A FINANCIALLY BALANCED BONUS-MALUS SYSTEM

GEERT COENE

UNISON Consultants Europe E.E.I.G., Brussels, Belgium

LOUIS G. DORAY[1]

Department of Mathematics and Statistics, University of Montreal, Montreal, Canada

ABSTRACT

The premiums for a bonus-malus system which stays in financial equilibrium over the years are calculated. This is done by minimizing a quadratic function of the difference between the premium for an optimal BMS with an infinite number of classes and the premium for a BMS with a finite number of classes, weighted by the stationary probability of being in a certain class, and by imposing various constraints on the system.

KEYWORDS

Bonus-malus; quadratic programming; stationary distribution; negative binomial.

1. INTRODUCTION

LEMAIRE and ZI (1994) analyze 30 bonus-malus systems (BMS) from around the world, with respect to four measures: the relative stationary average premium level, the coefficient of variation of the insured's premium, the efficiency of the system and the average optimal retention. They show that these measures are all positively correlated. They also conclude that "an apparently inescapable consequence of the implementation of a BMS is a progressive decrease of the observed average premium level, due to a concentration of policyholders in the high-discount classes".

At the end of 30 years, the average premium level for a policyholder goes from around 70% of the starting premium level for Belgium to a low of 40% for Japan. This financial imbalance results from penalties that are not severe enough for bad drivers.

[1] The second author gratefully acknowledges the financial support of the Natural Sciences and Engineering Research Council of Canada.

In this note, we report on the construction of three hypothetical BMS which have the property of staying financially balanced over the years. SAMMARTINI (1990) also considers the problem of constructing a BMS which is financially balanced. He does this by permitting a driver to move to a lower class only if the claim frequency of his preceding class is lower than a fixed value.

We achieve a bonus-malus system financially balanced over the long term by using the premiums as parameters of the model. The premium for each class will be determined in such a way that the total premiums received should be at least equal to 100% of the initial premium after a certain number of years.

This note is organized as follows. Section 2 introduces the notation and presents the model used. Section 3 reports the results of a simulation to calculate the number of policyholders in each class at periodic intervals, from which the stationary distribution can be estimated. Section 4 uses this information to construct an optimal BMS with a finite number of classes, with certain constraints imposed on the premiums so that the BMS is financially balanced at the end of a fixed time horizon. Finally, we present some concluding remarks.

2. DEFINITION OF THE BMS

Let n represent the number of classes of the system. The premium for class i, $i = 1, \ldots, n$, is equal to the product of a base premium P and a fraction $0.01 \times C_i$. A driver in class i pays a premium equal to $0.01 \times C_i \times P$.

Let N_t be the number of accidents a policyholder has during the period $[t-1, t)$. We will assume that N_t has a Negative Binomial (NB) distribution with parameters ($a = 1.0923183$, $b = 7.70077$). These parameter values were derived from the data of WEBER (1970).

In this note, we will consider three BMS:

A) BMS1 is a system with 18 classes ($n = 18$). The premium corresponding to class 10 is equal to the base premium P, so that $C_{10} = 100$. A new driver will start in this class. The class of a driver will be modified each year according to the following transition rules:
   1. a driver with no accident during a year goes down one class.
   2. a driver goes up by two classes for the first accident in a year and by three classes for each subsequent accident in that year.

   To model this system, let us denote by Y_t the class of a driver for the period $[t, t+1)$. This process Y_t is thus defined by the following equation

$$Y_0 = 10$$
$$Y_t = Y_{t-1} + 3N_t - 1, \ t \geqslant 1,$$

   with a minimum value of 1 and a maximum value of 18.

   BMS1 is similar to the old Belgian BMS except that this one had the additional restriction that a driver not responsible for any accident during 4 consecutive years, and whose class is higher than 10, will go back to class 10.

B) BMS2 is a system similar to BMS1, but with a higher penalty for an accident. A driver will go up by 3 classes for the first accident in a year and by 4 classes for each subsequent accident in that year, so that Y_t is defined by

$$Y_0 = 10$$

$$Y_t = Y_{t-1} + 4N_t - 1, \ t \geq 1.$$

C) BMS3 is a system with the same rules as BMS2 but with 24 classes.

3. STATIONARY DISTRIBUTION

When the random variables (r.v.) $N_1, \ldots, N_t$ (defined in section 1) are independent, the stationary distribution of a BMS can be computed using DUFRESNE's (1988) recursive procedure, as eigenvector of the transition matrix or as limit value of the transition matrix. In the model where the r.v. $N_1 \mid \lambda, \ldots, N_t \mid \lambda$ have a Poisson distribution and λ follows a gamma distribution, the unconditional r.v. $N_1, \ldots, N_t$, which follow a negative binomial distribution, are dependent.

