (ii). Therefore, unlike in \mathbb{R}^1 , the set of discontinuous points of f is still dense in I_0 for $\beta = 1$. \blacksquare

Corollary 5.2 *Let $n \geq 2$. Given $k \geq 0$ and a dyadic subcube $J \in \mathcal{D}_k(I_0)$, we have*

$$(5.1) \quad \sup_{x \in J} f(x) - \inf_{x \in J} f(x) = 2C_\beta 2^{-\beta k}.$$

6 Preliminaries of Fractal Geometry

We have seen that the function f is defined in a simple way, and it has a fine structure as it oscillates on arbitrarily small scales. In the following section, we will show that the graph of f is a self-affine set in \mathbb{R}^{n+1} with a non-integer fractal dimension for $0 < \beta < 1$. So, f is a fractal function for β in this range. (See [9, Introduction] for the definition of a fractal.)

We will need the following concepts (see [9, Section 9.4]).

Definition 6.1 *Self-affine set:* Let D be a closed subset of \mathbb{R}^n . Let τ_1, \dots, τ_m be affine contractive transformations from D to D . A non-empty compact set F is called self-affine with τ_1, \dots, τ_m if F is invariant for the τ_i , i.e., F satisfies $F = \bigcup_{i=1}^m \tau_i(F)$.

Remark The existence and uniqueness of such invariant set is guaranteed by [9, Theorem 9.1].

Definition 6.2 *Singular values of a contracting and non-singular mapping:* Assume that $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a contracting and non-singular linear mapping. The singular values $1 > \eta_1 \geq \eta_2 \geq \dots \geq \eta_n$ are defined as the positive square roots of the eigenvalues of T^*T , where T^* is the adjoint of T .

Definition 6.3 *Singular value function of T :* Let $0 \leq s \leq n$. The singular function of T is given by

$$(6.1) \quad \phi^s(T) = \eta_1 \eta_2 \cdots \eta_r^{s-r+1},$$

where r is the integer for which $r - 1 < s \leq r$.

We will use the following notion of dimension, due to Falconer (see [9, Theorem 9.12]):

$$(6.2) \quad d(T_1, \dots, T_m) = \inf \left\{ s : \sum_{k=1}^{\infty} \sum_{S_k} \phi^s(T_{i_1} \circ \cdots \circ T_{i_k}) < \infty \right\},$$

where S_k denotes the set of all k -term sequences $\{i_1, \dots, i_k\}$ with $1 \leq i_j \leq m$.

Following [12], we call (6.2) the ‘‘Falconer dimension’’ of the set $\{T_1, \dots, T_m\}$. It is related to the Hausdorff dimension ($\dim_H F$) and the Box dimension ($\dim_B F$) as follows:

Theorem 6.4 (Falconer) *Let $\tau_i = T_i + b_i$, ($i = 1, \dots, m$) be affine contractive transformations on \mathbb{R}^n , where T_i are linear contractive mappings and b_i are vectors in \mathbb{R}^n . If F is the affine invariant set satisfying*

$$F = \bigcup_{i=1}^m (T_i(F) + b_i),$$

then $\dim_H F = \dim_B F = d(T_1, \dots, T_m)$ for almost all $\{b_1, \dots, b_m\} \in \mathbb{R}^{nm}$ with respect to the nm -dimensional Lebesgue measure.

There are several equivalent definitions of the Box dimension. We adopt the following one which is convenient for our purpose (see [9, Section 3.1]).

Definition 6.5 *Box dimension of a set $F \subset \mathbb{R}^n$:* Let $\delta > 0$. A δ -mesh of \mathbb{R}^n refers to a collection of cubes of the form

$$[m_1\delta, (m_1 + 1)\delta] \times \cdots \times [m_n\delta, (m_n + 1)\delta],$$

where m_1, \dots, m_n are integers.

Let F be a non-empty bounded subset of \mathbb{R}^n and let $N_\delta(F)$ be the number of δ -mesh cubes that intersect F . The upper and lower Box dimensions of F are defined, respectively, as

$$\overline{\dim}_B(F) = \overline{\lim}_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta} \quad \text{and} \quad \underline{\dim}_B(F) = \underline{\lim}_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

If $\overline{\dim}_B(F) = \underline{\dim}_B(F)$, the common value is called the Box dimension of F and is denoted by $\dim_B(F)$.

In particular, the limit as δ tends to zero can be taken through $\delta_k = c^k$ with $0 < c < 1$.

