TWO-DIMENSIONAL LINEAR GROUPS OVER LOCAL RINGS

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Introduction. The problem of classifying the normal subgroups of the general linear group over a field was solved in the general case by Dieudonné (see **2** and **3**). If we consider the problem over a ring, it is trivial to see that there will be more normal subgroups than in the field case. Klingenberg (**4**) has investigated the situation over a local ring and has shown that they are classified by certain congruence groups which are determined by the ideals in the ring.

Klingenberg's solution roughly goes as follows. To a given ideal $\mathfrak{a}$, attach certain congruence groups $\mathrm{GC}(\mathfrak{a})$ and $\mathrm{SC}(\mathfrak{a})$. Next, assign a certain ideal (called the *order*) to a given subgroup G. The main result states that if G is normal with order $\mathfrak{a}$, then $\mathrm{GC}(\mathfrak{a}) \geqq G \geqq \mathrm{SC}(\mathfrak{a})$, that is, G satisfies the so-called *ladder relation at* $\mathfrak{a}$; conversely, if G satisfies the ladder relation at $\mathfrak{a}$, then G is normal and has order $\mathfrak{a}$. However, Klingenberg has restricted the local ring when dealing with two variables, by assuming that the residue class field is not $\mathbf{F}_3$ and does not have characteristic 2 (i.e., 2 is a unit in the ring). In this paper we are concerned with removing these restrictions. In view of the classical counterexamples of $\mathrm{GL}_2(\mathbf{F}_2)$ and $\mathrm{GL}_2(\mathbf{F}_3)$ we cannot expect to remove all of Klingenberg's restrictions. However, by a delicate analysis of the situation we can cut the exceptional behaviour down to size, i.e., to where the residue class field has exactly two or exactly three elements. Indeed, in the latter case of three elements we surprisingly find that the exceptional behaviour is confined to the top step of the ladder. Our *First Main Theorem*, then, states that when the residue class field has more than two elements, Klingenberg's solution carries over, but for one exception, namely, when the residue class field is $\mathbf{F}_3$ and the groups have the whole ring as order.

When the residue class field is $\mathbf{F}_2$, we lose the classification by means of the ladder relation and in fact we have no true classification. Our *Second Main Theorem* states that a normal subgroup satisfies a weak ladder relation, where we go down two "steps" in the ladder. However, not all groups satisfying the weak ladder relation need be normal.

1. Preliminaries.

1.1. *Definitions.* The inclusion sign $\subset$ for sets will mean strict containment; otherwise we use $\subseteq$. For groups, $H \leqq G$ means that H is a subgroup of G, and

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$H < G$ indicates that H is properly contained. The notation for arrows is as in O'Meara's book (**6**).

A *local ring* $\mathfrak{o}$ is a commutative ring with identity which has an absolutely maximal ideal $\mathfrak{p} \subset \mathfrak{o}$, i.e., $\mathfrak{a} \subseteq \mathfrak{p}$ for every ideal $\mathfrak{a} \subset \mathfrak{o}$. We let $\mathfrak{u} = \mathfrak{o} - \mathfrak{p}$ and easily see that $\mathfrak{u}$ consists of all the units of $\mathfrak{o}$ and is a group under multiplication, and that for $\alpha \in \mathfrak{u}$, $\omega \in \mathfrak{p}$, we have that $\alpha + \omega \in \mathfrak{u}$. A homomorphic image $\bar{\mathfrak{o}}$ of $\mathfrak{o}$, if not the $\{0\}$-ring, is again a local ring with maximal ideal $\bar{\mathfrak{p}}$. We note that every field is a local ring.

A *residue class field* of $\mathfrak{o}$ is any field isomorphic to the field $\mathfrak{o}/\mathfrak{p}$. We write $N\mathfrak{p}$ for the cardinality of $\mathfrak{o}/\mathfrak{p}$.

Let $M = M(\mathfrak{o})$ be a free module of n generators over $\mathfrak{o}$. The *general linear group* of M, $\mathrm{GL}_n(M)$, is the group of all linear automorphisms of $M = M(\mathfrak{o})$.

Let $\mathfrak{a} \subseteq \mathfrak{o}$ be an ideal. Then the canonical map $g_\mathfrak{a}\colon \mathfrak{o} \rightarrow \mathfrak{o}/\mathfrak{a}$ determines a canonical map $g_\mathfrak{a}\colon M(\mathfrak{o}) \rightarrow M(\mathfrak{o}/\mathfrak{a})$. When $\mathfrak{a} = \mathfrak{o}$, we understand $M(\mathfrak{o}/\mathfrak{a})$ to be the $\{0\}$-module. The map $g_\mathfrak{a}$ determines the canonical surjective homomorphism $h_\mathfrak{a}\colon \mathrm{GL}_n(M(\mathfrak{o})) \rightarrow \mathrm{GL}_n(M(\mathfrak{o}/\mathfrak{a}))$, with the following property:

$$h_\mathfrak{a}\sigma \circ g_\mathfrak{a} = g_\mathfrak{a} \circ \sigma \quad \forall\, \sigma \in \mathrm{GL}_n(M(\mathfrak{o})).$$

When $\mathfrak{a} = \mathfrak{o}$, we understand $\mathrm{GL}_n(M(\mathfrak{o}/\mathfrak{a}))$ to be the unit group. Let $\sigma \in \mathrm{GL}_n(M)$. We define the *order* $o(\sigma)$ of σ to be the smallest ideal $\mathfrak{a}$ such that $h_\mathfrak{a}\sigma \in \text{centre}\,\mathrm{GL}_n(M(\mathfrak{o}/\mathfrak{a}))$. The *order* $o(G)$ of a subgroup $G \leqq \mathrm{GL}_n(M)$ is defined to be the smallest ideal $\mathfrak{a}$ such that $h_\mathfrak{a} G \leqq \text{centre}\,\mathrm{GL}_n(M(\mathfrak{o}/\mathfrak{a}))$. We shall soon see that these ideals do exist.

To this effect, consider $\mathrm{GL}_n(\mathfrak{o})$, the group of all invertible $n \times n$ matrices over $\mathfrak{o}$. The subgroup of $\mathrm{GL}_n(\mathfrak{o})$ consisting of all matrices $\mathrm{diag}(\alpha)$, with $\alpha \in \mathfrak{u}$, is called the *group of radiations*. We denote by $E_{ij}(\lambda)$, where $i \neq j$, the matrix in $\mathrm{GL}_n(\mathfrak{o})$ with diagonal entries 1, and with all other entries 0 except the i, j position which contains λ.

PROPOSITION 1.1.1. centre $\mathrm{GL}_n(\mathfrak{o})$ *is the group of radiations.*

Proof. Every radiation falls in centre $\mathrm{GL}_n(\mathfrak{o})$. And if $P \in \text{centre}\,\mathrm{GL}_n(\mathfrak{o})$, then the equations $PE_{ij}(1) = E_{ij}(1)P$ will show that P is a radiation.

Let $\mathfrak{X}$ be a base for $M = M(\mathfrak{o})$. The choice of $\mathfrak{X}$ sets up an isomorphism $f_\mathfrak{X}$ from $\mathrm{GL}_n(M)$ to $\mathrm{GL}_n(\mathfrak{o})$. The homomorphism $h_\mathfrak{a}\colon \mathrm{GL}_n(M) \rightarrow \mathrm{GL}_n(M(\mathfrak{o}/\mathfrak{a}))$ gives rise to the homomorphism $h_\mathfrak{a}\colon \mathrm{GL}_n(\mathfrak{o}) \rightarrow \mathrm{GL}_n(\mathfrak{o}/\mathfrak{a})$, which is defined so that the diagram

$$\begin{tikzcd}
\mathrm{GL}_n(M) \arrow[r, "h_\mathfrak{a}"'] \arrow[d, tail, "f_\mathfrak{X}"'] & \mathrm{GL}_n(M(\mathfrak{o}/\mathfrak{a})) \arrow[d, tail, "\phi"] \\
\mathrm{GL}_n(\mathfrak{o}) \arrow[r, "h_\mathfrak{a}"'] & \mathrm{GL}_n(\mathfrak{o}/\mathfrak{a})
\end{tikzcd}$$

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is commutative. If the isomorphism ϕ is the one given by the base $g_{\mathfrak{a}}\mathfrak{X}$ for $M(\mathfrak{o}/\mathfrak{a})$, then $h_{\mathfrak{a}}\colon \mathrm{GL}_n(\mathfrak{o}) \to \mathrm{GL}_n(\mathfrak{o}/\mathfrak{a})$ is the map which consists of taking the entries of the matrices of $\mathrm{GL}_n(\mathfrak{o})$ modulo $\mathfrak{a}$, independently of the base $\mathfrak{X}$ which was chosen at the start. Whenever we shall make use of such a map $h_{\mathfrak{a}}$, we will implicitly give it the above interpretation.

Let $\sigma \in \mathrm{GL}_n(M)$ have matrix (α_{ij}) in the base $\mathfrak{X}$. By Proposition 1.1.1, we find that $o(\sigma)$ is the finitely generated ideal whose generators are $\alpha_{ii} - \alpha_{jj}$, $\forall\, i, j$, together with α_{ij}, $i \neq j$. We can check that $o(\sigma)$ does not depend on the base. And clearly $o(G)$ is the ideal generated by the ideals $o(\sigma)$, $\forall\, \sigma \in G \leqq \mathrm{GL}_n(\mathfrak{o})$.

