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MODULAR JORDAN NILALGEBRAS

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In this paper we give a classification up to isomorphism of Jordan nilalgebras whose lattices of subalgebras are modular when the ground field is algebraically closed.

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0. Introduction and preliminaries

Several authors have worked on the problem of finding out the consequences on the structure of an algebra of imposing classical lattice conditions on its lattice of subalgebras. Modularity is maybe the most popular of these conditions and we can find, for example, several papers on the structure of Lie and Malcev algebras whose lattices of subalgebras are modular or semimodular (see [**1**, **4**, **5**, **8**]) and, recently, on the same problem for Jordan algebras (see [**2**] and [**3**]).

In [**3**] it was shown that over algebraically closed fields Jordan algebras with modular lattices of subalgebras have nilradicals of codimension not bigger than two. In this paper we will extend the properties given in [**2**] for nilpotent Jordan algebras to Jordan nilalgebras and use such properties, when the field is algebraically closed, for obtaining a classification up to isomorphism of these nilradicals.

We will suppose that the reader knows the usual terms in lattice theory: sublattice, chain, lattice isomorphism, length of a lattice (the supremum of the lengths of all the chains, where the length of a chain is its cardinality minus one),...

A lattice $(L, \leqq)=(L, \wedge, \vee)$ is said to be *modular* when one of the three following equivalent conditions holds:

(1) If $x \geqq z$ then $x \wedge (y \vee z)=(x \wedge y) \vee z$.

(2) $x \wedge (y \vee z)=x \wedge [(y \wedge (x \vee z)) \vee z]$ ("shearing identity")

(3) L does not contain a pentagon (as a sublattice).

Given a, b in a lattice L, we define the *closed interval between a and b* and put $[a, b]$ for the sublattice of L given by $\{z \in L \mid a \leqq z \leqq b\}$.

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In a lattice $(L, \leqq)=(L, \wedge, \vee)$ we put $x -< y$ and say x, y are *consecutive* when $x \leqq y$, $x \neq y$ and $[x, y]=\{x, y\}$.

A lattice $(L, \leqq)=(L, \wedge, \vee)$ is said to be semimodular if:

$$a -< b \Rightarrow a \vee c -< b \vee c \text{ or } a \vee c = b \vee c \text{ for all } c \text{ in } L.$$

It is well known that a modular lattice is always semimodular and also:

Theorem 0.1. (Jordan–Hölder chain condition). *In a semimodular lattice with finite length all maximal chains have the same lenght.*

See Grätzer [6], for the terms and results mentioned on lattice theory.)

Let J be an algebra over a field. We will put $L(J)$ for its lattice of subalgebras, where $\leqq$, $\wedge$, $\vee$ are naturally defined ($\leqq$ is the inclusion relation). We define the *length* of J (and put $l(J)$) as the length of $L(J)$.

It is very easy to prove:

Proposition 0.2. *Let J be a F-algebra, F a field. Then $l(J) \leqq \dim_F J$. If J is solvable, then $l(J)=\dim_F J$.*

Definition 0.3. An algebra J is called *modular* (*semimodular*), if $L(J)$ is modular (semimodular).

The easiest examples of modular algebras are trivial algebras since the lattice of submodules of a module is a modular lattice.

Having in mind that closed intervals inherit the properties of modularity and semimodularity from the lattice where they are included, its immediate to prove the following result:

Proposition 0.4. *For any algebra J we have:*

(*i*) *If J is modular then it is semimodular.*

(*ii*) *If J is modular* (*semimodular*), *then all of its subalgebras and quotients are modular* (*semimodular*).

In a power associative algebra we define the *order* of a nilpotent element x (and put $o(x)$) as the number n such that $x^n=0$ and $x^{n-1} \neq 0$.

Given X a subset of an algebra J, we will put (X) for the subalgebra of J generated by X and $\langle X \rangle$ for the vector space spanned by X.

"$\leqq$" will mean "is a subalgebra of" when we deal with algebras.

"$+$" will mean sum of vector subspaces when we deal with vector spaces.

"$\dot{+}$" will mean direct sum of vector spaces.

"$\oplus$" will mean direct sum of ideals.