We have that $\overline{\dim}_B \bar{F} = \overline{\dim}_B F$ and $\underline{\dim}_B \bar{F} = \underline{\dim}_B F$, where \bar{F} is the closure of F (see [9, Proposition 3.4]).

7 Fractal Dimension of f

Let $G_f = \{(x, f(x)), x \in I_0\}$, the graph of the function f over cube I_0 . We will apply Theorem 6.4 to the closure \bar{G}_f instead of G_f because \bar{G}_f is compact, and $\dim_B \bar{G}_f = \dim_B G_f$.

In what follows, we show that \bar{G}_f is self-affine on $I_0 \times [-C_\beta, C_\beta]$, and we compute its Falconer dimension and Box dimension. We discuss the self-affinity of \bar{G}_f for the cases $n = 1$ and $n > 1$, respectively. Again, we consider $n = 1$ first.

Theorem 7.1 *Let τ_1 and τ_2 be affine transformations defined as follows:*

$$(7.1) \quad \tau_i(x, y) = T_i(x, y) + b_i, \quad i = 1, 2,$$

where

$$(7.2) \quad T_1 = T_2 = T = \begin{pmatrix} 2^{-1} & 0 \\ 0 & 2^{-\beta} \end{pmatrix}, b_1 = (0, 1) \text{ and } b_2 = \left(\frac{1}{2}, -1\right).$$

For $n = 1$, we have that \bar{G}_f is affine invariant of $\{\tau_1, \tau_2\}$, that is, it satisfies

$$G = \tau_1(G) \cup \tau_2(G) = (T_1(G) + b_1) \cup (T_2(G) + b_2).$$

In addition,

$$(7.3) \quad \dim_F \bar{G}_f = d(T_1, T_2) = \begin{cases} 2 - \beta & \text{if } 0 < \beta < 1, \\ 1 & \text{if } \beta \geq 1 \end{cases}$$

(See Figures 1, 2, and 3, which are provided by P. Góra.)

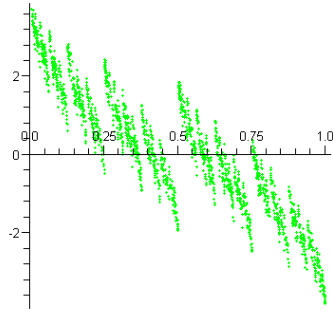


Figure 1: The affine invariant set satisfying (7.1) with $\beta = -\ln 0.77 / \ln 2$.

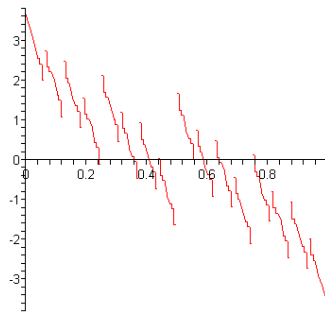


Figure 2: The partial sum of the first 7 terms of $f_\beta(x)$ with $\beta = -\ln 0.77 / \ln 2$.

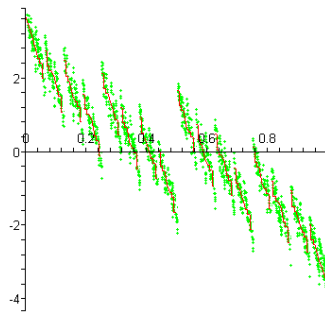


Figure 3: The two graphs above coincide well.

Proof of Theorem 7.1 Note that for $x \in I_0 = [0, 1]$, $f(x) = \sum_{l=0}^\infty f_l(x)$ can be written as:

$$f(x) = \begin{cases} f_0(x) + 2^{-\beta} f(2x) = 1 + 2^{-\beta} f(2x) & \text{if } 0 \leq x < \frac{1}{2}, \\ f_0(x) + 2^{-\beta} f(2(x - \frac{1}{2})) = -1 + 2^{-\beta} f(2(x - \frac{1}{2})) & \text{if } \frac{1}{2} \leq x < 1. \end{cases}$$

Hence, \bar{G}_f is affine invariant under the following two contracting mappings in \mathbb{R}^2 :

$$\begin{aligned} \tau_1: (x, y) &\longrightarrow (2^{-1}x, 2^{-\beta}y) + (0, 1), \\ \tau_2: (x, y) &\longrightarrow (2^{-1}x, 2^{-\beta}y) + (\frac{1}{2}, -1). \end{aligned}$$

The matrix forms of τ_i ($i = 1, 2$) are given by (7.1).

To prove (7.3), we consider $0 < \beta < 1$ and $\beta \geq 1$ separately.