For an ideal $\mathfrak{a} \subseteq \mathfrak{o}$, we set

$$\mathrm{GC}_n(M, \mathfrak{a}) = h_{\mathfrak{a}}^{-1}(\text{centre } \mathrm{GL}_n(M(\mathfrak{o}/\mathfrak{a}))),$$

which is a subgroup of $\mathrm{GL}_n(M)$, called the *general congruence group* mod $\mathfrak{a}$. We have that

$$\begin{aligned} \mathrm{GC}_n(M, \mathfrak{a}) &\lhd \mathrm{GL}_n(M), \\ \mathrm{GC}_n(M, \mathfrak{o}) &= \mathrm{GL}_n(M), \\ \mathrm{GC}_n(M, 0) &= \text{centre } \mathrm{GL}_n(M), \end{aligned}$$

where centre $\mathrm{GL}_n(M)$ consists of the radiations

$$\sigma x = \alpha x \quad \text{for some } \alpha \in \mathfrak{u} \text{ and } \forall\, x \in M.$$

For an ideal $\mathfrak{a} \subseteq \mathfrak{o}$, we define the *special congruence group* mod $\mathfrak{a}$ to be the subgroup $\mathrm{SC}_n(M, \mathfrak{a})$ of $\mathrm{GL}_n(M)$ consisting of the elements σ with

$$\det \sigma = 1 \quad \text{and} \quad h_{\mathfrak{a}}\sigma = \text{identity}.$$

We have that

$$\begin{aligned} &\mathrm{SC}_n(M, \mathfrak{a}) \lhd \mathrm{GL}_n(M), \qquad &&\mathrm{SC}_n(M, \mathfrak{a}) \leqq \mathrm{GC}_n(M, \mathfrak{a}), \\ &\mathrm{SC}_n(M, 0) = \{1\}, &&\mathrm{SC}_n(M, \mathfrak{o}) = \mathrm{SL}_n(M), \end{aligned}$$

and $\mathrm{SL}_n(M)$ is the *special linear group* of M. The same congruence groups $\mathrm{GC}_n(\mathfrak{a})$ and $\mathrm{SC}_n(\mathfrak{a})$ can de defined in $\mathrm{GL}_n(\mathfrak{o})$. The term "congruence" is justified from the fact that

$$\begin{aligned} S \in \mathrm{GC}_n(\mathfrak{a}) &\Leftrightarrow S \equiv \text{a radiation mod } \mathfrak{a}, \\ T \in \mathrm{SC}_n(\mathfrak{a}) &\Leftrightarrow T \equiv \text{identity mod } \mathfrak{a}, \text{ and } \det T = 1. \end{aligned}$$

And for any base $\mathfrak{X}$ for M, the isomorphism $f_{\mathfrak{X}}$ yields

$$f_{\mathfrak{X}}(\mathrm{GC}_n(M, \mathfrak{a})) = \mathrm{GC}_n(\mathfrak{a}), \qquad f_{\mathfrak{X}}(\mathrm{SC}_n(M, \mathfrak{a})) = \mathrm{SC}_n(\mathfrak{a}).$$

We can verify that the congruence groups mod $\mathfrak{a}$ have order $\mathfrak{a}$; the key observation is that $E_{12}(\lambda)$ is in $\mathrm{SC}_n(\mathfrak{a})$, $\forall\, \lambda \in \mathfrak{a}$.

1.2. *Notation and remarks.* Our study deals with dimension 2, although a few points will be discussed for dimension n. In the context of dimension 2, matrices are used extensively. In general, we implicitly understand that the entries of an element σ are given by

$$\sigma = \begin{pmatrix} a & c \\ d & b \end{pmatrix},$$

unless otherwise specified. If $\rho \in \mathrm{GL}_2(\mathfrak{o})$, then by σ_ρ and (ρ, σ) we mean the conjugate $\rho\sigma\rho^{-1}$ and the commutator $\rho\sigma\rho^{-1}\sigma^{-1}$, respectively.

For the matrices $E_{ij}(\lambda)$ mentioned earlier, we have the rule:

$$E_{ij}(\lambda)E_{ij}(\mu) = E_{ij}(\lambda + \mu).$$

The following standard matrices are used in forming conjugates and commutators:

$$\Gamma(\mu, \nu) = \begin{pmatrix} 0 & \mu \\ \nu & 0 \end{pmatrix}, \qquad \Theta(\alpha, \beta) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

$$\Phi(\alpha, \beta, \nu) = \begin{pmatrix} \alpha & 0 \\ \nu & \beta \end{pmatrix}, \qquad \Xi(\alpha, \beta, \mu) = \begin{pmatrix} \alpha & \mu \\ 0 & \beta \end{pmatrix}.$$

The usual facts about commutator and mixed commutator subgroups will be used; see Zassenhaus (**7**, pp. 78–81). We mention, in particular, that

$$B \lhd A, \qquad [A, B] \leqq G \leqq B \Rightarrow G \lhd A.$$

When we say that a subgroup $G \leqq \mathrm{GL}_2(\mathfrak{o})$ is invariant under $\mathrm{SL}_2(\mathfrak{o})$, we mean that G is stable under conjugation by the elements of $\mathrm{SL}_2(\mathfrak{o})$.

We list some lemmas to formalize some naturally expected facts.

LEMMA 1.2.1. *Let ρ and σ be elements of* $\mathrm{GL}_n(\mathfrak{o})$ *or* $\mathrm{GL}_n(M)$. *Then:* (i) $\sigma \in \mathrm{GC}_n(o(\sigma))$; (ii) $o(\sigma^{-1}) = o(\sigma)$; (iii) $o(\rho\sigma\rho^{-1}) = o(\sigma)$; (iv) $o(\rho\sigma) \subseteq o(\rho) + o(\sigma)$.

The proof is straightforward.

LEMMA 1.2.2. *Let $G \leqq \mathrm{GL}_2(\mathfrak{o})$ with $o(G) = \omega\mathfrak{o}$. Then G contains elements of order $\omega\mathfrak{o}$. If G is invariant under* $\mathrm{SL}_2(\mathfrak{o})$, *there are elements σ and σ' in G with*

$$d\mathfrak{o} = \omega\mathfrak{o} = c'\mathfrak{o}.$$

Proof. Clearly we cannot have $o(\sigma) \subseteq \omega\mathfrak{p}, \forall\, \sigma \in G$. If $\sigma \in G$ has $o(\sigma) = \omega\mathfrak{o}$, then at least one of $a - b$, c, or d generates $\omega\mathfrak{o}$. Depending on the cases occurring, form σ_Γ or σ_Φ with $\Gamma(-1, 1)$ and $\Phi(1, 1, 1)$, to obtain the required elements.

LEMMA 1.2.3. *Suppose that $N\mathfrak{p} > 2$ and let $G \leqq \mathrm{GL}_2(\mathfrak{o})$ be invariant under* $\mathrm{SL}_2(\mathfrak{o})$, *with $o(G) = \omega\mathfrak{o}$. Then*

$$o(G \cap \mathrm{SC}_2(\omega\mathfrak{o})) = \omega\mathfrak{o}, \qquad G \cap \mathrm{SC}_2(\omega\mathfrak{o}) \lhd \mathrm{SL}_2(\mathfrak{o}).$$

If $G \lhd \mathrm{GL}_2(\mathfrak{o})$, then $G \cap \mathrm{SC}_2(\omega\mathfrak{o}) \lhd \mathrm{GL}_2(\mathfrak{o})$.

Proof. Pick $\sigma \in G$ with $c\mathfrak{o} = \omega\mathfrak{o}$.

If $N\mathfrak{p} \geqq 4$, we can pick $\gamma \in \mathfrak{u}$ with

$$\gamma \not\equiv \pm 1 \bmod \mathfrak{p}, \quad \text{i.e. } \gamma^2 \not\equiv 1 \bmod \mathfrak{p}.$$

We may assume that σ has $a \in \mathfrak{u}$. We have $\det \sigma \in \mathfrak{u}$; hence, if $c\mathfrak{o} = \omega\mathfrak{o} \subseteq \mathfrak{p}$, then $a \in \mathfrak{u}$. Therefore, the case $a \in \mathfrak{p}$ can occur only when $\omega\mathfrak{o} = \mathfrak{o}$, and then $c, d \in \mathfrak{u}$ to preserve $\det \sigma \in \mathfrak{u}$. Now, one easily checks that σ_Φ with $\Phi(1, 1, 1)$ gives us a new σ with the desired property. Now, the lemma follows from the fact that $(\Theta(\gamma, \gamma^{-1}), \sigma)$ falls in $G \cap \mathrm{SC}_2(\omega\mathfrak{o})$ and has order $\omega\mathfrak{o}$, as we now verify.

We have that

$$(\Theta, \sigma) = \begin{bmatrix} ab - \gamma^2 cd & a(\gamma^2 - 1)c \\ b(\gamma^{-2} - 1)d & ab - \gamma^{-2}cd \end{bmatrix} \cdot (\det \sigma)^{-1} \in G.$$

The 1, 2 entry is a unit multiple of c, so that

$$c\mathfrak{o} \subseteq o((\Theta, \sigma)) \subseteq o(G) = \omega\mathfrak{o} = c\mathfrak{o};$$

hence, our commutator has order $\omega\mathfrak{o}$. It also has determinant 1; hence, if $\omega\mathfrak{o} = \mathfrak{o}$, then $(\Theta, \sigma) \in G \cap \mathrm{SL}_2(\mathfrak{o})$ and $o(G \cap \mathrm{SL}_2(\mathfrak{o})) = \mathfrak{o}$. And if $\omega\mathfrak{o} \subseteq \mathfrak{p}$, then we easily see that (Θ, σ) is congruent to identity mod $\omega\mathfrak{o}$, since c and d are in $\omega\mathfrak{o}$. Thus $(\Theta, \sigma) \in \mathrm{SC}_2(\omega\mathfrak{o})$, which establishes our assertion.

If $N\mathfrak{p} = 3$, then $2 \in \mathfrak{u}$ and the two classes of units are represented by 1 and -1. We can force our σ to have $a - b \in \omega\mathfrak{p}$, for if $a - b = \epsilon c$ with $\epsilon \in \mathfrak{u}$, we form σ_Φ, where $\Phi(1, 1, \nu)$ has

$$\begin{aligned} \nu &= -1 \quad \text{if } \epsilon \equiv 1 \bmod \mathfrak{p}, \\ \nu &= 1 \quad \text{if } \epsilon \equiv -1 \bmod \mathfrak{p}. \end{aligned}$$

Now, the commutator $(E_{21}(1), \sigma)$ yields what is required, with arguments similar to the above.

1.3. *Transvections*. A *hyperplane* of M is a free $(n - 1)$-dimensional direct summand of M. A *transvection* τ is an element of $\mathrm{GL}_n(M)$ which satisfies the following: (i) There is a hyperplane H of M such that $\tau | H$ is the identity mapping on H; (ii) $\tau x - x \in H, \forall\, x \in M$. If τ is a transvection, then τ^{-1} is a transvection with hyperplane H and $\rho\tau\rho^{-1}$ is a transvection with hyperplane $\rho H, \forall \rho \in \mathrm{GL}_n(M)$.

Transvections are important because they generate the special congruence groups used to classify normal subgroups of $\mathrm{GL}_n(M)$ or $\mathrm{GL}_n(\mathfrak{o})$. What follows are appropriate details, including the notion of matrix transvections in $\mathrm{GL}_n(\mathfrak{o})$.

We shall frequently mention matrices of type E,

$$E = \begin{bmatrix} 1 & & & & \alpha_{1n} \\ & \cdot & & 0 & \cdot \\ & & \cdot & & \cdot \\ & 0 & & \cdot & \cdot \\ & & & 1 & \alpha_{n-1,\,n} \\ & & & & 1 \end{bmatrix}.$$

Now fix an ideal $\mathfrak{a} \subseteq \mathfrak{o}$ and let $\{\tau_i\}_{i\in I}$, I a suitable index set, be the set of all transvections which are contained in $\mathrm{GC}_n(M, \mathfrak{a})$, that is, all transvections in $\mathrm{GL}_n(M)$ which have order $\subseteq \mathfrak{a}$. Fix a base $\mathfrak{X}$ for M, and let τ_i have matrix T_i in the base $\mathfrak{X}$. What does the set $\{T_i\}_{i\in I}$ actually consist of? Every matrix of type E with $o(E) \subseteq \mathfrak{a}$ falls in $\{T_i\}_{i\in I}$ since it defines some transvection τ_i by means of base $\mathfrak{X}$. Every conjugate $P^{-1}EP$ with $o(E) \subseteq \mathfrak{a}$ and $P \in \mathrm{GL}_n(\mathfrak{o})$, falls in $\{T_i\}_{i\in I}$, as is seen by defining some transvection τ_i on M with matrix E in a suitable base different from $\mathfrak{X}$, and then by taking the matrix of τ_i with respect to $\mathfrak{X}$. Now look at an arbitrary τ_i with hyperplane H, say. We can write $M = H \oplus \mathfrak{o}x_n$; hence, if we let $x_1, \ldots, x_{n-1}$ be a base for H, then $x_1, \ldots, x_{n-1}, x_n$ is a base for M. Since $\tau_i x_n - x_n$ is in H, then τ_i has some matrix E with $o(E) \subseteq \mathfrak{a}$, in this base. Then in the base $\mathfrak{X}$, τ_i has matrix $T_i = P^{-1}EP$ for some $P \in \mathrm{GL}_n(\mathfrak{o})$.

