Canad. Math. Bull. Vol. 16 (4), 1973

PARTIALLY SELF-INJECTIVE REGULAR RINGS

BY
ANDREW B. CARSON*

ABSTRACT. It is proved, for any uncountable cardinal λ, that a λ-complete Boolean ring is λ-self-injective. An example shows that the converse need not hold.

1. **Introduction.** In this paper all rings are commutative and have a nonzero identity, and all ring homomorphisms preserve the identity. By a *regular* ring we mean a ring R such that, for each $r \in R$, there exists $r' \in R$ such that $rr'r=r$. We call a topological space *Boolean* if it is compact, Hausdorff, and totally disconnected. We say that a subset of a Boolean space X is *clopen* if it is both open and closed in X, and that X is *extremally disconnected* if, for each open subset U of X, $\mathrm{Cl}(U)$ is clopen in X. For any ring $\langle R, +, \cdot\rangle$ let $B(R)$ denote $\{e \in R : e^2=e\}$. It is easily verified that $\langle B(R), \vee, \wedge, ^-\rangle$ is a Boolean algebra and $\langle B(R), +', \cdot\rangle$ is a Boolean ring, where $e\vee f=e+f-e\cdot f$, $e\wedge f=e\cdot f$, $e^-=1-e$, and

$$e+'f = e+f-2e\cdot f = (e\vee f)\wedge(e\wedge f)^-.$$

In particular, any Boolean ring may be viewed as a Boolean algebra, and vice versa. Under both of these viewpoints, the maximal ideals of $B(R)$ are the same.

Lambeck has shown, in [4, Section 2.4], that a Boolean ring is self-injective if and only if it is complete as a Boolean algebra. It is known (see [6, 22.4]) that a Boolean algebra is complete if and only if its Stone space is extremally disconnected. We generalize these concepts with the following definitions.

DEFINITION 1.1. Let R be a ring and λ a cardinal.

(i) An ideal in R is a *λ-ideal* if it is generated by some set containing fewer than λ elements.

(ii) The ring R is *λ-self-injective* if, for each λ-ideal I of R and $f \in \mathrm{Hom}_R(I, R)$, there exists $f' \in \mathrm{Hom}_R(R, R)$ such that $f'|_I=f$.

(iii) A Boolean algebra A is *λ-complete* if each subset of A with cardinality less than λ has a supremum in A. Under these circumstances we also say that the Boolean ring A is *λ-complete*.

Received by the editors February 22, 1971 and, in revised form, July 24, 1972.

* This material is from the author's doctoral dissertation, completed at the University of British Columbia, Vancouver 8, Canada. The author was partially supported, as a graduate student, by a MacMillan Family Fellowship.

https://doi.org/10.4153/CMB-1973-081-6 Published online by Cambridge University Press

In these terms, the Injective Test Theorem (see [3, p. 49]) states that the ring R is self-injective if and only if it is μ-self-injective for each cardinal μ.

DEFINITION 1.2. Let X be a Boolean space and λ a cardinal number.

(i) Suppose that $U \subseteq X$. Then U is a *λ-subset* of X if it can be expressed as a union of fewer than λ clopen subsets of X.

(ii) The space X is *λ-extremally disconnected* if, for each λ-subset U of X, $\mathbf{Cl}(U)$ is clopen in X.

(iii) The space X has the *λ-disjointness property* if, for any λ-subsets U and V of X, $U \cap V = \phi$ implies that $\mathbf{Cl}(U) \cap \mathbf{Cl}(V) = \phi$.

To study these concepts we use the representation theory developed by R. S. Pierce in [5]. This associates with each commutative regular ring R a unique Boolean space $X(R)$, and a unique sheaf $k(R)$ of fields over $X(R)$, such that $R \simeq \Gamma(X(R), k(R))$, the ring of all continuous sections of $k(R)$ over $X(R)$. The space $X(R)$ is actually the set of all maximal ideals in $B(R)$ (viewed either as a Boolean ring or as a Boolean algebra) with the hull-kernel topology, or equivalently, the Stone space of the Boolean algebra $B(R)$. For each $M \in X(R)$, the stalk of $k(R)$ over M is given by $k_M(R) = (R/_{R \cdot M})$.