Throughout this paper we will deal with algebras over fields of characteristic not two.

We will also suppose that the reader knowns the usual definitions and properties concerning Jordan algebras which can be found in [7] and [9].

1. Modular Jordan nilalgebras: the ideal T

In [3] it was proved:

Proposition 1.1. *Let J be a power associative algebra. If J is semimodular then it is algebraic and for any nilpotent element x in J, we have $o(x) \leqq 5$.*

Corollary 1.2. *If a Jordan nilalgebra is semimodular, then it is locally nilpotent.*

Theorem 1.3. *Let N be a Jordan nilagebra. Then the following are equivalent:*

(i) *N is modular.*

(ii) *N is semimodular.*

(iii) *Given any pair of subalgebras of N, S and T, $S+T$ is always a subalgebra of N.*

We will use characterization (iii) instead of the very definition of modularity when we deal with Jordan nilalgebras. In particular, when N is a modular Jordan nilalgebra and a, b are elements in N, the subalgebra $(a,b)=(a) \vee (b)$ generated by a and b is exactly $(a)+(b)$.

If N is a nilpotent modular algebra, $T(N)=\{x \in N \mid x^2=0\}$ is the biggest trivial subalgebra of N and it is also and ideal of N (see [2]). Since these facts can be written in terms of pairs of elements, and using (1.2), the following is clear:

Proposition 1.4. *Let N be a modular Jordan nilalgebra. Then $T(N)=\{x \in N \mid x^2=0\}$ is an ideal of N, and it is also the biggest trivial subalgebra of N.*

Now, for N any modular Jordan nilalgebra, we can construct inductively a chain of ideals of N, $T_i(N)$, in the following way:

$$T_1(N)=T(N)$$

$$T_n(N)/T_{n-1}(N)=T(N/T_{n-1}(N)).$$

It is clear that we obtain an ascending chain of ideals with the property:

$$T_n(N)^2 \subseteq T_{n-1}(N).$$

With the same proofs as those given in [2] for nilpotent algebras and using (1.2) we obtain:

Theorem 1.5. *Let N be a modular Jordan nilalgebra. Then $N=T_3(N)$.*

Corollary 1.6. *Let N be a modular Jordan nilalgebra. Then the third solvable power of N is zero.*

Proposition 1.7. *Let N be a modular Jordan nilalgebra. Then $(N^3)^2=0$.*

We will also point out a Lemma, used in the proof of (1.4), which will be explicitly used in the next section.

Lemma 1.8. *Let N be a modular Jordan nilalgebra. Let a, y be elements in N such that $y^2=0$. We have:*

$$\text{if } o(a)=3 \text{ then } y\,a=0,$$

$$\text{if } o(a)=4 \text{ then } y\,a\in\langle a^3\rangle,$$

$$\text{if } o(a)=5 \text{ then } y\,a\in\langle a^3,a^4\rangle.$$

2. Modular Jordan nilalgebras over algebraically closed fields

Throughout this section N will be a modular Jordan nilalgebra over F, an algebraically closed field.

Lemma 2.1. *Let a, b be elements in N, with $o(a)=o(b)=3$. Then*

$$b=\lambda a+s,$$

where λ is in $F^*=F-\{0\}$ and s is in $T(N)$.

Proof. We know $(a)=\langle a,a^2\rangle$, $(b)=\langle b,b^2\rangle$.

Case 1. if $(a)=(b)$, then $b=\lambda a+\mu a^2$ with $\lambda\neq 0$ and μa^2 is obviously in $T(N)$.

Case 2. If $\dim_F((a)\cap(b))=1$, then it is clear that $(a)\cap(b)=\langle a^2\rangle=\langle b^2\rangle$. Since F is algebraically closed we can suppose, changing a or b if necessary, that

$$a^2=b^2.$$

From (1.3), ab belongs to $(a)+(b)=\langle a,b,a^2\rangle$ and has no component in a or b since (a,b) is nilpotent from (1.2). That is to say $ab=\gamma a^2$, where γ is in F. Now we have for all x in F:

$$(a+xb)^2=(1+2\gamma x+x^2)a^2.$$

Let x_1, x_2 be the roots of the polynominal $1+2\gamma x+x^2$. If these roots are different, then $\langle a+x_1b\rangle$, $\langle a+x_2b\rangle$ are subalgebras of N, $(a,b)=\langle a+x_1b\rangle\vee\langle a+x_2b\rangle=$

$\langle a+x_1b\rangle+\langle a+x_2b\rangle$, which is a contradiction since we are supposing that $(a,b)=(a)+(b)$ has dimension three. Hence $x_1=x_2$ and $\gamma=1$ or -1.