Thus, $\{T_i\}_{i\in I}$ consists of all matrices of type E with $o(E) \subseteq \mathfrak{a}$, together with their conjugates in $\mathrm{GL}_n(\mathfrak{o})$. Clearly, this set remains the same if the base $\mathfrak{X}$ is changed. If we agree to call the elements of $\{T_i\}_{i\in I}$ *transvections in* $\mathrm{GL}_n(\mathfrak{o})$ *of order* $\subseteq \mathfrak{a}$, then we can state the following proposition.

PROPOSITION 1.3.1. *For any ideal* $\mathfrak{a} \subseteq \mathfrak{o}$, $\mathrm{SC}_n(M, \mathfrak{a})$ *and* $\mathrm{SC}_n(\mathfrak{a})$ *are the subgroups of* $\mathrm{GL}_n(M)$ *and* $\mathrm{GL}_n(\mathfrak{o})$, *respectively, generated by the transvections of order* $\subseteq \mathfrak{a}$.

Proof. See Klingenberg (**4**, p. 138) for $\mathrm{GL}_n(M)$.

When $n = 2$, the order of every transvection is a principal ideal since the transvections are the matrices $E_{12}(\lambda)$ together with their conjugates in $\mathrm{GL}_2(\mathfrak{o})$. $E_{21}(\lambda)$ is a transvection, being the conjugate of $E_{12}(\lambda)$ by $\Gamma(1, 1)$. We refer to the matrices $E_{ij}(\lambda)$ as *elementary transvections*.

For any n we can likewise consider the $E_{ij}(\lambda)$ as elementary transvections. This becomes clear when we see that a transvection of type E factors into

$$E = E_{1n}(\alpha_{1n}) \ldots E_{n-1,n}(\alpha_{n-1,n})$$

and that $E_{ij}(\lambda)$ is conjugate to $E_{in}(\lambda)$. We may consider elementary transvections in $\mathrm{GL}_n(M)$, *relative to a base*: they are the transformations which have the matrices $E_{ij}(\lambda)$ in that base.

A *line* of M is a one-dimensional direct summand of M. When $n = 2$, hyperplanes of M are lines.

LEMMA 1.3.2. $\mathrm{SL}_2(M)$ *is transitive on the lines of* M.

Proof. Given a line L in M and $\sigma \in \mathrm{SL}_2(M)$, σL is also a line of M. Now, is there a $\sigma \in \mathrm{SL}_2(M)$ which carries a given line $\mathfrak{o}x$ to a given line $\mathfrak{o}y$? We can write

$$M = \mathfrak{o}x \oplus \mathfrak{o}x_2, \qquad M = \mathfrak{o}y \oplus \mathfrak{o}y_2.$$

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The matrix expressing the change of base is in $\mathrm{GL}_2(\mathfrak{o})$; hence, if $y = \alpha x + \beta x_2$, at least one of α, β is a unit. Then σ is defined either by

$$\sigma x = \alpha x + \beta x_2, \qquad \sigma x_2 = \alpha^{-1} x_2,$$

or by

$$\sigma x = \alpha x + \beta x_2, \qquad \sigma x_2 = -\beta^{-1} x.$$

LEMMA 1.3.3. *Any transvection is conjugate in* $\mathrm{SL}_2(M)$, *or* $\mathrm{SL}_2(\mathfrak{o})$, *to some elementary transvection.*

Proof. Pick a base x_1, x_2 for M. If τ is a transvection with line L, there is a $\sigma \in \mathrm{SL}_2(M)$ such that $\sigma L = \mathfrak{o}x_1$ (by Lemma 1.3.2), so that $\sigma\tau\sigma^{-1}$ is a transvection with line $\mathfrak{o}x_1$, hence is elementary. Therefore, we have that

$$\tau = \sigma^{-1}(\sigma\tau\sigma^{-1})\sigma.$$

LEMMA 1.3.4. $\mathrm{SC}_2(M, \mathfrak{a})$ *and* $\mathrm{SC}_2(\mathfrak{a})$ *are the normal subgroups of* $\mathrm{SL}_2(M)$ *and* $\mathrm{SL}_2(\mathfrak{o})$, *respectively, generated by the elementary transvections of order* $\subseteq \mathfrak{a}$, *in fact, generated by either the transvections* $E_{12}(\lambda)$ *or* $E_{21}(\lambda)$, *where* $\lambda \in \mathfrak{a}$.

Proof. Note that $E_{21}(\lambda) = \Gamma E_{12}(-\lambda)\Gamma^{-1}$, where $\Gamma(-1, 1)$ is in $\mathrm{SL}_2(\mathfrak{o})$. The remainder is trivial, using Lemma 1.3.3.

PROPOSITION 1.3.5. *Let I be an index set and let $\{\tau_i\}_{i\in I}$ be a set of transvections of orders $\mathfrak{a}_i \subseteq \mathfrak{o}$. Let $\mathfrak{a}$ be the ideal generated by the $\mathfrak{a}_i$, $i \in I$. Then the normal subgroup of* $\mathrm{GL}_n(\mathfrak{o})$ $[\mathrm{GL}_n(M)]$ *which is generated by the set $\{\tau_i\}_{i\in I}$ is the group* $\mathrm{SC}_n(\mathfrak{a})$ $[\mathrm{SC}_n(M, \mathfrak{a})]$.

Proof. For $n \geqq 3$, the result is found in (**4**, p. 139); in this case it is sufficient to assume invariance under SL_n to get the conclusion.

Thus, we assume that $n = 2$. Consider a single transvection τ and let G be the normal subgroup of $\mathrm{GL}_2(\mathfrak{o})$ generated by τ. We may assume that $\tau = E_{12}(\lambda)$. Then $G \leqq \mathrm{SC}_2(\lambda\mathfrak{o})$. For each $\epsilon \in \mathfrak{u}$, form τ_Θ with $\Theta(\epsilon, 1)$ to obtain $E_{12}(\epsilon\lambda)$ in G. If $\zeta \in \mathfrak{p}$, then $-1 + \zeta \in \mathfrak{u}$, and we easily deduce that G contains $E_{12}(\gamma\lambda)\ \forall\ \gamma \in \mathfrak{o}$. Hence, by Lemma 1.3.4, $G \geqq \mathrm{SC}_2(\lambda\mathfrak{o})$.

Now let G be the normal subgroup of $\mathrm{GL}_2(\mathfrak{o})$ generated by $\{\tau_i\}_{i\in I}$. We have that $G \leqq \mathrm{SC}_2(\mathfrak{a})$. We wish to show that every transvection τ whose order $\subseteq \mathfrak{a}$ falls in G. We may assume that $\tau = E_{12}(\mu)$. By the first part of the proof,

$$\mathrm{SC}_2(\mathfrak{a}_i) \leqq G \quad \forall\ i \in I.$$

Since $\mu \in \mathfrak{a}$, there is an expression

$$\mu = \sum_{\mathrm{fin}} \mu_i, \qquad \mu_i \in \mathfrak{a}_i,$$

and we also have that

$$E_{12}(\mu_i) \in \mathrm{SC}_2(\mathfrak{a}_i) \leqq G,$$

so that

$$\tau = E_{12}(\mu) = \prod_{\mathrm{fin}} E_{12}(\mu_i) \in G.$$

When we step down to a homomorphic image of $\mathfrak{o}$, everything behaves naturally as expected. More precisely, if $\mathfrak{b} \subseteq \mathfrak{o}$ is an ideal and if the bar "$\overline{\quad}$" stands for both homomorphisms

$$g_{\mathfrak{b}}\colon \mathfrak{o} \to \mathfrak{o}/\mathfrak{b} \quad \text{and} \quad h_{\mathfrak{b}}\colon \mathrm{GL}_2(\mathfrak{o}) \to \mathrm{GL}_2(\mathfrak{o}/\mathfrak{b}),$$

then $o(\bar{\sigma}) = \overline{o(\sigma)}$ for elements, $o(\bar{G}) = \overline{o(G)}$ for subgroups; the local ring $\bar{\mathfrak{o}}$ has maximal ideal $\bar{\mathfrak{p}}$ and $N\mathfrak{p} = N\bar{\mathfrak{p}}$; for any ideal $\mathfrak{a} \subseteq \mathfrak{o}$,

$$\overline{\mathrm{SC}_2(\mathfrak{a})} = \mathrm{SC}_2(\bar{\mathfrak{a}}),$$

which follows from Proposition 1.3.1 (i.e., look at elementary generators).

PROPOSITION 1.3.6. *Let $\mathfrak{a} \subseteq \mathfrak{o}$ be an ideal. Then*

$$\mathrm{SC}_n(\mathfrak{a}) = [\mathrm{GL}_n(\mathfrak{o}), \mathrm{GC}_n(\mathfrak{a})]$$

but for one exception: when $n = 2$, $\mathfrak{o}/\mathfrak{p} = \mathbf{F}_2$, $\mathfrak{a} = \mathfrak{o}$, we have that

$$\mathrm{DGL}_2(\mathfrak{o}) < \mathrm{SL}_2(\mathfrak{o}).$$

The same holds for $\mathrm{GL}_n(M)$.

Proof. In view of Klingenberg (**4**, pp. 138–139), we only have to consider $n = 2$, $\mathfrak{o}/\mathfrak{p} = \mathbf{F}_2$.

(1) Suppose that $\mathfrak{a} = \mathfrak{o}$. Let $\overline{\quad}$ denote the homomorphism which factors by $\mathfrak{p}$. Then

$$\overline{\quad}\colon \mathrm{GL}_2(\mathfrak{o}) \to \mathrm{GL}_2(\mathfrak{o}/\mathfrak{p}) = \mathrm{GL}_2(\mathbf{F}_2) \cong S_3,$$

the symmetric group; see Artin (**1**, p. 170). Then

$$\overline{\mathrm{DGL}_2(\mathfrak{o})} = \mathrm{D}\overline{\mathrm{GL}_2(\mathfrak{o})} < \mathrm{GL}_2(\mathbf{F}_2) = \mathrm{SL}_2(\mathbf{F}_2) = \overline{\mathrm{SL}_2(\mathfrak{o})};$$

hence, we must have that $\mathrm{DGL}_2(\mathfrak{o}) < \mathrm{SL}_2(\mathfrak{o})$.