In §2 we note that a characterization of self-injectivity for a regular ring R in terms of the pair $(X(R), k(R))$, given by Pierce in [5, Lemma 23.1], can be modified for λ-self-injectivity, and relate λ-completeness and λ-self-injectivity for a Boolean ring with properties of its Stone space. Using these facts we show that, for any cardinal λ, a λ-complete Boolean ring is λ-self-injective. The converse of this is false, we show in §3, if $\lambda \geq \aleph_1$.

In a later paper we shall apply these results to show that $R[[X]]$, the ring of formal power series with coefficients from R, is coherent if and only if R is $\aleph_1$-self-injective and $B(R)$ is $\aleph_1$-complete.

2. **Theorems.** In this section R denotes an arbitrary commutative regular ring with unity, k denotes $k(R)$, X denotes $X(R)$, and λ denotes an arbitrary cardinal. Identify $R = \Gamma(X, k)$. We begin by relating the concepts introduced in Definition 1.2.

LEMMA 2.1. *Suppose that X is λ-extremally-disconnected. Then it has the λ-disjointness property.*

Proof. Suppose that U and V are open subsets of X such that U is a λ-subset of X and $U \cap V = \phi$. Then $V \subseteq (X - U)$ so that

$$V \subseteq \text{Interior}[(X-U)] = [X - \mathbf{Cl}(U)].$$

Thus, by hypothesis, $\mathbf{Cl}(V) \subseteq [X - \mathbf{Cl}(U)]$. Therefore $\mathbf{Cl}(U) \cap \mathbf{Cl}(V) = \phi$. ∎

The above proof can be reversed (when $V = [X - \mathbf{Cl}(U)]$) to show that X is extremally disconnected if it has the μ-disjointness property for each cardinal μ. However this does not establish the converse of Lemma 2.1, for it might be that U

https://doi.org/10.4153/CMB-1973-081-6 Published online by Cambridge University Press

is a λ-subset of X while $(X-\mathbf{Cl}(U))$ is not. In §3 we show that the converse of Lemma 2.1 is false, if $\lambda>\aleph_0$.

The following concept will be used to characterize λ-self-injectivity for R in terms of X and k.

DEFINITION 2.2. (i) For a subset Y of X let $\Gamma(Y, k)$ denote the ring of all continuous sections of k over Y.

(ii) The pair (X, k) has the *λ-extension property* if, for each λ-subset U of X and $\sigma \in \Gamma(U, k)$, there exists $\sigma' \in \Gamma(X, k)$ such that $\sigma'|_U=\sigma$.

LEMMA 2.3. *Suppose that (X, k) has the λ-extension property. Then X has the λ-disjointness property.*

Proof. Suppose that U and V are λ-subsets of X such that $U \cap V=\phi$ yet there exists $\mathrm{x} \in \mathbf{Cl}(U) \cap \mathbf{Cl}(V)$. Define $\sigma \in \Gamma(U \cup V, k)$ by $\sigma(u)=0 \in k_u$ for $u \in U$ and $\sigma(v)=1 \in k_v$ for $v \in V$. Since U and V are open and disjoint in X, therefore $\sigma \in \Gamma(U \cup V, k)$. Thus there exists $\sigma' \in \Gamma(X, k)$ satisfying $\sigma'|_{U\cup V}=\sigma$. Then $\sigma'(\mathrm{x})=0$ since $\mathrm{x} \in \mathbf{Cl}(U)$ and $\sigma'(\mathrm{x})=1$ since $\mathrm{x} \in \mathbf{Cl}(V)$. This contradiction establishes that $\mathbf{Cl}(U) \cap \mathbf{Cl}(V)=\phi$. ■

THEOREM 2.4. (i) *The regular ring R is λ-self-injective if and only if (X, k) has the λ-extension property.*

Now suppose that R is a Boolean ring. Then:

(ii) *R is λ-self-injective if and only if X has the λ-disjointness property.*

(iii) *R is λ-complete if and only if X is λ-extremally disconnected.*

Proof. (i) This is a straightforward modification of [5, Lemma 23.1].