First, for $0 < \beta < 1$, the singular values of T in (7.2) are $\eta_1 = 2^{-\beta} > \eta_2 = 2^{-1}$. Correspondingly, the singular values of

$$T^k = \underbrace{T \circ \dots \circ T}_{k \text{ times}}$$

are $\eta_1^k = 2^{-\beta k} > \eta_2^k = 2^{-k}$.

We consider $1 \leq s \leq 2$ instead of $0 \leq s \leq 2$ in Definition 6.3 since as a graph of a function, the dimension of G_f is greater than or equal to that of I_0 , its projection on \mathbb{R}^1 . (See [9, Chapter 6].)

We calculate the singular function $\phi^s(T^k)$ given by (6.1) in three cases as follows:

Case 1 $s = 2$: then $r = 2$ and

$$\phi^s(T^k) = \eta_1^k (\eta_2^k)^{s-r+1} = 2^{-\beta k} 2^{-k} = 2^{-(1+\beta)k}.$$

Case 2 $1 < s < 2$: also $r = 2$, and

$$\phi^s(T^k) = \eta_1^k (\eta_2^k)^{s-r+1} = 2^{-\beta k} 2^{-(s-1)k} = 2^{(1-\beta-s)k}.$$

Case 3 $s = 1$: then $r = 1$, and $\phi^s(T^k) = (\eta_1^k)^{s-r+1} = 2^{-\beta k}$.

Now, calculate $d(T_1, T_2)$ in (6.2):

Since for all k -term sequences $\{i_1, \dots, i_k\}$ with $i_j = 1$ or 2 ($j = 1, \dots, k$) we have $T_{i_j} = T$ and $T_{i_1} \circ \dots \circ T_{i_k} = T^k$, so

$$\sum_{S_k} \phi^s(T_{i_1} \circ \dots \circ T_{i_k}) = 2^k \phi^s(T^k).$$

Case 1 $s = 2$:

$$\sum_{k=1}^\infty 2^k \phi^s(T^k) = \sum_{k=1}^\infty 2^k 2^{-(1+\beta)k} = \sum_{k=1}^\infty 2^{-\beta k} = \frac{2^{-\beta}}{1 - 2^{-\beta}} < \infty.$$

Case 2 $1 < s < 2$:

$$\sum_{k=1}^\infty 2^k \phi^s(T^k) = \sum_{k=1}^\infty 2^k 2^{(1-\beta-s)k} = \sum_{k=1}^\infty 2^{k(2-\beta-s)} < \infty \iff s > 2 - \beta.$$

Case 3 $s = 1$:

$$\sum_{k=1}^{\infty} 2^k \phi^s(T^k) = \sum_{k=1}^{\infty} 2^k 2^{-\beta k} = \sum_{k=1}^{\infty} 2^{(1-\beta)k} = \infty.$$

Consequently,

$$d(T_1, T_2) = \inf\{s : s > 2 - \beta\} = 2 - \beta.$$

Second, for $\beta \geq 1$, the singular values of T are $\eta_1 = 2^{-1} \geq \eta_2 = 2^{-\beta}$, and the singular values of T^k are $\eta_1^k = 2^{-k} \geq \eta_2^k = 2^{-\beta k}$.

We only need to look at $1 < s < 2$. Thus, for $r = 2$, the singular function

$$\phi^s(T^k) = \eta_1^k (\eta_2^k)^{s-r+1} = 2^{-k} 2^{-(s-1)\beta k} = 2^{-k-(s-1)\beta k},$$

and

$$\sum_{k=1}^{\infty} 2^k \phi^s(T^k) = \sum_{k=1}^{\infty} 2^k 2^{-k-(s-1)\beta k} = \sum_{k=1}^{\infty} 2^{-(s-1)\beta k} < \infty.$$

Therefore, $d(T_1, T_2) = \inf\{s : s > 1\} = 1$. ■

Now, we generalize Theorem 7.1 to \mathbb{R}^n ($n > 1$).

Let $x = (x_1, \dots, x_n) \in I_0 = [0, 1]^n$, and consider the graph

$$G_f = \{(x, f(x)), x \in I_0\} \subset \mathbb{R}^n \times \mathbb{R}^1.$$

Consider the contracting mappings $\tau_i: \mathbb{R}^n \times \mathbb{R}^1 \rightarrow \mathbb{R}^n \times \mathbb{R}^1$ given by $\tau_i(x, y) = T_i(x, y) + b_i$, where T_i are $n + 1$ dimension diagonal matrices, with

$$T_i = T = \text{diag}\{2^{-1}, \dots, 2^{-1}, 2^{-\beta}\} \forall i, b_i = (b_{i,1}, \dots, b_{i,n}, b_{i,n+1}),$$

with $b_{i,j} = 0$ or $\frac{1}{2}$ for $j = 1, \dots, n$ and $b_{i,n+1} = (-1)^{P(b_i)}$, where $P(b_i)$ denotes the number of times $\frac{1}{2}$ occurs in $b_{i,1}, \dots, b_{i,n}$. There are in total 2^n such distinct τ_i .