(2) Now let $\mathfrak{a} \subseteq \mathfrak{p}$. Write the mixed commutator group as $K(\mathfrak{a})$. We have that

$$K(\mathfrak{a}) \lhd \mathrm{GL}_2(\mathfrak{o}), \qquad K(\mathfrak{a}) \leqq \mathrm{SC}_2(\mathfrak{a}),$$

by looking at generators. If $\mathfrak{a} = \omega\mathfrak{o}$, then

$$(E_{12}(1), \Theta(1, 1 + \omega)) = E_{12}(\omega(1 + \omega)^{-1})$$

and belongs to $K(\omega\mathfrak{o})$; hence, by Proposition 1.3.5,

$$K(\omega\mathfrak{o}) = \mathrm{SC}_2(\omega\mathfrak{o}).$$

For an arbitrary $\mathfrak{a}$, pick a typical transvection τ of order $\subseteq \mathfrak{a}$. Since $o(\tau) = \omega\mathfrak{o} \subseteq \mathfrak{a}$, then

$$\tau \in \mathrm{SC}_2(\omega\mathfrak{o}) = K(\omega\mathfrak{o}) \leqq K(\mathfrak{a});$$

hence $\mathrm{SC}_2(\mathfrak{a}) \leqq K(\mathfrak{a})$, that is, $K(\mathfrak{a}) = \mathrm{SC}_2(\mathfrak{a})$.

We give some criteria for finding transvections in a group.

LEMMA 1.3.7. *Let* $G \leqq \mathrm{GL}_2(\mathfrak{o})$ *be invariant under* $\mathrm{SL}_2(\mathfrak{o})$.

(i) *If* G *contains* $\Xi(a, b, *)$ *or* $\Phi(a, b, *)$, *then* G *contains the transvection* $E_{12}(a - b)$.

(ii) *If* $N\mathfrak{p} > 2$ *and if* G *contains* $\Xi(*, *, \omega)$ *or* $\Phi(*, *, \omega)$, $\omega \in \mathfrak{p}$, *then* G *contains a (elementary) transvection of order* $\omega\mathfrak{o}$.

If in particular $G \lhd \mathrm{GL}_2(\mathfrak{o})$, *then* $G \geqq \mathrm{SC}_2((a - b)\mathfrak{o})$ *and* $G \geqq \mathrm{SC}_2(\omega\mathfrak{o})$, *respectively*.

Proof. The very last statement follows from Proposition 1.3.5. The problem for Φ is reduced to that for Ξ upon conjugation with $\Gamma(-1, 1) \in \mathrm{SL}_2(\mathfrak{o})$.

(i) The problem is solved immediately by taking $(E_{12}(-b), \Xi)$.

(ii) Write our Ξ as $\Xi(a, b, \omega)$. If $N\mathfrak{p} = 3$, then for the units a, b, one of $a + b$, $a - b$ is in $\mathfrak{u}$ while the other is in $\mathfrak{p}$. Use this fact and use step (i) after taking the following commutator in G: with $\alpha = 1 + \omega$, $\beta = (1 + \omega)^{-1}$, $\nu = -(1 + \beta)a$, we have that $(\Phi(\alpha, \beta, \nu), \Xi) = \Phi(x, y, *)$, where

$$x = (ab - \alpha\nu b\omega)\det \sigma^{-1}, \qquad y = (ab + \alpha\nu a\omega - \beta\nu\omega(a - b) + \nu^2\omega^2)\det \sigma^{-1}.$$

If $N\mathfrak{p} \geqq 4$, we pick $\gamma \in \mathfrak{u}$ such that

$$\gamma \not\equiv \pm 1 \bmod \mathfrak{p}, \quad \text{i.e., } \gamma^2 \not\equiv 1 \bmod \mathfrak{p}.$$

Taking $(\Xi, \Theta(\gamma, \gamma^{-1})) = E_{12}((1 - \gamma^2)b^{-1}\omega)$ yields the result.

LEMMA 1.3.8. *Let* $N\mathfrak{p} = 2$ *and* $G \lhd \mathrm{GL}_2(\mathfrak{o})$. *If* G *contains* $\Xi(*, *, \omega)$ *or* $\Phi(*, *, \omega)$, *then* $G \geqq \mathrm{SC}_2(\omega\mathfrak{p})$.

Proof. Again we only need to work with $\Xi = \Xi(*, b, \omega)$. For each $\zeta \in \mathfrak{p}$, we have that

$$(\Xi(*, b, \omega), \Theta(1 - b\zeta, 1)) = E_{12}(\zeta\omega),$$

which falls in G. Now apply Proposition 1.3.5.

1.4. *The problem of classification*. Given a subgroup $G \leqq \mathrm{GL}_n(\mathfrak{o})$, are there necessary and sufficient conditions for $G \lhd \mathrm{GL}_n(\mathfrak{o})$? To the group G there corresponds the ideal $o(G) \subseteq \mathfrak{o}$. Trivial considerations show that

$$\begin{aligned} G \text{ has order } 0 &\Leftrightarrow G \leqq \text{centre } \mathrm{GL}_n(\mathfrak{o}), \\ &\Leftrightarrow \mathrm{GC}_n(0) \geqq G \geqq \mathrm{SC}_n(0), \\ &\Leftrightarrow G \lhd \mathrm{GL}_n(\mathfrak{o}) \text{ with } o(G) = 0, \end{aligned}$$

which is the same situation as over fields. Klingenberg (**4**) has shown that

$$G \lhd \mathrm{GL}_n(\mathfrak{o}) \text{ with } o(G) = \mathfrak{a} \Leftrightarrow \mathrm{GC}_n(\mathfrak{a}) \geqq G \geqq \mathrm{SC}_n(\mathfrak{a}),$$

provided that $n \geqq 3$ or that $n = 2$ together with $2 \in \mathfrak{u}$, $N\mathfrak{p} > 3$, that is, $\mathrm{char}(\mathfrak{o}/\mathfrak{p}) \neq 2$ and $\mathfrak{o}/\mathfrak{p} \neq \mathbf{F}_3$. The implication "$\Leftarrow$" is easy, if we recall Proposition 1.3.6. For the implication "$\Rightarrow$", the step $\mathrm{GC}_n(\mathfrak{a}) \geqq G$ is trivial, and the hard part is to show that $G \geqq \mathrm{SC}_n(\mathfrak{a})$. In his situation, Klingenberg has done this by using only invariance over $\mathrm{SL}_n(\mathfrak{o})$, as it is done for fields. This provides a simultaneous classification of the normal subgroups of $\mathrm{SL}_n(\mathfrak{o})$.

We shall investigate the cases left open by Klingenberg. Again "$\Leftarrow$" is easy, by Proposition 1.3.6, and we shall show that $G \geqq \mathrm{SC}_2(\mathfrak{a})$ holds most of the time when $N\mathfrak{p} > 2$. But invariance under $\mathrm{SL}_2(\mathfrak{o})$ is too weak and special considerations (Property T) are made. When $N\mathfrak{p} = 2$, we discover many aberrations from the usual classification. In each instance of bad behaviour, we point out how far we have to step down before finding a congruence group in G, and these results are used to reduce hard cases to workable cases, in measuring the deviations.

2. The case $N\mathfrak{p} > 2$.

2.1. *Groups of order* $\mathfrak{o}$. When $N\mathfrak{p} > 2$, there are units $\epsilon \not\equiv 1 \bmod \mathfrak{p}$. If $2 \in \mathfrak{p}$, then $\epsilon^2 \not\equiv 1 \bmod \mathfrak{p}$, and so on.

THEOREM 2.1.1. *Let* $2 \in \mathfrak{p}$, $N\mathfrak{p} > 2$. *If* $G \leqq \mathrm{GL}_2(\mathfrak{o})$ *is invariant under* $\mathrm{SL}_2(\mathfrak{o})$, *with* $o(G) = \mathfrak{o}$, *then* $G \geqq \mathrm{SL}_2(\mathfrak{o})$.

Proof. In view of Lemma 1.2.3, it is enough to prove the theorem for $G \leqq \mathrm{SL}_2(\mathfrak{o})$. By Lemma 1.2.2, pick $\sigma \in G$ with $o(\sigma) = d\mathfrak{o} = \mathfrak{o}$. We may assume that $d \not\equiv 1 \bmod \mathfrak{p}$, for we could replace σ by σ_Θ, where $\Theta(\epsilon^{-1}, \epsilon) \in \mathrm{SL}_2(\mathfrak{o})$ has $\epsilon \not\equiv 1 \bmod \mathfrak{p}$, since σ_Θ has as 2, 1 entry, $\epsilon^2 d$. With the elements $\Gamma(-1, 1)$, $\tau = E_{12}(-ad^{-1})$, $\Theta(d, d^{-1})$ of $\mathrm{SL}_2(\mathfrak{o})$, we find that

$$\sigma_1 = \Gamma(\Theta, \sigma_\tau)\Gamma^{-1} = \Xi(d^{-2}, d^2, *) \in G.$$

Then for each $\alpha \in \mathfrak{o}$,

$$(\sigma_1, E_{12}(\alpha)) = E_{12}(\alpha(d^{-4} - 1)) \in G,$$

and this is an elementary transvection of order $\alpha\mathfrak{o}$. Hence, we can obtain enough transvections to apply Lemma 1.3.4.

We turn to groups of order $\mathfrak{o}$ when $N\mathfrak{p} = 3$.

LEMMA 2.1.2. *Let* $N\mathfrak{p} = 3$. *Let I be an index set and let* $\{\tau_i\}_{i\in I}$ *be a set of transvections of orders* $\mathfrak{a}_i$. *Let* $\mathfrak{a}$ *be the ideal generated by the* $\mathfrak{a}_i$, $i \in I$. *Then the normal subgroup of* $\mathrm{SL}_2(\mathfrak{o})$ *generated by* $\{\tau_i\}_{i\in I}$ *is* $\mathrm{SC}_2(\mathfrak{a})$.

Proof. Consider a single transvection τ and let G be the normal subgroup of $\mathrm{SL}_2(\mathfrak{o})$ generated by τ. By Lemma 1.3.3 we may assume that $\tau = E_{12}(\omega) \in G$. Its inverse $E_{12}(-\omega)$ is in G.

