Dear Editor,

Mean passage times for tridiagonal transition matrices

We consider homogeneous finite-state Markov chains with either a discrete or continuous time parameter, whose transition probability/intensity matrices have tridiagonal form. Krafft and Schaefer (1993) derived formulae for the mean transition times in the discrete-time case. Earlier, different formulae have been derived by Blom (1989) in the particular case of the Ehrenfest urn model. For the latter model Palacios (1993) uses an electric network approach to derive a formula for the mean of a special passage time. The methods of all authors mentioned above are different and the structure of their formulae are different too. In this note we suggest another approach which works in both the discrete-time and continuous-time cases. This is based on Stefanov's (1991) results for finite-state Markov chains. Actually, we derive the same formulae as those given by Krafft and Schaefer (1993) in the discrete-time case, although we write them in a slightly different way. Moreover, we prove that the same formulae hold true in the continuous-time case too. One should just replace the transition probabilities by the respective transition intensities. This results in a unified approach.

Consider a homogeneous $(m+1)$-state Markov chain $\{X(t)\}_{t\geqq 0}$ with either a discrete or continuous time parameter, defined on the probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$. Let τ be a finite stopping time, i.e. $\boldsymbol{P}(\tau<+\infty)=1$. It is well known that the Radom–Nikodym derivative of the measure generated by the trajectories of the process $\{X(t)\}_{t\geqq 0}$ up to the time τ, with respect to a σ-finite measure ν_τ equals (see for example Stefanov (1991)):

$$\exp\left\{\sum_{i,\,j=0}^{m} N_{i,\,j}(\tau)\ln p_{ij}\right\} \tag{1}$$

in the discrete time case, where $\{p_{ij}\}_{i,\,j=0}^{m}$ is the transition probability matrix, $N_{i,\,j}(\tau)$ is the number of one-step transitions from state i to state j in the time interval $[0,\tau]$;

$$\exp\left\{\sum_{i,\,j=0,i\neq j}^{m} N_{i,\,j}(\tau)\ln\lambda_{i,\,j}-\sum_{i=1}^{m}\lambda_{ii}T_i(\tau)\right\} \tag{2}$$

in the continuous-time case, where $\{\lambda_{ij}\}_{i,\,j=0}^{m}$, $\lambda_{ii}=\Sigma_{j=0,\,i\neq j}^{m}\lambda_{ij}$, is the intensity matrix, $N_{i,\,j}(\tau)$ is as above, and $T_i(\tau)$ is the sojourn time at state i in the time interval $[0,\tau]$.

We shall consider first the discrete-time case. Assume that for each i, $i=0,1,\cdots,m$

(3)
$$p_{ij}=\begin{cases} a_i & \text{if } j=i-1 \\ b_i & \text{if } j=i \\ c_{i+1} & \text{if } j=i+1 \\ 0 & \text{otherwise} \end{cases}$$

where $a_0=c_{m+1}=0$, $b_i \geqq 0$, $i=0,\cdots,m$, and the remaining a_i's, c_i's are all positive. Define

$$\tau_{n,k} \overset{\text{def}}{=} \inf\{t>0 : X(t)=k,\ X(t-1)\neq k \mid X(0)=n\}.$$

We shall find formulas for $E\tau_{n,n+1}$ and $E\tau_{n,n-1}$. Of course, the remaining mean transition times are simple expressions of these.

Assume first that $b_i>0$, $i=0,\cdots,m$. In view of Stefanov's (1991) results the family given by (1) is a non-curved exponential family if either $\tau=\tau_{n,n+1}$ or $\tau=\tau_{n,n-1}$. Therefore, in view of well known analytical properties of non-curved exponential families, Barndorff–Nielsen ((1978), p. 106), which allow differentiation, with respect to p_{ij}, under the integral sign in the equality

$$\int \exp\left\{\sum_{i,\,j=0}^{m} N_{i,\,j}(\tau_{n,n+1})\ln p_{ij}\right\} dv_{\tau_{n,n+1}}=1,$$

we derive the following equalities:

(4)
$$\frac{EN_{0,1}(\tau_{n,n+1})}{c_1}=\frac{EN_{0,0}(\tau_{n,n+1})}{b_0}$$
$$\frac{EN_{i,i-1}(\tau_{n,n+1})}{a_i}=\frac{EN_{i,i}(\tau_{n,n+1})}{b_i}=\frac{EN_{i,i+1}(\tau_{n,n+1})}{c_{i+1}}, \qquad i=1,2,\cdots,n.$$