Thus, taking $-a$ instead of a if necessary we can suppose $ab=a^2$.

Now $(a-b)^2=0$, that is to say $a-b$ is in $T(N)$, which is what we wanted to prove (λ would appear from the changes we may have done).

Case 3. If $(a)\cap(b)=0$, then $(a,b)=(a)\vee(b)=(a)+(b)=\langle a,a^2,b,b^2\rangle$ has dimension four (using (1.2)). From (1.8) we know:

$$ab^2=ba^2=a^2b^2=0.$$

If $ab=0$, then $(a+b)^2=a^2+b^2=(a-b)^2$, $(a+b)^3=(a-b)^3=0$. Hence, using (1.3), $(a,b)=(a+b)\vee(a-b)=(a+b)+(a-b)=\langle a+b,a-b,a^2+b^2\rangle$, which is a contradiction since we are supposing that (a,b) has dimension four.

If $ab\neq 0$, since ab belongs to $\langle a,b,a^2,b^2\rangle$ and has no component in a or b using (1.2), we can put without loss of generality

$$ab=a^2+\mu b^2,\text{ where } \mu \text{ is in } F.$$

Now $(a-1/2b)^2=(1/4-\mu)b^2$, $(a-1/2b)^3=0$ using (1.8) and the subalgebra $(a,b)=(a-1/2b)\vee(b)=(a-1/2b)+(b)=\langle a-1/2b,b,b^2\rangle$, which is a contradiction since (a,b) is supposed to have dimension four. □

Corollary 2.2. *Let N have elements all of which have order less than or equal to three. Then either N is trivial or $N\cong(a)\oplus S$ with $o(a)=3$ and S a trivial algebra.*

Proof. If N is not trivial then there exists a, an element in N such that $o(a)=3$. Using (2.1), $N=\langle a\rangle+T(N)$. From (1.8) $aT(N)=0$. Let us take B a basis of $T(N)$ containing a^2. Take $S=\langle B-\{a^2\}\rangle$. □

Lemma 2.3. *Let a, b be elements in N such that $o(a)=4$, $o(b)=3$. Then the elements b^2 and ab are in $\langle a^3\rangle$.*

Proof. *Case 1.* Suppose $(a)\cap(b)\neq 0$. In (a) there is no element of order three, hence

$$(a)\cap(b)=\langle b^2\rangle.$$

Thus $b^2=\lambda a^2+\mu a^3$.

If $\lambda\neq 0$, we can put

$$\sqrt{\lambda}a+\frac{\mu}{\sqrt{\lambda}}a^2$$

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instead of a and suppose, without loss of generality that $a^2=b^2$ (use (1.8)). Now ab is in $(a,b)=(a)+(b)=\langle a,a^2,a^3,b\rangle$ (from (1.3)) and has no component in either a or b because (a,b) is nilpotent. Put $ab=\lambda a^2+\mu a^3$, where λ,μ are in F. Hence $(a+xb)^2=(1+2\lambda x+x^2)a^2+2\mu x a^3$, for all x in F. Take x_1 one of the roots of $1+2\lambda x+x^2$ and $b'=a+x_1 b$. Using (1.8) it is easy to check that $b'^3=0$. If $o(b')=2$, then $(a,b)=(b',b)=(b')+(b)=\langle b',b,b^2\rangle$, which is a contradiction since (a,b) has at least dimension four. If $o(b')=3$, then b and b' are elements of order three and $b'-\varepsilon b$ has square different form zero for any element ε in F; this contradicts (2.1).

We have proved $\lambda=0$ and $b^2=\mu a^3$. Using that F is algebraically closed we can change b if necessary and suppose with loss of generality $b^2=a^3$.