For each $\epsilon \in \mathfrak{u}$, we have that $E_{12}(\pm\epsilon^2\omega) \in G$, as seen by taking $(\tau^{\pm 1})_\Theta$ with $\Theta(\epsilon, \epsilon^{-1}) \in \mathrm{SL}_2(\mathfrak{o})$. Then, for $\gamma \in \mathfrak{u}$ with $\gamma \not\equiv 1 \bmod \mathfrak{p}$, we have that $E_{12}(\pm(\gamma - 1)^2\omega) \in G$, from which we obtain $E_{12}(\pm 2\gamma\omega) \in G$. Since 1 and -1 represent the two classes of units, this shows that $E_{12}(2\gamma\omega)$ is in G for all $\gamma \in \mathfrak{u}$. Since $2 \in \mathfrak{u}$, this means that $E_{12}(\gamma\omega) \in G$ for all $\gamma \in \mathfrak{u}$. If $\zeta \in \mathfrak{p}$, then $-1 + \zeta \in \mathfrak{u}$, and we easily arrive at $E_{12}(\zeta\omega) \in G$. An application of Lemma 1.3.4 shows that $G = \mathrm{SC}_2(\omega\mathfrak{o})$.

The remainder of the proof is just the same as in Proposition 1.3.5.

We consider the situation over the field $\mathbf{F}_3$, and subsequently the results over our local ring will be readily obtained.

LEMMA 2.1.3. $\mathrm{DSL}_2(\mathbf{F}_3) \lhd \mathrm{GL}_2(\mathbf{F}_3)$, $\mathrm{DSL}_2(\mathbf{F}_3) < \mathrm{SL}_2(\mathbf{F}_3)$, $o(\mathrm{DSL}_2(\mathbf{F}_3)) = \mathbf{F}_3$.

Proof. The order is $\mathbf{F}_3$ since $(E_{12}(1), E_{21}(2))$ has off-diagonal elements 1. Now, $\mathrm{PSL}_2(\mathbf{F}_3)$ is the alternating group A_4 (**1**, p. 170). If we had $\mathrm{DSL}_2(\mathbf{F}_3) = \mathrm{SL}_2(\mathbf{F}_3)$, then we would have

$$A_4 = \mathrm{PSL}_2(\mathbf{F}_3) = \mathrm{DPSL}_2(\mathbf{F}_3) = \mathrm{D}A_4,$$

which is not true.

PROPOSITION 2.1.4. *If* $G \leqq \mathrm{GL}_2(\mathbf{F}_3)$ *is invariant under* $\mathrm{SL}_2(\mathbf{F}_3)$, *with* $o(G) = \mathbf{F}_3$, *then either*

$$G \geqq \mathrm{SL}_2(\mathbf{F}_3) \quad \textit{or} \quad G \cap \mathrm{SL}_2(\mathbf{F}_3) = \mathrm{DSL}_2(\mathbf{F}_3),$$

which is a quaternion group, of index 3 *in* $\mathrm{SL}_2(\mathbf{F}_3)$.

Proof. In view of Lemma 1.2.3, we may assume that $G \leqq \mathrm{SL}_2(\mathbf{F}_3)$. By Lemma 1.2.2, pick $\sigma \in G$ with d as a unit. If necessary, we form σ_τ, where $\tau = E_{12}(-ad^{-1})$, so that we may take

$$\sigma = \begin{pmatrix} 0 & -d^{-1} \\ d & b \end{pmatrix} \in G.$$

Suppose that $b \neq 0$. Then, with $\Gamma(-1, 1) \in \mathrm{SL}_2(\mathbf{F}_3)$, we obtain $(\sigma_\Gamma \cdot \sigma)^2 = E_{12}(bd) \in G$; hence, $G = \mathrm{SL}_2(\mathbf{F}_3)$ by Lemma 2.1.2.

Suppose that $G < \mathrm{SL}_2(\mathbf{F}_3)$. Then b must be 0. If $d = 1$, we have the element $\Gamma(-1, 1) \in G$. The event $d = 2$ yields Γ^3, but since Γ^4 is the identity, we still have $\Gamma \in G$. The conjugate of Γ^3 in G by $E_{21}(1)$ gives us the element

$$\rho = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \in G.$$

But $\Gamma(-1, 1)$ and ρ generate a quaternion group Q, which is then a subgroup of G. Since $\mathrm{SL}_2(\mathbf{F}_3)$ has 24 elements, then by Lagrange's theorem,

$$G = Q = \mathrm{DSL}_2(\mathbf{F}_3).$$

The following lemma is the key fact required to reduce the situation from $\mathfrak{o}$ to $\mathbf{F}_3$ when $N\mathfrak{p} = 3$.

LEMMA 2.1.5. *Let* $N\mathfrak{p} = 3$ *and* $G \leqq \mathrm{GL}_2(\mathfrak{o})$ *be invariant under* $\mathrm{SL}_2(\mathfrak{o})$, *with* $o(G) = \mathfrak{o}$. *Then* $G \geqq \mathrm{SC}_2(\mathfrak{p})$.

Proof. As in the above proof, we take $G \leqq \mathrm{SL}_2(\mathfrak{o})$ and

$$\sigma = \begin{pmatrix} 0 & -d^{-1} \\ d & b \end{pmatrix} \in G.$$

Let ζ be an arbitrary element of $\mathfrak{p}$, put $\gamma = d^{-1} + \zeta$, a unit, and let

$$\rho = \begin{pmatrix} \gamma^{-1}bd^{-1} & \gamma \\ -\gamma^{-1} & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}).$$

Then $(\rho, \sigma) = \Xi(\gamma^2 d^2, \gamma^{-2}d^{-2}, *) \in G$ and Lemma 1.3.7 (i) gives us an elementary transvection of order $\zeta\mathfrak{o}$ in G. Now, Lemma 2.1.2 completes the proof.

THEOREM 2.1.6. *Let* $N\mathfrak{p} = 3$ *and* $G \leqq \mathrm{GL}_2(\mathfrak{o})$ *be invariant under* $\mathrm{SL}_2(\mathfrak{o})$, *with* $o(G) = \mathfrak{o}$. *Then either*

$$G \geqq \mathrm{SL}_2(\mathfrak{o}) \quad \text{or} \quad G \cap \mathrm{SL}_2(\mathfrak{o}) = \mathrm{DSL}_2(\mathfrak{o}),$$

which is of index 3 *in* $\mathrm{SL}_2(\mathfrak{o})$.

Proof. Reduce the problem to $G \leqq \mathrm{SL}_2(\mathfrak{o})$, by Lemma 1.2.3. Consider the map

$$h_\mathfrak{p}\colon \mathrm{GL}_2(\mathfrak{o}) \to \mathrm{GL}_2(\mathfrak{o}/\mathfrak{p}) = \mathrm{GL}_2(\mathbf{F}_3),$$

and let $\overline{}$ stand for the restriction of $h_\mathfrak{p}$ to $\mathrm{SL}_2(\mathfrak{o})$ and for the map

$$\overline{}: \mathfrak{o} \to \mathfrak{o}/\mathfrak{p} = \mathbf{F}_3.$$

The kernel of $\overline{}$ is $\mathrm{SC}_2(\mathfrak{p})$. In view of Lemma 2.1.3, we must have that $o(\mathrm{DSL}_2(\mathfrak{o})) = \mathfrak{o}$. Our above lemma yields

$$\mathrm{DSL}_2(\mathfrak{o}) \geqq \mathrm{SC}_2(\mathfrak{p}), \qquad G \geqq \mathrm{SC}_2(\mathfrak{p}).$$

Hence, our situation in

$$\mathrm{SL}_2(\mathfrak{o})/\mathrm{SC}_2(\mathfrak{p}) \cong \overline{\mathrm{SL}_2(\mathfrak{o})} = \mathrm{SL}_2(\mathbf{F}_3)$$

is that of Proposition 2.1.4 and a few logical manipulations prove our theorem.

Our above theorem does not have a converse, as is shown by the following example.

Example 2.1.7. The element $\phi = \Gamma(1, 1) \in \mathrm{GL}_2(\mathbf{F}_3)$ does not belong to $\mathrm{DSL}_2(\mathbf{F}_3)$ (consider $\det \phi$), and ϕ^2 is the identity; hence,

$$G = \mathrm{DSL}_2(\mathbf{F}_3) \cup \phi\, \mathrm{DSL}_2(\mathbf{F}_3)$$

is a subgroup of $\mathrm{GL}_2(\mathbf{F}_3)$ with

$$G \geqq \mathrm{DSL}_2(\mathbf{F}_3), \qquad o(G) = \mathbf{F}_3.$$

With $\tau = E_{12}(1)$, we find that

$$(\tau, \phi) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \notin \mathrm{DSL}_2(\mathbf{F}_3),$$

which is described by its generators in the proof of Proposition 2.1.4. Hence, $\phi_\tau \notin G$, and thus G is not invariant under $\mathrm{SL}_2(\mathbf{F}_3)$.

For a local ring with $N\mathfrak{p} = 3$, use

$$G = \mathrm{DSL}_2(\mathfrak{o}) \cup \phi\, \mathrm{DSL}_2(\mathfrak{o})$$

again, and show that it is not invariant under $\mathrm{SL}_2(\mathfrak{o})$ by factoring mod $\mathfrak{p}$, and thus obtaining the above situation over $\mathbf{F}_3$.

2.2. *Property* T. We have considered normal subgroups (in $\mathrm{GL}_2(\mathfrak{o})$ and in $\mathrm{SL}_2(\mathfrak{o})$) generated by transvections, and each time we have obtained a corresponding congruence group. When $2 \in \mathfrak{p}$, do we get $\mathrm{SC}_2(\mathfrak{a})$ as the normal subgroup of $\mathrm{SL}_2(\mathfrak{o})$ generated by a transvection of order $\mathfrak{a}$? Not always, as we will show.

Suppose that $2 \in \mathfrak{p}$. Let $\omega \in \mathfrak{p}$ with $\omega \notin \omega^2\mathfrak{o} + 2\mathfrak{o}$. We define $\mathfrak{z}(\omega)$ as the ideal of $\mathfrak{o}$ which is the inverse image of the ideal

$$\mathfrak{z}(\bar{\omega}) \subseteq \bar{\mathfrak{o}} = \mathfrak{o}/(\omega^2\mathfrak{o} + 2\mathfrak{o}),$$

consisting of the zero divisors of $\bar{\omega} \neq \bar{0}$. We have that

$$\omega^2\mathfrak{o} + 2\mathfrak{o} \subseteq \mathfrak{z}(\omega) \quad \text{and} \quad \omega\mathfrak{o} \subseteq \mathfrak{z}(\omega) \subseteq \mathfrak{p}.$$

PROPOSITION 2.2.1. *Suppose that* $2 \in \mathfrak{p}$. *Let* $\omega \in \mathfrak{p}$ *with* $\omega \notin \omega^2\mathfrak{o} + 2\mathfrak{o}$. *If there exists a unit* γ *such that*

$$\gamma \not\equiv \textit{a square} \bmod \mathfrak{z}(\omega),$$

then the normal subgroup G *of* $\mathrm{SL}_2(\mathfrak{o})$ *generated by the transvection* $E_{12}(\omega)$ *satisfies* $G < \mathrm{SC}_2(\omega\mathfrak{o})$.