In view of Stefanov ((1991); (3) on p. 355) we have

(5)
$$N_{i,i+1}(\tau_{n,n+1})=N_{i+1,i}(\tau_{n,n+1}), \qquad i=0,1,\cdots,n.$$

Bearing in mind that

$$EN_{n,n+1}(\tau_{n,n+1})=1$$

and using (4) and (5) we find

$$EN_{n,n}(\tau_{n,n+1})=\frac{b_n}{c_{n+1}}$$

$$EN_{i,i}(\tau_{n,n+1})=\frac{a_n\cdots a_{i+1}b_i}{c_{n+1}\cdots c_{i+1}}, \qquad i=0,1,\cdots,n-1$$

$$EN_{i,i+1}(\tau_{n,n+1})=EN_{i-1,i}(\tau_{n,n+1})=\frac{a_n\cdots a_i}{c_{n+1}\cdots c_{i+1}}, \qquad i=1,2,\cdots,n.$$

Thus, replacing b_i by $1-a_i-c_{i+1}$, we derive after an easy simplification that

$$E\tau_{n,n+1}=\sum_{i=0}^{n}\sum_{j=i-1}^{i+1}EN_{i,j}(\tau_{n,n+1})$$
$$=\frac{1}{c_{n+1}}\left(1+\sum_{i=1}^{n}\prod_{k=i}^{n}\left(\frac{\alpha_k}{c_k}\right)\right)$$

where $EN_{0,-1}(\tau_{n,n+1})\overset{\text{def}}{=}0$. Similarly, we can find an expression for $E\tau_{n,n-1}$. Also, in view of Stefanov (1991), it is easy to see that these formulas are valid in the case when some of b_i's are equal to zero. Therefore, we have proved the following.

Theorem 1. If the transition probability matrix of the $(m+1)$-state Markov chain is given by (3), then for the mean transition times we have

$$E\tau_{n,n+1}=\frac{1}{c_{n+1}}\left(1+\sum_{i=1}^{n}\prod_{k=i}^{n}\left(\frac{a_k}{c_k}\right)\right)$$

$$E\tau_{n,n-1}=\frac{1}{a_n}\left(1+\sum_{i=n+1}^{m}\prod_{k=n+1}^{i}\left(\frac{c_k}{a_k}\right)\right)$$

$$E\tau_{n,k}=\sum_{j=n}^{k-1}E\tau_{j,j+1},\qquad n<k$$

$$E\tau_{n,k}=\sum_{j=k+1}^{n}E\tau_{j,j-1},\qquad n>k$$

$$E\tau_{n,n}=(1+a_nE\tau_{n-1,n}+c_{n+1}E\tau_{n+1,n})/(a_n+c_{n+1}).$$

Consider now an $(m+1)$-state Markov chain with continuous time parameter, whose transition intensities are given as follows:

$$\lambda_{ij}=\begin{cases}a_i & \text{if } j=i-1\\ b_i & \text{if } j=i\\ c_{i+1} & \text{if } j=i+1\\ 0 & \text{otherwise}\end{cases}\tag{6}$$

where $i=0,1,\cdots,m$, $b_i=a_i+c_{i+1}$, $a_0=c_{m+1}=0$, and the remaining a_i's, b_i's, c_i's are all positive. Actually, this is an $(m+1)$-state birth–death process. The stopping time $\tau_{n,k}$ is defined as follows:

$$\tau_{n,k}\overset{\text{def}}{=}\inf\{t>0: X(t)=k,\ X(t-)\neq k\mid X(0)=n\}.$$

In view of Stefanov (1991), Proposition 2, the same methodology is applicable in this case. In particular the equalities given by (4) hold true, where $EN_{i,i}(\tau_{n,n+1})/b_i$ should be replaced by $ET_i(\tau_{n,n+1})$, $i=1,2,\cdots,n$. Also the equalities given by (5) hold true and

$$EN_{n,n+1}(\tau_{n,n+1})=1.$$

Therefore, proceeding along the same lines as in the discrete-time case and bearing in mind that $E\tau_{n,n+1}=\Sigma_{i=0}^{n}ET_i(\tau_{n,n+1})$, we derive the following.

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Theorem 2. If the intensity matrix of the $(m+1)$-state chain is given by (6), then for the mean transition times we have exactly the same expressions as those given in Theorem 1.

Remark 1. The formulas for $E\tau_{n,n+1}$, derived in the above two theorems, hold true also in the case when the number of states is countably infinite. Actually, it is straightforward to see that the derivation of these formulas does not depend on the number of the states up to countably many. However the same does not apply for $E\tau_{n,n-1}$.

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Yours sincerely

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