Put $ab=\gamma a^2+\delta a^3$, using (a,b) is nilpotent. Now, if $\gamma\neq 0$, then

$$b'=a-\frac{1}{2\gamma}b$$

satisfies $b'^2=\alpha a^3, b'^3=0$ and $(a,b)=(b',b)=(b')+(b)=\langle b,b^2=a^3,b'\rangle$, which is a contradiction since (a,b) has dimension at least four. We have proved $\gamma=0$ and $ab\in\langle a^3\rangle$.

Case 2. Suppose $(a)\cap(b)=0$. From (1.8), $\langle a^3\rangle$ is an ideal in (a,b). In the quotient $(a,b)/\langle a^3\rangle$, the elements $a+\langle a^3\rangle$ and $b+\langle a^3\rangle$ have order three and the subalgebras generated by them have zero intersection. This contradicts (2.1). □

Corollary 2.4. *Let a, b' be elements in N such that $o(a)=4, o(b')=3$. There exists b, an element in N such that $(a,b')=(a,b)=\langle a,a^2,a^3=b^2,b\rangle$, $o(b)=3$, $ab=a^2b=a^3b=0$.*

Proof. We have proved in (2.3) that $b'^2=\lambda a^3$ where λ is in F^*.

Take

$$b''=\frac{1}{\sqrt{\lambda}}b'.$$

From (2.3) $ab''=\mu a^3$. Now $b=b''-\mu a^3$ is the element we wanted (to check that, use (1.8)). □

Lemma 2.5. *Let a, b be elements in N such that $o(a)=o(b)=4$. Then $(a)=(b)$ or $(a)\cap(b)=\langle a^2,a^3\rangle=\langle b^2,b^3\rangle$.*

Proof. If $(a)\cap(b)=0$ then, considering $\langle a^3,b^3\rangle$, which is an ideal in $(a,b)=(a)+(b)$ by (1.8), we get the quotient $(a,b)/\langle a^3,b^3\rangle$, where the elements $a+\langle a^3,b^3\rangle$ and $b+\langle a^3,b^3\rangle$ generate subalgebras with zero intersection, contradicting (2.1).

Thus $(a)\cap(b)\neq 0$. Let us see that $\dim_F((a)\cap(b))=1$ is impossible. It is clear that one-dimensional subalgebras of (a) are $\langle a^2+\mu a^3\rangle=\langle(a+\frac{1}{2}\mu a^2)^2\rangle$, where μ is in F, and

$\langle a^3 \rangle$. Making changes of the form $a+\frac{1}{2}\mu a^2$ instead of a, permuting a and b, and multiplying by elements in F^* if necessary, the possibilities can be reduced to:

Case 1. $(a)\cap(b)=\langle a^2\rangle=\langle b^2\rangle$, where $a^2=b^2$. The product ab is in $(a)\vee(b)=(a)+(b)$ and has "zero component" in a and b, since $(a)\vee(b)=(a,b)$ is nilpotent. That is to say $ab=\alpha a^2+\beta a^3+\gamma b^3$, where $\alpha,\beta,\gamma\in F$.

We have $(a+xb)^2=(1+1\alpha x+x^2)a^2+2\beta a^3+2\gamma x b^3$ for all x in F.

Let x_1 be a root of the polynomial $1+2\alpha x+x^2$ (in particular $x_1\neq 0$). Take $b'=a+x_1 b$. The element b'^2 belongs to $\langle a^3,b^3\rangle$ and $b'^3=0$, since $ab^3=ba^3=0$ ($ab^3=a(bb^2)=a(ba^2)=(ab)a^2=0$ using the Jordan identity and the fact that ab and b^2 are in $T(N)$). If $b'^2=0$ then $(a)+(b)=(a,b)=(b',b)=(b')+(b)=\langle b',b,b^2,b^3\rangle$, which is a contradiction since we are supposing $(a)+(b)$ has dimension five. If $b'^2\neq 0$ then $o(b')=3$ and using (2.3) $b'a\in\langle a^3\rangle$ and $b'b\in\langle b^3\rangle$. But $b'a=a^2+x_1 ab$ and thus $ab\in\langle a^2,a^3\rangle$. Similarly $b'b=ab+x_1 b^2$ and $ab\in\langle b^2,b^3\rangle$. We know $\langle a^2,a^3\rangle\cap\langle b^2,b^3\rangle=\langle b^2\rangle$. Hence $ab\in\langle a^2\rangle$, $ab=\alpha a^2$ and $b'^2=0$, which is a contradiction.