Proof. Let $\overline{}$ denote the homomorphism which factors by $\omega^2\mathfrak{o} + 2\mathfrak{o}$. Then $\bar{\gamma} \not\equiv$ a square mod $\mathfrak{z}(\bar{\omega})$, and $\bar{G}$ is the normal subgroup of $\mathrm{SL}_2(\bar{\mathfrak{o}}) = \overline{\mathrm{SL}_2(\mathfrak{o})}$ generated by $E_{12}(\bar{\omega})$. With

$$\rho_i = \begin{pmatrix} \alpha_i & \mu_i \\ \nu_i & \beta_i \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}),$$

a typical element of $\bar{G}$ has the form

$$\prod_{i=1}^{r} \bar{\rho}_i E_{12}(\bar{\omega}) \bar{\rho}_i^{-1},$$

that is,

$$\begin{pmatrix} \bar{1} + (\bar{\alpha}_1\bar{\nu}_1 + \ldots + \bar{\alpha}_r\bar{\nu}_r)\bar{\omega} & (\bar{\alpha}_1^{\,2} + \ldots + \bar{\alpha}_r^{\,2})\bar{\omega} \\ (\bar{\nu}_1^{\,2} + \ldots + \bar{\nu}_r^{\,2})\bar{\omega} & \bar{1} + (\bar{\alpha}_1\bar{\nu}_1 + \ldots + \bar{\alpha}_r\bar{\nu}_r)\bar{\omega} \end{pmatrix}.$$

Since $\bar{2} = \bar{0}$ in $\bar{\mathfrak{o}}$, we cannot have $E_{12}(\bar{\gamma}\bar{\omega}) \in \bar{G}$, since this would contradict the property of $\bar{\gamma}$. Hence $\bar{G} < \mathrm{SC}_2(\bar{\omega}\bar{\mathfrak{o}})$, so that $G < \mathrm{SC}_2(\omega\mathfrak{o})$.

Example 2.2.2. Take a local field with ring of integers $\mathfrak{o}$, prime element π, and suppose that $2\mathfrak{o} \subseteq \mathfrak{p}^4$. Then $2\mathfrak{o} = \mathfrak{p}^k$ with $k \geqq 4$. There are integers m such that

$$2 \leqq m \leqq k/2.$$

Pick ω such that $\omega\mathfrak{o}$ is any one of the ideals $\mathfrak{p}^m$. We claim that the normal subgroup of $\mathrm{SL}_2(\mathfrak{o})$ generated by $E_{12}(\omega)$ is strictly contained in $\mathrm{SC}_2(\omega\mathfrak{o})$. Here, $\omega^2\mathfrak{o} + 2\mathfrak{o} = \mathfrak{p}^{2m}$ and $\pi \notin \mathfrak{z}(\omega)$; that is,

$$\mathfrak{z}(\omega) \subset \mathfrak{p}.$$

Now the unit $1 + \pi$ has quadratic defect $\mathfrak{p}$ as shown by O'Meara (**6**, Proposition 63.6); hence

$$1 + \pi \not\equiv \text{a square mod } \mathfrak{z}(\omega),$$

and Proposition 2.2.1 proves our claim.

It is now apparent that when $2 \in \mathfrak{p}$ we cannot classify the normal subgroups of $\mathrm{GL}_2(\mathfrak{o})$ by working only with invariance under $\mathrm{SL}_2(\mathfrak{o})$. We make the following definition.

Definition 2.2.3. *The local ring* $\mathfrak{o}$ *is said to have Property* T *if for any* $\omega \in \mathfrak{p}$ *the normal subgroup of* $\mathrm{SL}_2(\mathfrak{o})$ *generated by the transvection* $E_{12}(\omega)$ *is* $\mathrm{SC}_2(\omega\mathfrak{o})$.

Of course, Property T is quite strong. But instead of attacking the classification of the normal subgroups of $\mathrm{GL}_2(\mathfrak{o})$ in direct fashion, we give a unified treatment which will show that invariance under $\mathrm{SL}_2(\mathfrak{o})$ is enough if Property T holds. The ring of Example 2.2.2 does not have Property T, but there are interesting rings which do, as in the examples below.

Example 2.2.4. Any local ring with $2 \in \mathfrak{p}$, $N\mathfrak{p}$ finite, and $\mathfrak{p}^2 = 0$, has Property T, for instance, the local rings $\mathbf{Z}/4\mathbf{Z}$ and $\mathbf{F}_4[x]/(x^2)$.

For $\delta \in \mathfrak{u}$, conjugate $E_{12}(\omega)$ by $\Theta(\delta, \delta^{-1})$ to obtain $E_{12}(\delta^2\omega)$. Using $\mathfrak{p}^2 = 0$ and perfectness of the residue class field we see that $E_{12}(\gamma\omega)$ can be reached for all $\gamma \in \mathfrak{o}$; hence Lemma 1.3.4 yields the result.

Example 2.2.5. A less trivial example is a local ring with finite $N\mathfrak{p} > 2$ and with $\mathfrak{p} = 2\mathfrak{o}$; an illustration is the valuation ring of $\mathbf{Q}(\sqrt{5})$ under the prolongated 2-adic valuation from $\mathbf{Q}$; indeed, $\mathfrak{p} = 2\mathfrak{o}$ and $N\mathfrak{p} = 4$.

Let $\omega \in \mathfrak{p}$ and let G be the normal subgroup of $\mathrm{SL}_2(\mathfrak{o})$ generated by $E_{12}(\omega)$. Proceed in a way somewhat similar to the proof of Lemma 2.1.2 to show that

$$G \geqq \mathrm{SC}_2(2\omega\mathfrak{o}), \quad \text{i.e., } G \geqq \mathrm{SC}_2(\omega\mathfrak{p}).$$

By perfectness of $\mathfrak{o}/\mathfrak{p}$, every unit γ has the form

$$\gamma = \delta^2 + 2\alpha, \quad \text{for some } \delta \in \mathfrak{u}, \alpha \in \mathfrak{o};$$

hence $E_{12}(\gamma\omega) = E_{12}(\delta^2\omega)E_{12}(2\alpha\omega) \in G$. Now Lemma 1.3.4 completes the proof.

It will be convenient to refer to the following remark.

Remark 2.2.6. (i) Suppose that $N\mathfrak{p} = 3$ or that $2 \in \mathfrak{p}$, $N\mathfrak{p} > 2$, Property T holds. Let I be an index set and let $\{\tau_i\}_{i\in I}$ be a set of transvections of orders $\mathfrak{a}_i$.

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Let $\mathfrak{a}$ be the ideal generated by the $\mathfrak{a}_i$, $i \in I$. Then the normal subgroup of $\mathrm{SL}_2(\mathfrak{o})$ generated by $\{\tau_i\}_{i \in I}$ is $\mathrm{SC}_2(\mathfrak{a})$.

This is a combination of Proposition 1.3.5 and Lemma 2.1.2.

(ii) Suppose that $N\mathfrak{p} = 3$ or that $2 \in \mathfrak{p}$, $N\mathfrak{p} > 2$, Property T holds. Let $G \leqq \mathrm{GL}_2(\mathfrak{o})$ be invariant under $\mathrm{SL}_2(\mathfrak{o})$. Then: (a) if G contains $\Xi(a, b, *)$ or $\Phi(a, b, *)$, then $G \geqq \mathrm{SC}_2((a-b)\mathfrak{o})$; (b) if G contains $\Xi(*, *, \omega)$ or $\Phi(*, *, \omega)$, where $\omega \in \mathfrak{p}$, then $G \geqq \mathrm{SC}_2(\omega\mathfrak{o})$.

This is a restatement of Lemma 1.3.7.

Moreover, the type of ring which we consider here is preserved under homomorphism.

2.3. The Main Theorem. We consider ideals $\mathfrak{a} \subseteq \mathfrak{p}$ and the solution of the classification problem is reached by successively taking $\mathfrak{a}$ with $a\mathfrak{p} = 0$, then principal, then finitely generated, and at last arbitrary.

Lemma 2.3.1. *Suppose that $N\mathfrak{p} = 3$ or that $2 \in \mathfrak{p}$, $N\mathfrak{p} > 2$, Property* T *holds. Let $G \leqq \mathrm{GL}_2(\mathfrak{o})$ be invariant under $\mathrm{SL}_2(\mathfrak{o})$, with $o(G) = \mathfrak{a} \subseteq \mathfrak{p}$. If $\mathfrak{a}\mathfrak{p} = 0$, then $G \geqq \mathrm{SC}_2(\mathfrak{a})$.*

Proof. Fix a typical element $\sigma \in G$. We shall show that G contains $\mathrm{SC}_2((a-b)\mathfrak{o})$, $\mathrm{SC}_2(c\mathfrak{o})$, and $\mathrm{SC}_2(d\mathfrak{o})$. If $d = 0$, then $\mathrm{SC}_2(d\mathfrak{o}) \leqq G$ trivially, and the same for $c = 0$ and $a - b = 0$.

(1) Let $d \neq 0$. If $N\mathfrak{p} = 3$, apply Remark 2.2.6 (ii) to

$$(E_{12}(1), \sigma) = \Xi(ab + ad, ab - ad, *)(\det \sigma)^{-1} \in G$$

to obtain $\mathrm{SC}_2(d\mathfrak{o}) \leqq G$.

When $2 \in \mathfrak{p}$, we can assume our σ has $a = b$, for if not, we just replace it by

$$(\Theta(\beta^{-1}, \beta), \sigma) = \begin{bmatrix} ab & a(\beta^{-2} - 1)c \\ b(\beta^2 - 1)d & ab \end{bmatrix} \cdot (\det \sigma)^{-1} \in G,$$

where $\beta \in \mathfrak{u}$ with $\beta \not\equiv 1 \bmod \mathfrak{p}$. Then $(E_{12}(1), \sigma) = \Xi(*, *, -ad) \in G$, and Remark 2.2.6 (ii) yields $\mathrm{SC}_2(d\mathfrak{o}) \leqq G$.

(2) If $c \neq 0$, reduce the situation to step (1) by forming σ_Γ with $\Gamma(-1, 1)$. If $a - b \neq 0$, then we can make d "become" 0 as follows: if $d \neq 0$, then by step (1),

$$E_{21}(-a^{-1}d) \in \mathrm{SC}_2(d\mathfrak{o}) \leqq G;$$

hence $E_{21}(-a^{-1}d) \cdot \sigma = \Xi(a, b, *) \in G$; then Remark 2.2.6 (ii) applies.

(3) For every $\mu \in o(\sigma)$ we have that $E_{12}(\mu) \in G$ since

$$\mu = \alpha(a - b) + \beta c + \gamma d \quad \text{for some } \alpha, \beta, \gamma \in \mathfrak{o},$$

and since the product

$$E_{12}(\alpha(a - b)) \cdot E_{12}(\beta c) \cdot E_{12}(\gamma d)$$

falls in G. For each $\mu \in \mathfrak{a}$, we proceed as in Proposition 1.3.5 to obtain $E_{12}(\mu) \in G$. Then Lemma 1.3.4 can be applied.