To estimate the number of drivers in each class after a large number of years, we must therefore resort to simulation. We start with an hypothetical portfolio of 100,000 drivers initially in class 10 and simulate for each of them their claim experience over the next 40 years, using the NB (1.0923183, 7.70077) distribution. Table 1 contains the simulated number of drivers in each class for BMS1 after 10, 20, 30 and 40 years. The number of drivers in each class in year 40 is divided by 100,000 to estimate the stationary probability of being in class i, denoted f_i (last column of Table 1). Tables 2 and 3 contain the same information for BMS2 and BMS3 respectively. The period of 40 years was chosen as the approximate average driving career of a policyholder.

TABLE 1

NUMBER OF DRIVERS AT TIME t FOR BMS1

Class \ t	10	20	30	40	f_i
1	40544	61141	66085	66229	0.66229
2	0	9024	5471	5658	0.05658
3	24855	5431	6140	7675	0.07675
4	0	1197	4049	2246	0.02246
5	0	6146	1681	1985	0.01985
6	14526	745	1104	2487	0.02487
7	0	326	2780	992	0.00992
8	0	4104	666	823	0.00823
9	8518	227	523	1703	0.01703
10	14	261	2151	648	0.00648
11	36	2986	482	661	0.00661
12	4759	289	543	1433	0.01433
13	98	461	1759	676	0.00676
14	238	2203	682	840	0.00840
15	2885	663	886	1349	0.01349
16	520	1086	1627	1078	0.01078
17	1131	1907	1398	1492	0.01492
18	1876	1803	1973	2025	0.02025

https://doi.org/10.2143/AST.26.1.563236 Published online by Cambridge University Press

TABLE 2

NUMBER OF DRIVERS AT TIME t FOR BMS2

Class \ t	10	20	30	40	f_i
1	40320	51217	56205	56351	0.56351
2	0	8798	3738	4881	0.04881
3	0	3731	4173	4744	0.04744
4	24881	3974	7682	5314	0.05314
5	0	1007	1828	2056	0.02056
6	0	7465	1622	3218	0.03218
7	0	1099	1477	1800	0.01800
8	14587	758	3867	1488	0.01488
9	5	620	1072	1238	0.01238
10	79	5480	1037	2453	0.02453
11	193	615	1062	1227	0.01227
12	8510	799	2949	1264	0.01264
13	277	1043	1135	1398	0.01398
14	542	3766	1353	2257	0.02257
15	1168	1336	1782	1650	0.01650
16	4668	1876	2903	2164	0.02164
17	1795	2656	2573	2797	0.02797
18	3065	3760	3542	3700	0.03700

TABLE 3

NUMBER OF DRIVERS AT TIME t FOR BMS3

Class \ t	10	20	30	40	f_i
1	40435	51266	56289	56019	0.56019
2	0	8896	3638	5137	0.05137
3	0	3660	3958	4488	0.04488
4	24747	4019	7784	4998	0.04998
5	0	831	1595	1754	0.01754
6	0	7503	1440	3540	0.03540
7	0	880	1076	1497	0.01497
8	14514	497	4090	1079	0.01079
9	0	220	654	732	0.00732
10	0	5810	497	2772	0.02772
11	0	248	402	596	0.00596
12	8675	131	3338	542	0.00542
13	0	91	338	494	0.00494
14	0	4259	325	2256	0.02256
15	1	101	332	497	0.00497
16	4973	144	2764	494	0.00494
17	1	234	412	591	0.00591
18	29	3094	489	1973	0.01973
19	71	298	676	776	0.00776
20	2850	515	2276	1034	0.01034
21	198	854	985	1296	0.01296
22	425	2447	1486	2239	0.02239
23	998	1512	2087	2143	0.02143
24	2083	2490	3069	3053	0.03053

4. OPTIMAL BMS

LEMAIRE (1985) has shown that in an optimal BMS, with an infinite number of classes, the premium for a driver who had N accidents in t years was given by

$$\frac{b}{a} \frac{a+N}{b+t} \times P.$$

He also showed that this optimal BMS was financially balanced, i.e. the average premium received each year by the insurer was $100\% P$. With our estimated values for the parameters a and b, we find in Table 4 the percentages of P for the optimal premium for $N \leq 4$ and $t \leq 9$.