Case 2. $(a)\cap(b)=\langle a^2\rangle=\langle b^3\rangle$ and $a^2=b^3$. Using (1.8), $\langle a^3,b^3\rangle=\langle a^2,a^3\rangle$ is an ideal of (a,b). In the quotient $(a,b)/\langle a^3,b^3\rangle$, the elements $a+\langle a^3,b^3\rangle$, and $b+\langle a^3,b^3\rangle$ have orders two and three respectively. From (1.8) $ab\in\langle a^3,b^3\rangle$. Put $ab=\lambda b^3+\mu a^3=\lambda a^2+\mu a^3$.

Let us note that $(b-\mu a^2)^2=b^2$, $(b-\mu a^2)^3=b^3$, $a(b-\mu a^2)=\lambda a^2$. Considering $b-\mu a^2$ instead of b, we can suppose:

$$ab=\lambda a^2,\, a^2=b^3.$$

If $\lambda\neq 0$ then

$$\left(a-\frac{1}{2\lambda}b\right)^2=\left(\frac{1}{2\lambda}\right)^2 b^2.$$

Using Case 1, either

$$o\left(a-\frac{1}{2\lambda}b\right)\neq 4$$

or

$$\left(a-\frac{1}{2\lambda}b\right)\cap(b)$$

has dimension at least two; in either case,

$$(a,b)=\left(a-\frac{1}{2\lambda}b\right)\vee(b)=\left(a-\frac{1}{2\lambda}b\right)+(b)$$

has dimension four, which contradicts our assumption. Hence $ab=0$.

Now $(a+b)^2=b^2+b^3=(b+1/2b^2)^2$.

It is clear that $(b)=(b+1/2b^2)$. Hence $(a,b)=(a+b)\vee(b+1/2b^2)=(a+b)+(b+1/2b^2)=\langle a+b,(a+b)^2,(a+b)^3,b+1/2b^2,(b+1/2b^2)^3\rangle$ (orders of $a+b$ and $b+1/2b^2$ are clearly less than or equal to four). From our assumption, the dimension of (a,b) is five and thus $(a+b)^3\neq 0$. Now, the existence of elements $a+b$ and $b+1/2b^2$ contradicts Case 1.

Case 3. $(a)\cap(b)=\langle a^3\rangle=\langle b^3\rangle$. Evidently $\langle a^3\rangle$ is an ideal in (a,b). In the quotient $(a,b)/\langle a^3\rangle$, the elements $a+\langle a^3\rangle$ and $b+\langle b^3\rangle$ have order three and generate subalgebras with zero intersection, contradicting Lemma (2.1).

We have shown that $(a)\cap(b)$ has dimension 2 or 3, which proves the Lemma since for any element c of order three the only 2-dimensional subalgebra of (c) is $\langle c^2,c^3\rangle$.

□

Corollary 2.6. *Let a, b be elements in N such that $o(a)=o(b)=4$. Then either $(a)=(b)$ or $(a,b)=(a,c)$ where c is an element in N such that $o(c)\leqq 3$.*

Proof. Let us suppose $(a)\neq(b)$. From (2.5) $\langle a^2,a^3\rangle=\langle b^2,b^3\rangle$.

If $a^3=\lambda b^2+\gamma b^3$ and $\lambda\neq 0$ (λ,γ in F), replacing b by

$$\sqrt{\lambda}b+\frac{1}{2\sqrt{\lambda}}\gamma b^2$$

we can assume that $a^3=b^2$ and $b^3=\alpha a^2+\beta a^3$, where α, β are in F and $\alpha\neq 0$. Now the products

$$(\dots((((a^3b)a)b)a)b\dots)a$$

are never zero, which is impossible since (a,b) is nilpotent.