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Formula 2.3.2. Let σ have $c, d \in \mathfrak{p}$ and $\det \sigma = 1$ and let $\Xi = \Xi(\alpha, \beta, \mu)$, where

$$\beta = 1 - b^{-1}d \in \mathfrak{u}, \quad \alpha = \beta^{-1}, \quad \mu = 1 + \alpha.$$

Thus $\Xi \in \mathrm{SL}_2(\mathfrak{o})$, $\alpha = 1 + \epsilon d$, some $\epsilon \in \mathfrak{u}$, and $\mu = 2 + \epsilon d$. And (Ξ, σ) has

$$\begin{aligned} 1, 2 \text{ entry } &= ac(\alpha^2 - 1) - \alpha\mu a(a - b) - \mu d(\beta c + \mu a), \\ 2, 1 \text{ entry } &= 0. \end{aligned}$$

LEMMA 2.3.3. *Suppose that* $2 \in \mathfrak{p}$, $N\mathfrak{p} > 2$, *Property* T *holds. Let* $G \leqq \mathrm{GL}_2(\mathfrak{o})$ *be invariant under* $\mathrm{SL}_2(\mathfrak{o})$, *with* $o(G) = \omega\mathfrak{o} \subseteq \mathfrak{p}$. *Then for any given* $m \geqq 1$, *G contains elements*

$$\begin{pmatrix} 1 + x & c + z \\ w & 1 + y \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 + x & z \\ d + w & 1 + y \end{pmatrix},$$

with x, y, z, *and* w *in* $2^m\omega\mathfrak{o}$ *and* c *and* d *any pre-assigned elements of* $\omega\mathfrak{o}$.

Proof. We may assume that $G \leqq \mathrm{SC}_2(\omega\mathfrak{o})$ by Lemma 1.2.3. Then an element $\sigma \in G$ has

$$a = 1 + x, \quad b = 1 + y, \qquad \text{with } x, y \in \omega\mathfrak{o}.$$

(1) We first establish the following fact: *if* $\omega^m = 0$ *for some* $m \geqq 1$, *then* $G \geqq \mathrm{SC}_2(\omega\mathfrak{o})$.

Suppose that $\omega^2 = 0$. If $\omega\mathfrak{p} \neq 0$, let $\overline{}$ denote the homomorphism which factors by $\omega\mathfrak{p}$, and apply Lemma 2.3.1 to obtain $E_{21}(\bar{\omega}) \in \bar{G}$. Pull back to get an inverse image $\sigma \in G$ with $x, y \in \omega\mathfrak{p}$ and $d\mathfrak{o} = \omega\mathfrak{o}$. Then

$$(E_{12}(1), \sigma) = \Xi(*, *, \eta d) \in G \quad \text{for some } \eta \in \mathfrak{u};$$

hence, $G \geqq \mathrm{SC}_2(\omega\mathfrak{o})$ by Remark 2.2.6 (ii).

Suppose that $\omega^3 = 0$, $2\omega^2 = 0$. If $\omega^2 \neq 0$, factor by $\omega^2\mathfrak{o}$; hence, the above case yields $E_{12}(\bar{\omega}) \in \bar{G}$, and hence we have a $\sigma \in G$ with $x, y, d \in \omega^2\mathfrak{o}$ and $c\mathfrak{o} = \omega\mathfrak{o}$. We force d to be 0 as follows: we form

$$(E_{12}(1), \sigma) = \Xi(*, *, \eta d) \in G,$$

where $\eta \in \mathfrak{u}$, as given by combining $\det \sigma = 1$ and $2\omega^2 = 0$; then Remark 2.2.6 (ii) yields $E_{21}(-a^{-1}d) \in \mathrm{SC}_2(d\mathfrak{o}) \leqq G$. Then d becomes 0 in

$$E_{21}(-a^{-1}d) \cdot \sigma = \Xi(*, *, c) \in G,$$

and Remark 2.2.6 (ii) yields $G \geqq \mathrm{SC}_2(\omega\mathfrak{o})$.

When $\omega^3 = 0$ and $2\omega^2 \neq 0$, factor by $2\omega^2\mathfrak{o}$ and apply the above case to obtain

$$\begin{pmatrix} \bar{1} & \bar{\omega} \\ \bar{\omega}^2 & \bar{1} \end{pmatrix} \in \bar{G}.$$

Pull back to G and do as above, where $\sigma \in G$ now has $x, y \in 2\omega^2\mathfrak{o}$, $d\mathfrak{o} = \omega^2\mathfrak{o}$, and $c\mathfrak{o} = \omega\mathfrak{o}$.

Now we proceed by induction to m, with $m \geqq 4$. If $\omega^{m-1} \neq 0$, factor by $\omega^{m-1}\mathfrak{o}$ and use the induction hypothesis to obtain

$$\begin{pmatrix} \bar{1} & \bar{\omega} \\ \bar{\omega}^{m-2} & \bar{1} \end{pmatrix} \in \bar{G}.$$

Pull back to get $\sigma \in G$ with $x, y \in \omega^{m-1}\mathfrak{o}$, $d\mathfrak{o} = \omega^{m-2}\mathfrak{o}$, and $c\mathfrak{o} = \omega\mathfrak{o}$. Again, force d to be 0, to reach $G \geqq \mathrm{SC}_2(\omega\mathfrak{o})$. In forming the commutator $(E_{12}(1), \sigma)$ we get d^2 for the 2, 1 entry and $d^2 \in (\omega^{m-2})^2\mathfrak{o} \subseteq \omega^m\mathfrak{o} = 0$ only if $m \geqq 4$, which explains why $m = 3$ was treated separately.

(2) We easily deduce that for any given $m \geqq 1$, G contains elements

$$\begin{pmatrix} 1+x & c+z \\ w & 1+y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1+x & z \\ d+w & 1+y \end{pmatrix},$$

with x, y, z, and w in $\omega^m\mathfrak{o}$ and c and d any pre-assigned elements of $\omega\mathfrak{o}$. For, fixing m and factoring by $\omega^m\mathfrak{o}$ if $\omega^m \neq 0$, we obtain, from step (1),

$$E_{12}(\bar{c}),\, E_{21}(\bar{d}) \in \mathrm{SC}_2(\overline{\omega\mathfrak{o}}) \leqq \bar{G},$$

and the assertion follows by pulling back to G.

(3) We are now able to prove that *if* $2^m\omega = 0$ *for some* $m \geqq 1$, *then* $G \geqq \mathrm{SC}_2(\omega\mathfrak{o})$. This immediately proves our lemma, with arguments as in step (2).

Suppose that $2\omega = 0$, i.e., $m = 1$. By steps (1) and (2), respectively, we may assume that $\omega^3 \neq 0$ and can pick

$$\sigma = \begin{pmatrix} 1+x & c \\ d & 1+y \end{pmatrix} \in G$$

with $x, y, c \in \omega^4\mathfrak{o}$ and $d\mathfrak{o} = \omega\mathfrak{o}$. Then Formula 2.3.2 gives us an element

$$\Xi(*, *, \eta\omega^3) \in G \quad \text{for some } \eta \in \mathfrak{u};$$

hence, $\mathrm{SC}_2(\omega^3\mathfrak{o}) \leqq G$ by Remark 2.2.6 (ii). Re-use step (2) to pick

$$\phi = \begin{pmatrix} a & c \\ d & b \end{pmatrix} \in G,$$

where $d \in \omega^3\mathfrak{o}$ and $c\mathfrak{o} = \omega\mathfrak{o}$. Since

$$E_{21}(-a^{-1}d) \in \mathrm{SC}_2(\omega^3\mathfrak{o}) \leqq G,$$

then

$$E_{21}(-a^{-1}d) \cdot \phi = \Xi(*, *, c) \in G,$$

and Remark 2.2.6 (ii) shows that $G \geqq \mathrm{SC}_2(\omega\mathfrak{o})$.

The proof is completed using induction on m. We factor by $2^{m-1}\omega\mathfrak{o}$ and pull back to G the element $E_{12}(\bar{\omega}) \in \bar{G}$, and the details are familiar.

We have now enough tools to complete the main theorem in the case of principal ideals.

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PROPOSITION 2.3.4. *Suppose that* $N\mathfrak{p} = 3$ *or that* $2 \in \mathfrak{p}$, $N\mathfrak{p} > 2$, *Property* T *holds. Let* $G \leqq \mathrm{GL}_2(\mathfrak{o})$ *be invariant under* $\mathrm{SL}_2(\mathfrak{o})$, *with* $o(G) = \omega\mathfrak{o} \subseteq \mathfrak{p}$. *Then* $G \geqq \mathrm{SC}_2(\omega\mathfrak{o})$.

Proof. We may assume that $G \leqq \mathrm{SC}_2(\omega\mathfrak{o})$, by Lemma 1.2.3.

If $N\mathfrak{p} = 3$, factor by $\omega\mathfrak{p}$, use Lemma 2.3.1 to get $E_{21}(\bar{\omega}) \in \bar{G}$, pull back to G, and use Formula 2.3.2 and Remark 2.2.6; the details are the usual ones.

When $2 \in \mathfrak{p}$, we use Lemma 2.3.3 to pick

$$\sigma = \begin{pmatrix} 1+x & c \\ d & 1+y \end{pmatrix} \in G,$$

with $x, y, c \in 2^4\omega\mathfrak{o}$ and $d\mathfrak{o} = 2\omega\mathfrak{o}$. The remainder of the proof consists of obtaining $\mathrm{SC}_2(2^3\omega\mathfrak{o}) \leqq G$ and re-using Lemma 2.3.3 to pick

$$\phi = \begin{pmatrix} a & c \\ d & b \end{pmatrix} \in G,$$

with $d \in 2^3\omega\mathfrak{o}$ and $c\mathfrak{o} = \omega\mathfrak{o}$, with manipulations as in step (3) of Lemma 2.3.3.

COROLLARY 2.3.4a. *Let* $\mathfrak{o}$ *be as in the proposition and* G *be invariant under* $\mathrm{SL}_2(\mathfrak{o})$. *If* G *contains an element of order* $\omega\mathfrak{o} \subseteq \mathfrak{p}$, *then* $G \geqq \mathrm{SC}_2(\omega\mathfrak{o})$.

Proof. Let $\sigma \in G$ have $o(\sigma) = \omega\mathfrak{o}$ and let $G(\sigma)$ be the subgroup, invariant under $\mathrm{SL}_2(\mathfrak{o})$, generated by σ. Then the proposition yields

$$\mathrm{SC}_2(\omega\mathfrak{o}) \leqq G(\sigma) \leqq G.$$

COROLLARY 2.3.4b. *Let* $\mathfrak{o}$ *and* G *be as above, with* $o(G) = \mathfrak{a} \subseteq \mathfrak{p}$, *where* $\mathfrak{a}$ *is a finitely generated ideal. Then* $G \geqq \mathrm{SC}_2(\mathfrak{a})$.

Proof. By the proposition, we may assume that $\mathfrak{a}$ has $r \geqq 2$ generators $a_1, \ldots, a_r$.