TABLE 4

OPTIMAL WEIGHT $100 \times C_{(N,\, t)}$

$t \setminus N$	0	1	2	3	4
0	100.00				
1	88.51	169.53	250.56	331.59	412.61
2	79.38	152.06	224.73	297.41	370.08
3	71.96	137.85	203.73	269.61	335.50
4	65.81	126.07	186.32	246.57	306.82
5	60.63	116.14	171.65	227.16	282.66
6	56.21	107.66	159.12	210.58	262.03
7	52.38	100.34	148.30	196.25	244.21
8	49.05	93.95	138.35	183.75	228.65
9	46.11	88.32	130.54	172.75	214.96

We can approximate each BMS introduced in section 2 by a table of the same form as Table 4. Those tables give, for each value of $N = 0, 1, 2, 3, 4$ and $t = 1, \ldots, 9$ the percentage C_i of the premium P paid by a driver who had N accidents by time t and who is in class i. Note that the initial class for a driver is always class 10.

Table 5 approximates BMS1. For certain values of N and t, many classes are possible, since the class in which a driver is, depends not only on the total number of accidents N but also on the way these accidents are distributed among the t years. For example, a driver who has 4 accidents in the first two years can be in class 17 or 18 at the end of the second year. He will be in class 17 if these accidents are distributed as $(N_1, N_2) = (4, 0)$, but he will be in class 18, if these accidents are distributed any other way. This problematic situation can occur when there are many accidents ($N \geq 4$), but its probability is very small.

TABLE 5

APPROXIMATION FOR SBM1

$t \setminus N$	0	1	2	3	4
0	C_{10}				
1	C_9	C_{12}	C_{15}	C_{18}	C_{18}
2	C_8	C_{11}	C_{14}	C_{17}	C_{18}
3	C_7	C_{10}	C_{13}	C_{16}	C_{18}
4	C_6	C_9	C_{12}	C_{15}	C_{18}
5	C_5	C_8	C_{11}	C_{14}	C_{17}
6	C_4	C_7	C_{10}	C_{13}	C_{16}
7	C_3	C_6	C_9	C_{12}	C_{15}
8	C_2	C_5	C_8	C_{11}	C_{14}
9	C_1	C_4	C_7	C_{10}	C_{13}

As a general rule, we will then choose the highest class in which a driver can be. It also corresponds to the most probable class. With our estimated values for the parameters of the NB distribution, we find, for a driver with 4 accidents in 2 years, that the probability of being in class 17 is 0.000105747 (only when $(N_1, N_2) = (4, 0)$), while that of being in class 18 is 20 times higher (0.00222068). Class 18 will also be the class of all drivers with more than 4 accidents in the first year and none in the second year.

A problematic situation can also occur when the bottom class is reached for longer driving periods. In this study, we limited the time horizon to $t = 9$ years. But with BMS1, a policyholder who has only one accident in 12 years could be in class 3, 2 or 1 the following year, depending on whether this accident occurred in year 12, 11 or before. This situation will occur more frequently than that discussed previously, if we consider long driving periods.

Similarly, we can construct Table 6 to approximate BMS2 and Table 7 for BMS3. To find the optimal values of C_i of Tables 5, 6 and 7, we will minimize for all (N, t), the quadratic error between the premium in a system with an infinite number of classes,

$$\frac{b}{a} \frac{a+N}{b+t} \times P,$$

and that paid in the approximating BMS, $0.01 \times C_i \times P$, weighted by f_i, the stationary probability of being in class i.

It is equivalent to minimize the quadratic function

$$\sum_{(N,\, t)} f_i \times [C_i - 100\, C_{(N,\, t)}]^2$$

on the variables C_i; we will impose certain constraints on the BMS:

1. The constraints $C_{i+1} - C_i \geqslant 0$, $i = 1, \ldots, n-1$ will ensure that the premium increases with the class.

TABLE 6

APPROXIMATION FOR SBM2

$t \setminus N$	0	1	2	3	4
0	C_{10}				
1	C_9	C_{13}	C_{17}	C_{18}	C_{18}
2	C_8	C_{12}	C_{16}	C_{18}	C_{18}
3	C_7	C_{11}	C_{15}	C_{18}	C_{18}
4	C_6	C_{10}	C_{14}	C_{18}	C_{18}
5	C_5	C_9	C_{13}	C_{17}	C_{18}
6	C_4	C_8	C_{12}	C_{16}	C_{18}
7	C_3	C_7	C_{11}	C_{15}	C_{18}
8	C_2	C_6	C_{10}	C_{14}	C_{18}
9	C_1	C_5	C_9	C_{13}	C_{17}

TABLE 7

APPROXIMATION FOR SBM3

$t \setminus N$	0	1	2	3	4
0	C_{10}				
1	C_9	C_{13}	C_{17}	C_{21}	C_{24}
2	C_8	C_{12}	C_{16}	C_{20}	C_{24}
3	C_7	C_{11}	C_{15}	C_{19}	C_{23}
4	C_6	C_{10}	C_{14}	C_{18}	C_{22}
5	C_5	C_9	C_{13}	C_{17}	C_{21}
6	C_4	C_8	C_{12}	C_{16}	C_{20}
7	C_3	C_7	C_{11}	C_{15}	C_{19}
8	C_2	C_6	C_{10}	C_{14}	C_{18}
9	C_1	C_5	C_9	C_{13}	C_{17}

2. We will set C_{10} equal to 100.
3. To ensure that the portfolio is financially balanced in the long term, we will impose the inequality constraint $\Sigma_i f_i C_i \geqslant 100$.