(ii) Since R is a Boolean ring, each k_{x}, for $\mathrm{x} \in X$, is a field satisfying the polynomial identity $X^2-X=0$, and thus is the two element field. In view of (i) and Lemma 2.3, it suffices to show that (X, k) has the λ-extension property if X has the λ-disjointness property. Let X have the λ-disjointness property. Suppose that U is a λ-subset of X and $\sigma \in \Gamma(U, k)$. Let

$$V = \{\mathrm{x} \in U: \sigma(\mathrm{x}) = 0\} \quad \text{and} \quad W = \{\mathrm{x} \in U: \sigma(\mathrm{x}) = 1\}.$$

Then $V \cap W=\phi$, $V \cup W=U$, and V and W are open in X. Thus V and W are disjoint λ-subsets of X so that, since X has the λ-disjointness property, there exists a clopen subset C of X such that $\mathbf{Cl}(V) \cap C=\phi$ and $\mathbf{Cl}(W)\subseteq C$. Then $\sigma' \in \Gamma(X, k)$ and $\sigma'|_U=\sigma$, where $\sigma(\mathrm{x})=0$ when $\mathrm{x} \notin C$ and $\sigma(\mathrm{x})=1$ when $\mathrm{x} \in C$.

(iii) A trivial variation of the proof of [6, 22.4], required due to differing concepts of λ-completeness, establishes this result. ■

We now use Theorem 2.4 to relate the concepts of λ-completeness and λ-self-injectivity.

THEOREM 2.5. (i) *If R is λ-self-injective, then so is the ring $B(R)$.*

Now suppose that R is a Boolean ring. Then:

(ii) *If R is λ-complete, then R is λ-self-injective.*

https://doi.org/10.4153/CMB-1973-081-6 Published online by Cambridge University Press

Proof. (i) Note that $B(B(R))=B(R)$ so that $X(B(R))=X(R)=X$. If R is λ-self-injective, then, by Theorem 2.4(i) and Lemma 2.3, X has the λ-disjointness property. Thus, by 2.4(ii) with $B(R)$ in place of R, $B(R)$ is λ-self-injective.

(ii) If R is λ-complete then, by 2.4(iii), X is λ-extremally disconnected so that, by Lemma 2.1 and Theorem 2.4(ii), R is λ-self-injective. ∎

The following property of spaces with the λ-disjointness property will be used in a later paper determining when $R[[X]]$ is coherent.

LEMMA 2.6. *Let X have the λ-disjointness property. Then, for any λ-subsets U and V of X,*

$$\mathbf{Cl}(U \cap V) = \mathbf{Cl}(U) \cap \mathbf{Cl}(V).$$

Proof. That $\mathbf{Cl}(U \cap V) \subseteq \mathbf{Cl}(U) \cap \mathbf{Cl}(V)$, is true in general. Suppose that $x \in \mathbf{Cl}(U) \cap \mathbf{Cl}(V)$. Let F_x denote the family of all clopen neighbourhoods of x. Note that, for any $N \in F_x$, $N \cap U$ and $N \cap V$ are λ-subsets of X such that

$$x \in \mathbf{Cl}(N \cap U) \cap \mathbf{Cl}(N \cap V).$$

Thus

$$N \cap (U \cap V) = (N \cap U) \cap (N \cap V) \neq \phi.$$

Since F_x is a filter converging to x, this yields $x \in \mathbf{Cl}(U \cap V)$. ∎

3. **Examples.** In this section λ denotes an infinite cardinal. We construct a Boolean space X with the λ-disjointness property that is not $\aleph_1$-extremally disconnected. It follows from 2.4 that the Boolean ring of clopen subsets of X is λ-self-injective but not $\aleph_1$-complete. Since such an X also has the λ'-disjointness property for each $\lambda' < \lambda$, and since there are arbitrarily large regular cardinals, we assume without loss of generality that λ is regular.

The space X is constructed to be a one point union of the form $Y \cup W/p=q$ where Y is the Stone-Cech compactification of the discrete space N of natural numbers, $p \in Y-N$, and W is a suitable Boolean space.

The space W is now constructed. Let Λ be a set of cardinality λ and let T be the Boolean algebra of all subsets of Λ of cardinality or cocardinality less than λ. Let W be the Stone space of T, or equivalently $X(T)$, where T is viewed as a ring. That is to say, W is the family of all maximal proper ideals of T with the hull-kernel topology. Let the ideal $q=\{t \in T: |t| < \lambda\}$ of T be viewed as a point in W.