We have shown that $a^3=\gamma b^3$. Consequently $a^2=\delta b^2+\varepsilon b^3$, where δ, ε are in F and $\delta\neq 0$. Putting

$$\sqrt{\delta}b+\frac{1}{2\sqrt{\delta}}\varepsilon b^2$$

instead of b, we can suppose:

$$a^3=\gamma b^3,\ a^2=b^2.$$

The product ab is in $(a,b)=\langle a,b,a^2,a^3\rangle$ and has zero component in a and b since (a,b) is nilpotent. We can write $ab=\phi a^2+\eta a^3$.

In these conditions $(a+xb)^2=(1+2\phi x+x^2)a^2+2\eta xa^3$ for all x in F.
Take $c=a+x_1b$ for x_1 some root of the polynomial $1+2\phi x+x^2$. $\square$

Corollary 2.7. *If all elements in N have order less than five and there is some element of order four, then N is isomorphic to one of the following algebras:*

(i) $(b)\oplus S$,

(ii) $\langle b,b^2,b^3,a\rangle\oplus S$,

where S is trivial, $o(b)=4$ and, in (ii), $o(a)=3$, $a^2=b^3$, $ab=ab^2=ab^3=0$, $\dim_F\langle b,b^2,b^3,a\rangle=4$.

Proof. If we cannot find elements of order three, then, from the previous result, $N=(a)\vee T_1(N)=(a)+T_1(N)$, where $o(a)=4$. Let $\{a^2,a^3,s_i \mid i\in I\}$ be a basis of $T_1(N)$. We know from (1.8) that a $s_i=\mu_i a^3$, where $\mu_i\in F$. Define $s_i'=s_i-\mu_i a^2$ for all i in I. Taking $S=\langle s_i' \mid i\in I\rangle$, we get (i).

Suppose x, b are elements in N such that $o(x)=3$, $o(b)=4$. From (2.6), N is generated by B and some elements of order less than or equal to three. Using (2.1) all elements in N of order less than or equal to three are in $(x)\vee T_1(N)$. Thus $N=(b)\vee(x)\vee T_1(N)=(b,x)+T_1(N)$. The subalgebra (b,x) is equal, using (2.4), to $(b,a)=<a,a^2,a^3=b^2,b\rangle$, where b is an element in N such that $o(b)=3$, $ba=ba^2=ba^3=0$, $dim_F(a,b)=4$. Consider $\{a^2,a^3,s_i; i\in I\}$ a basis of $T_1(N)$ and proceed as above to get S such that $aS=0$. $bS=0$ (from (1.8)) and we get (ii). $\square$

Lemma 2.8. *If N possesses elements of order five then it does not possess elements of order four.*

Proof. Let a,b be elements in N such that $o(a)=5$, $o(b)=4$. Since $o(a^2)=3$, from Lemma (2.3), we know that $a^4=(a^2)^2$ is in $\langle b^3\rangle$. It is clear that, in these conditions, $\langle a^4\rangle=\langle b^3\rangle$ is an ideal of $(a,b)=(a)+(b)$. In the quotient $(a,b)/\langle a^4\rangle$, the elements $a+\langle a^4\rangle$, $b+\langle a^4\rangle$ have orders four and three respectively. Using Lemma (2.3) it is readily seen that $(a+\langle a^4\rangle)\cap(b+\langle a^4\rangle)=\langle b^2+\langle a^4\rangle\rangle=\langle a^3+\langle a^4\rangle\rangle$, which can be lifted to $(a)\cap(b)=\langle b^2,b^3\rangle=\langle a^3,a^4\rangle$, and $[a+\langle a^4\rangle][b+\langle a^4\rangle]$ is in $\langle a^3+\langle a^4\rangle\rangle$, which leads to ab is in $\langle a^3,a^4\rangle$.

We can suppose $a^4=b^3$, since F is algebraically closed.