Solving this quadratic programming problem for the three BMS with the IMSL software, we find the optimal solutions appearing in Table 8. For the reason discussed above on the length of the period of analysis, optimal premium levels may depend on the maximum value of t used in the calculations. For this analysis, we limited ourselves to a period of $t = 9$ years.

TABLE 8

OPTIMAL PREMIUMS FOR THE BMS

Class	BMS1	BMS2	BMS3	Belgian BMS
0				54
1	79.2	70.1	54.1	54
2	82.2	73.1	57.0	54
3	85.5	76.4	60.4	57
4	88.8	80.2	64.2	60
5	93.8	86.5	78.5	63
6	99.6	91.9	83.9	66
7	100.0	98.2	90.1	69
8	100.0	100.0	97.5	73
9	100.0	100.0	100.0	77
10	100.0	100.0	100.0	81
11	180.2	155.1	147.1	85
12	195.1	167.6	159.6	90
13	220.8	179.3	174.0	95
14	237.9	197.0	189.0	100
15	258.1	212.0	204.0	105
16	282.4	229.7	221.7	111
17	306.6	250.9	233.6	117
18	357.9	294.4	241.6	123
19			260.9	130
20			283.7	140
21			311.1	160
22			314.8	200
23			343.5	
24			395.3	

With BMS1, we note that a single claim increases the premium by 95%, while four claim-free years reduce the premium by only 0.4%. BMS2 is a better system since it produces larger bonuses than BMS3. The same conclusion is reached when comparing BMS3 to BMS2. With BMS3, a good driver can receive a decrease of 45% of his premium after 9 years without any accident. This bonus is financed by the bad drivers, who may end up paying up to 4 times the initial premium. BMS3 would be preferred to BMS1 and BMS2.

The last column of Table 8 presents the percentage of the premium charged in the current Belgian BMS. The transition rules are somewhat different, however: a driver goes up by four classes for the first accident in a year and by five classes for each subsequent accident in that year. After four consecutive claim-free years, a policyholder can not be in a class higher than 14, complicating the Markovian structure of the model. It is interesting to note that, like BMS3, the Belgian system has a maximum bonus of 46%; however, the maximum malus in the Belgian system is only 100%, compared to 295% for BMS3.

BMS3 also has the interesting property that it stays financially balanced over the years. Table 9 shows the average percentage C of the premium received by the insurer for t = 1, ..., 40. As can be seen, the average premium fluctuates between 96% and 115% of the initial premium over the next 40 years.

TABLE 9

FINANCIAL EQUILIBRIUM FOR SBM3

t	C	t	C	t	C	t	C	t	C
1	110.2	9	99.3	17	98.0	25	98.9	33	100.1
2	115.1	10	98.7	18	98.2	26	99.2	34	100.4
3	114.3	11	98.9	19	98.4	27	99.5	35	100.5
4	105.3	12	95.9	20	96.8	28	98.5	36	99.7
5	107.2	13	97.8	21	98.3	29	99.6	37	100.4
6	103.0	14	97.9	22	98.6	30	99.8	38	100.6
7	102.6	15	98.3	23	98.9	31	99.9	39	100.6
8	97.4	16	96.2	24	97.7	32	99.2	40	100.0

5. CONCLUSION

In this note, we have shown how to construct a BMS which stays in financial equilibrium over the years, using a quadratic minimization procedure, with linear equality or inequality constraints. We could add more constraints on the premiums of the BMS. For example, if it is deemed desirable that the differential in premium between classes be at least k%, we would add the constraint

$$C_{i+1} - C_i \geqslant \mathrm{k}.$$

This could alleviate some of the difficulties encountered with BMS1 (high premium increase following a first accident and many claim-free years needed before receiving a bonus). The maximum bonus given to a driver and the maximum penalty a bad driver could receive could be limited with the constraints

$$C_1 \geqslant A$$
$$C_n \leq B.$$

Some regulatory requirements as well as constraints of insurance executives could also be accommodated in the system.

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Geert Coene
UNISON Consultants Europe E.E.I.G.
Avenue Louise 228/7
B-1050 Brussels
Belgium

Louis G. Doray
Department of Mathematics and Statistics
University of Montreal
P.O. Box 6128, Station Centre-ville
Montreal, P.Q.
Canada H3C 3J7