LEMMA 3.1 (i). *Y has the λ-disjointness property.*

(ii) *W has the λ-disjointness property.*

(iii) *There is no λ-subset U of W such that $q \in \bar{U}-U$.*

Proof. (i) This follows from 2.1 since (see [2, ex. 6M]) Y is extremally disconnected.

(ii) This follows from 2.4 (iii) and 2.1 since T is clearly λ-complete.

(iii) The clopen subsets of W have the form $N_t=\{w \in W: t \notin w\}$, where $t \in T$. Let $U = \cup \{N_{t(\alpha)}: \alpha < \lambda'\}$, where $\lambda' < \lambda$, be an arbitrary λ-subset of W such that

https://doi.org/10.4153/CMB-1973-081-6 Published online by Cambridge University Press

$q \notin U$. Hence $t(\alpha) \in q$ so that $|t(\alpha)| < \lambda$, for each $\alpha < \lambda'$. Let $t = \cup \{t(\alpha) : \alpha < \lambda'\}$. Then $|t| < \lambda$, since λ is regular. Hence $t \in q$ so that $q \notin N_t$. The result will follow by establishing that $\bar{U} \subseteq N_t$. Since N_t is closed, it suffices to show that $U \subseteq N_t$. Suppose that $w \in N_{t(\alpha)}$ for some $\alpha < \lambda'$. Then $t(\alpha) \notin w$ so that, since w is a maximal ideal in T, $(\Lambda - t(\alpha)) \in w$. Then $w \in N_t$, for otherwise $t \in w$ so that $\Lambda = (\Lambda - t(\alpha)) \vee t \in w$, contradicting $w \nsubseteq \Lambda$. ∎

THEOREM 3.2. *There exists a Boolean space X that has the λ-disjointness property (where λ is a cardinal) but is not $\aleph_1$-extremally disconnected.*

Proof. Let X be the one point union $Y \cup W/p=q$ where Y, W, p, and q are as above. It follows by standard topological arguments and Lemma 3.1 that X is a Boolean space with the λ-disjointness property. However X is not $\aleph_1$-extremally disconnected since N is an $\aleph_1$-subset of X yet $\mathbf{Cl}_X(N) = Y$ is not open in X since q is not isolated in W. ∎

THEOREM 3.3. *There exists a Boolean ring R that is λ-self-injective (where λ is a cardinal) but is not $\aleph_1$-complete.*

Proof. Let X be the Boolean space from Theorem 3.2 and let R be the Boolean ring of all clopen subsets of X. As is well known, the Stone space $X(R)$ of R (where R is viewed as a Boolean algebra) is homeomorphic to X. The Theorem now follows from 3.2 and 2.4. ∎

REMARK Let N' be an infinite discrete space and Y' be its Stone-Cech compactification. T. Cramer has remarked in private communication that $(Y' - N')$, as a subspace of Y', is a Boolean space with the $\aleph_1$-disjointness property that is not $\aleph_1$-extremally disconnected. However if the continuum hypothesis holds, then $(Y' - N')$ fails to have the $\aleph_2$-disjointness property, no matter how large the cardinality of N' is.

The author would like to thank the referee for alterations which have greatly simplified construction of the space X in section 3.

REFERENCES

1. P. Eklof and G. Sabbagh, *Model-completions and modules*, Ann. Math. Logic 2 (1971), 251–295.
2. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton N.J., 1960.
3. J. P. Jans, *Rings and Homology*, Holt, Rinehart, and Winston, New York, 1964.
4. J. Lambek, *Lectures on Rings and Modules*, Blaisdell, Waltham Mass., 1966.
5. R. S. Pierce, *Modules over commutative regular rings*, Mem. Amer. Math. Soc. no. 70 (1967).
6. R. Sikorski, *Boolean Algebras* (second edition) Springer-Verlag, Berlin (1964).

UNIVERSITY OF SASKATCHEWAN
SASKATOON, CANADA
S7N OWO

3

https://doi.org/10.4153/CMB-1973-081-6 Published online by Cambridge University Press