Now, $a^3=\lambda a^4+\alpha b^2$, where α, $\lambda\in F$, $\alpha\neq 0$. Using (2.3) again we know $a^2b=\delta b^3=\delta a^4$, $\delta\in F$. For all x in F we can write the following identities where the Greek letters represent elements in F:

$$(a+xb)^2=a^2+x^2b^2+2xab=a^2+\varepsilon a^3+\eta a^4,$$

$$(a+xb)^3=a^3+\rho a^4,\ \text{since } a^3b\in\langle b^3\rangle, \text{ by (1.8), and } a^4b=b^3b=0,$$

$$(a+xb)^4=a^4+xba^3=a^4+x\alpha bb^2=a^4+x\alpha b^3=a^4+x\alpha a^4=(1+x\alpha)a^4.$$

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Taking $x_1=-1/\alpha$, we get $o(a+x_1b)=4$ and so $a+x_1b$ is in $T_2(N)$. But b is also in $T_2N)$ and we get that a is in $T_2(N)$, which is a contradiction since $o(a)=5$ and $(T_2(N)^2)^2=0$. □

Proposition 2.9. *If N possesses an element a of order five then N is one of the following algebras:*

(i) $(a)\oplus S$

(ii) $[(a)+\langle t\rangle]\oplus S$,

where S is a trivial ideal, $o(t)=2$, $ta=a^3$, $ta^2=ta^3=ta^4=0$.

Proof. Using (2.8) we know that there is no element in N of order four. Besides that a^2 is an element of order three and, using (2.1), all the elements of order three are in $(a^2)+T_1(N)$.

In the quotient $N/T_1(N)$, $a+T_1(N)$ is an element of order three and, from (2.1), we obtain that $N/T_1(N)=(a+T_1(N))+T_1(N/T_1(N))=(a+T_1(N))+T_2(N)/T_1(N)$. Thus $N=(a)+T_2(N)$. We proved above that $T_2(N)$ (which consists of the set of elements of order less then five) is contained in $(a^2)+T_1(N)$. We obtain that $N=(a)+T_1(N)$.

Let $\{a^3,a^4,s_i \mid i\in I\}$ be a basis of $T_1(N)$. We know (see (1.8)) that $s_ia=\mu_ia^3+\lambda_ia^4$, where μ_i and λ_i are in F. Define $s_i'=s_i-\lambda_ia^4$. We get another basis $\{a^3,a^4,s_i' \mid i\in I\}$, of $T_1(N)$. The elements s_i' satisfy $s_i'a=\mu_ia^3$.

If $s_i'a=0$ for all i in I, we are in the situation described in (i), taking $S=\langle\{s_i' \mid i\in I\}\rangle$, since $Sa^2=0$ using (1.8).

Otherwise there exists i_0 in I such that $s_{i_0}'a=\mu_{i_0}a^3$ where $\mu_{i_0}\neq 0$. Without loss of generality we can suppose $s_{i_0}'a=a^3$. Define $t_j=s_j-\mu_js_{i_0}'$ for all j in I different from i_0, $t=s_{i_0}'$ and $S=\langle\{t_j \mid j\neq i_0, j\in I\}\rangle$. It is clear that $St=Sa=0$. Besides that $Sa^2=0$ using (1.8) and $Sa^3=Sa^4=0$ since S, a^3 and a^4 are in $T_1(N)$. Using the definition of t, $ta=s_{i_0}'a=a^3$, $ta^2=0$ (use (1.8)) and $ta^3=ta^4=0$, since t, a^3, a^4 are in $T_1(N)$. We are in the situation described in (ii).

Proposition 2.10. *Let M be a modular Jordan nilagebra. Let S be a trivial algebra. Then $M\oplus S$ is modular. (Of course it is a Jordan nilalgebra.)*

Proof. We just need to check (1.3) (iii), since $M\oplus S$ is clearly a Jordan nilalgebra. This is equivalent to proving that $ab\in(a)+(b)$ for any pair of elements a,b in $M\oplus S$.

Put $a=m+s$, $b=n+t$, where m, n are in M, s, t are in S. Now $ab=mn$.