Theorem 7.2 \bar{G}_f is the affine invariant set on $I_0 \times [-C_\beta, C_\beta]$, which satisfies

$$G = \bigcup_{i=1}^{2^n} \tau_i(G),$$

where τ_i ($i = 1, \dots, 2^n$) are 2^n distinct affine contractions as described above. Moreover,

$$\dim_F \bar{G}_f = d(T_1, \dots, T_{2^n}) = \begin{cases} n + 1 - \beta & \text{if } 0 < \beta < 1, \\ n & \text{if } \beta \geq 1. \end{cases}$$

The proof of Theorem 7.2 is analogous to that of Theorem 7.1 and will be omitted.

Since Theorem 6.4 gives no clue for which b_1, \dots, b_m , the Hausdorff dimension and the Box dimension agree with the Falconer dimension (6.2), we calculate the Box dimension of the graph G_f directly in what follows.

Theorem 7.3 *The Box dimension of the graph of f is $n + 1 - \beta$ for $0 < \beta < 1$, and n for $\beta \geq 1$, respectively.*

Proof Let $\delta_k = 2^{-k}$. By (5.1), the number of δ_k -mesh cubes in \mathbb{R}^{n+1} in the column over each $J \in \mathcal{D}_k(I_0)$ that intersect G_f is at most $2C_\beta 2^{-\beta k} / 2^{-k} + 2$. Since there are in total 2^{nk} many such J , $N_{\delta_k} \leq 2^{nk} \cdot (2C_\beta 2^{-\beta k} / 2^{-k} + 2)$. So

$$\overline{\dim}_B G_f \leq \lim_{k \rightarrow \infty} \frac{\log[2^{nk} \cdot (2C_\beta 2^{(1-\beta)k} + 2)]}{-\log 2^{-k}} = \begin{cases} n + 1 - \beta & \text{if } 0 < \beta < 1; \\ n & \text{if } \beta \geq 1. \end{cases}$$

When $\beta \geq 1$, since $\underline{\dim}_B G_f \geq \underline{\dim}_B(\text{Proj}_{\mathbb{R}^n} G_f) = \dim_B I_0 = n$ (see [9, Chapter 6]), we get $\dim_B G_f = n$.

When $0 < \beta < 1$, since the part of G_f over $J \in \mathcal{D}_k$ is affine to G_f , f restricted to J is onto from J to $[\inf_{x \in J} f(x), \sup_{x \in J} f(x)]$ by Theorem 2.1. It follows that the number of δ_k -mesh cubes in \mathbb{R}^{n+1} in the column over each $J \in \mathcal{D}_k(I_0)$ intersecting G_f is at least $2C_\beta 2^{-\beta k} / 2^{-k}$. Therefore, $N_{\delta_k} \geq 2^{nk} \cdot 2C_\beta 2^{-\beta k} / 2^{-k}$, and

$$\underline{\dim}_B G_f \geq \lim_{k \rightarrow \infty} \frac{\log[2C_\beta \cdot 2^{(n+1-\beta)k}]}{-\log 2^{-k}} = n + 1 - \beta,$$

Hence, $\dim_B G_f = n + 1 - \beta$. ■

Remark As a counterexample, in [17], the function f is interesting only for $0 < \beta < \frac{1}{2}$ due to the properties of Q_α spaces. However, f is well defined for all $\beta > 0$. Moreover, the related mappings $\{\tau_1, \dots, \tau_{2^n}\}$ is a system of affine contractions for all $\beta > 0$ (a system of similar contractions when $\beta = 1$). Theorems 7.1 and 7.2 tell us that f is a fractal function if $0 < \beta < 1$, since for β in this range, $\dim_B G_f = n + 1 - \beta > n$. As a corollary of Theorems 3.1 and 7.2, the fractal dimension of $f = f_\beta$ is related to the space $Q_\alpha(\mathbb{R}^n)$ by the fact that $f_\beta \in Q_\alpha$ ($0 < \alpha < 1$) if and only if $n \leq \dim_B G_{f_\beta} < n + 1 - \alpha$.

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