M is a modular Jordan nilalgebra, hence $(m)\vee(n)=(m)+(n)$ is a nilpotent algebra and it is not difficult to see that mn is in $\langle m^2,m^3,\ldots,n^2,n^3,\ldots\rangle$, which is an ideal in $(m)+(n)$. Thus $ab=mn$ is in $\langle m^2,m^3,\ldots,n^2,n^3,\ldots\rangle=\langle a^2,a^3,\ldots,b^2,b^3,\ldots\rangle$ which is contained in $(a)+(b)$. □

Theorem 2.11. *Modular Jordan nilalgebras over an algebraically closed field F are, up to isomorphism, the following commutative, power-associative algebras:*

$$M_{1,k}=S, \text{ where } S^2=0, \dim_F S=k,$$

$M_{2,k}=(a)\oplus S$, *where* a *is nilpotent*, $o(a)=3, S^2=0, \dim_F S=k$,

$M_{3,k}=(a)\oplus S$, *where* a *is nilpotent*, $o(a)=4$, $S^2=0$, $\dim_F S=k$,

$M_{4,k}=\langle a,a^2,a^3,b\rangle\oplus S$, *where* $\dim_F\langle a,a^2,a^3,b\rangle=4$, a, b *nilpotent*, $o(a)=4$, $o(b)=3$, $b^2=a^3$, $ba=ba^2=ba^3=0$, $S^2=0$, $\dim_F S=k$,

$M_{5,k}=(a)\oplus S$, *where* a *is nilpotent*, $o(a)=5$, $S^2=0$, $\dim_F S=k$,

$M_{6,k}=[(a)\dotplus\langle t\rangle]\oplus S$, *where* a, t *nilpotent*, $o(a)=5$, $o(t)=2$, $ta=a^3$, $ta^2=ta^3=ta^4=0$, $S^2=0$, $\dim_F S=k$

(*k could be an infinite cardinal number*).

Moreover $M_{i,k}\cong M_{j,r}$ *if and only if* $i=j, k=r$.

Proof. Use (2.2), (2.7), (2.9) and (1.1) to see that all modular Jordan nilalgebras must be isomorphic to one of the models described in the theorem.

To see all the algebras $M_{i,k}$ are Jordan nilalgebras is straightforward. Using (2.10) the task of checking that they are modular is reduced to seeing that $M_{i,0}$ is modular for $i=2$, 3, 4, 5, 6, which is also straightforward.

The following properties are immediate:

$M_{1,k}$ does not possess elements of orders 3, 4, or 5.

$M_{2,k}$ does not possess elements of orders 4 or 5, but it does possess elements of order 3.

$M_{3,k}$ does not possess elements of orders 3 or 5, but it does posses elements of order 4.

$M_{4,k}$ does not possess elements of order 5, but it does possess elements of order 3 or 4.

$M_{5,k}$ possesses elements of order 5 and $\dim_F(T_1(M_{5,k})M_{5,k})=\dim_F\langle a^4\rangle=1$.

$M_{6,k}$ possesses elements of order 5 and $\dim_F(T_1(M_{6,k})M_{6,k})=\dim_F\langle a^3,a^4\rangle=2$.

From these properties it is clear that $M_{i,k}\cong M_{j,r}$ implies $i=j$. But now k must be equal to r, since $\dim_F M_{i,k}=\dim_F M_{i,r}$. □

Corollary 2.12. *A modular Jordan nilalgebra over an algebraically closed field is nilpotent (of order five).*

Corollary 2.13. *A modular Jordan nilalgebra over an algebraically closed field is special.*

Proof. Use Shirshov's Theorem (see [9]) since $M_{i,0}$ for $i \in \{1,2,3,4,5,6\}$ is generated by two elements and $M_{i,j}$ is the direct sum (of ideals) of $M_{i,0}$ and a trivial algebra.

□

Note. It would be interesting to know if (2.12) and (2.13) can be generalized to the case of arbitrary ground fields (of characteristic not two). A direct proof of (2.12) and (2.13), without going through the classification obtained in (2.11) would be useful to provide ideas to attack the same problem in arbitrary fields.

On the other hand, as it was mentioned in the introduction, in [3] it is shown that all modular Jordan algebras are extensions of dimension two of their nilradicals. It could be interesting to get a classification of modular Jordan algebras (not necessarily nil) over algebraically closed fields, extending (2.11).

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