(1) We first prove that if there is some s, $2 \leqq s \leqq r$, such that $a_i\mathfrak{p} = 0$ for $i = s, \ldots, r$, then $G \geqq \mathrm{SC}_2(\mathfrak{a})$.

We start with $s = 2$. Factor by $a_1\mathfrak{p}$, apply Lemma 2.3.1 to find $E_{12}(\bar{a}_1) \in \bar{G}$ and pull back to obtain

$$\sigma = \begin{pmatrix} 1+x & a_1+z \\ w & 1+y \end{pmatrix} \in G,$$

with $x, y, z, w \in a_1\mathfrak{p}$. Then $o(\sigma) = a_1\mathfrak{o}$; hence, the above corollary yields $\mathrm{SC}_2(a_1\mathfrak{o}) \leqq G$. For $i = 2, \ldots, r$, we also find elements

$$\tau_i = \begin{pmatrix} 1+x_i & a_i+z_i \\ w_i & 1+y_i \end{pmatrix} \in G,$$

where $x_i, y_i, z_i, w_i \in a_1\mathfrak{p}$. Then

$$E_{21}(-(1+x_i)^{-1}w_i),\ E_{12}(-z_i) \in \mathrm{SC}_2(a_1\mathfrak{o}) \leqq G.$$

Now $E_{21}(-(1+x_i)^{-1}w_i)$. $\tau_i = \Xi(*, *, a_i + z_i) \in G$, so that Remark 2.2.6 (ii) yields $E_{12}(a_i + z_i) \in \mathrm{SC}_2((a_i + z_i)\mathfrak{o}) \leqq G$. Then

$$E_{12}(a_i) = E_{12}(-z_i)E_{12}(a_i + z_i) \in G,$$

and we have enough transvections in G (see Remark 2.2.6 (i)) to obtain

$$G \geqq \mathrm{SC}_2(\mathfrak{a}).$$

For $s > 2$, proceed by induction in the same way (i.e., factor by $a_{s-1}\mathfrak{p}$), using the induction hypothesis instead of Lemma 2.3.1.

(2) Now, if all $a_i\mathfrak{p} \neq 0$, factor by $a_r\mathfrak{p}$, apply step (1), and pull back to G using similar arguments all along.

THEOREM 2.3.5. *Suppose that* $N\mathfrak{p} = 3$ *or that* $2 \in \mathfrak{p}$, $N\mathfrak{p} > 2$, *Property* T *holds. Let* $G \leqq \mathrm{GL}_2(\mathfrak{o})$ *be invariant under* $\mathrm{SL}_2(\mathfrak{o})$, *with* $o(G) = \mathfrak{a} \subseteq \mathfrak{p}$. *Then* $G \geqq \mathrm{SC}_2(\mathfrak{a})$.

Proof. Fix $\mu \in \mathfrak{a}$. Since $\mathfrak{a}$ is generated by the ideals $o(\sigma)$, for all $\sigma \in G$, there is an expression

$$\mu = \mu_1 + \ldots + \mu_r,$$

where $\mu_i \in o(\sigma_i)$, $i = 1, \ldots, r$.

Let G' be the subgroup, invariant under $\mathrm{SL}_2(\mathfrak{o})$, generated by $\sigma_1, \ldots, \sigma_r$. We have that

$$G' \leqq G, \qquad o(G') = \mathfrak{b} = o(\sigma_1) + \ldots + o(\sigma_r).$$

Since each $o(\sigma_i)$ is finitely generated, so is $\mathfrak{b}$. Then Corollary 2.3.4b provides us with

$$E_{12}(\mu) \in \mathrm{SC}_2(\mathfrak{b}) \leqq G' \leqq G,$$

and the theorem follows by Lemma 1.3.4, since we have enough transvections.

THEOREM 2.3.6. *Suppose that* $2 \in \mathfrak{p}$ *and* $N\mathfrak{p} > 2$. *Let* $G \lhd \mathrm{GL}_2(\mathfrak{o})$, *with* $o(G) = \mathfrak{a} \subseteq \mathfrak{p}$. *Then* $G \geqq \mathrm{SC}_2(\mathfrak{a})$.

Proof. Starting with Lemma 2.3.1, one can obtain, for each result listed, a corollary for the situation where $2 \in \mathfrak{p}$, $N\mathfrak{p} > 2$ and $G \lhd \mathrm{GL}_2(\mathfrak{o})$, with very simple changes in the proofs. Then follow the proof of the above theorem, using for G' the normal subgroup of $\mathrm{GL}_2(\mathfrak{o})$ generated by $\sigma_1, \ldots, \sigma_r$.

Let us recapitulate the essential facts which emerge from our discussion and from the particular case of $2 \in \mathfrak{u}$ and $N\mathfrak{p} \neq 3$ which was done in (**4**).

FIRST MAIN THEOREM 2.3.7. *Let* $\mathfrak{o}$ *be a local ring with* $N\mathfrak{p} > 2$ *and let* G *be a subgroup of* $\mathrm{GL}_2(\mathfrak{o})$. *Then*

$$G \lhd \mathrm{GL}_2(\mathfrak{o}) \textit{ with } \quad o(G) = \mathfrak{a} \Leftrightarrow \mathrm{GC}_2(\mathfrak{a}) \geqq G \geqq \mathrm{SC}_2(\mathfrak{a}),$$

but for one exception: when $N\mathfrak{p} = 3$ *and* $o(G) = \mathfrak{o}$, *the ladder relation at* $\mathfrak{o}$ *is not necessarily satisfied by* $G \lhd \mathrm{GL}_2(\mathfrak{o})$. *An example is* $\mathrm{DSL}_2(\mathfrak{o}) < \mathrm{SL}_2(\mathfrak{o})$.

3. The case $N\mathfrak{p} = 2$. We shall encounter unusual behaviour in this case and our aim will not be to seek a general solution. It is rather to show how these particular problems arise and how certain answers can be formulated. Thus, we shall specify our local ring considerably. From now on, $\mathfrak{o}$ will be a *local ring with principal maximal ideal* $\mathfrak{p} = \pi\mathfrak{o}$ *and with* $N\mathfrak{p} = 2$, *that is*, $\mathfrak{o}/\mathfrak{p} = \mathbf{F}_2$. Every proper ideal of $\mathfrak{o}$ which is not contained in $\bigcap_{i=1}^{\infty}\mathfrak{p}^i$ is of the form $\mathfrak{p}^n$ and we shall only consider such ideals. Our ring retains some interesting cases from number theory; examples are the 2-adic integers of $\mathbf{Q}$ and of $\mathbf{Q}_2$.

We have $2 \in \mathfrak{p}$ and $\epsilon \equiv 1 \bmod \mathfrak{p}$, $\delta \pm \epsilon \in \mathfrak{p}$, for all ϵ, δ in $\mathfrak{u}$. If α and β both generate $\mathfrak{p}^n$, then $\alpha \pm \beta$ is in $\mathfrak{p}^{n+1}$. We also have the following:

(i) If $\mathfrak{p}^n \supset \mathfrak{p}^{n+1}$, then $(\mathfrak{p}^n : \mathfrak{p}^{n+1}) = N\mathfrak{p} = 2$;

(ii) If $\mathfrak{p} \supset \mathfrak{p}^2 \supset \ldots \supset \mathfrak{p}^n$, then $(\mathfrak{o} : \mathfrak{p}^n) = (N\mathfrak{p})^n = 2^n$;

(iii) If $\mathfrak{p}^n = 0$ and $\mathfrak{p}^{n-1} \neq 0$, where $n \geqq 2$, then $\mathfrak{o}$ is finite with $(N\mathfrak{p})^n = 2^n$ elements and $\mathfrak{p}^{n-1} = \{0, \pi^{n-1}\}$.

An element of $\mathrm{GL}_2(\mathfrak{o})$ cannot have all entries in $\mathfrak{u}$, hence there can only be six types of elements in $\mathrm{GL}_2(\mathfrak{o})$. For $G \lhd \mathrm{GL}_2(\mathfrak{o})$, with $o(G) = \mathfrak{o}$ or $o(G) = \mathfrak{p}^n$, we can always pick an element $\sigma \in G$ with $d \in \mathfrak{u}$ or $d\mathfrak{o} = \mathfrak{p}^n$, respectively, by Lemma 1.2.2.

3.1. Basic Lemmas. We devote a brief section to useful observations before studying the normal subgroups of $\mathrm{GL}_2(\mathfrak{o})$.

The groups $\mathrm{DGL}_2(\mathfrak{o})$ and $\mathrm{DSL}_2(\mathfrak{o})$ both have order $\mathfrak{o}$, as we can see by forming the commutator $(E_{12}(1), E_{21}(1))$, which has 1 as its 2, 1 entry.

Lemma 3.1.1. *Let* $G \lhd \mathrm{GL}_2(\mathfrak{o})$ *with* $o(G) = \mathfrak{o}$. *Then:*

(i) $G \cap \mathrm{DGL}_2(\mathfrak{o}) \lhd \mathrm{GL}_2(\mathfrak{o})$, $o(G \cap \mathrm{DGL}_2(\mathfrak{o})) = \mathfrak{o}$;

(ii) *G contains the two types of elements given by*

$$\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & b \end{pmatrix}.$$

Proof. (i) Pick $\sigma \in G$ with $d \in \mathfrak{u}$. Then $(E_{12}(1), \sigma)$ has as 2, 1 entry $d^2(\det \sigma)^{-1} \in \mathfrak{u}$.

(ii) Pick $\sigma \in G \cap \mathrm{DGL}_2(\mathfrak{o})$ with $d \in \mathfrak{u}$. Then σ_Ξ is one of the required elements, where $\Xi(d, 1, -a)$ is used. The other element is simply $(\sigma_\Xi)_\Theta$, with $\Theta(1, -1)$.

Formula 3.1.2. If $\sigma \in \mathrm{GL}_2(\mathfrak{o})$ with $b \in \mathfrak{u}$ and if $\Xi(1, 1 - \mu b^{-1}d, \mu) \in \mathrm{GL}_2(\mathfrak{o})$, that is, μd must be in $\mathfrak{p}$, then

$$(\Xi, \sigma) = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \cdot (\det \Xi\sigma)^{-1},$$

where

$$A - B = (1 - \mu b^{-1}d)[\mu d(a + b) + cd(1 - \mu b^{-1}d)] + d[\mu(a - b) - c + \mu^2 d],$$

$$C = \mu cd[ab^{-1} - 1 + \mu b^{-1}d] - \mu a(a - b) - \mu^2 ad.$$

LEMMA 3.1.3. *Let* $\mathfrak{p}^n \neq 0$ *and* $\mathfrak{p}^{n+1} = 0$, $n \geqq 1$. *Then*

$$SC_2(\mathfrak{p}^n) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2,$$

which is a group of order 8.

Proof. Examine an arbitrary element of $SC_2(\mathfrak{p}^n)$; we can find only eight possibilities. These elements all have order 2; hence, the result follows; see (**5**, p. 51).

COROLLARY 3.1.3a. *Suppose that* $\mathfrak{p}^n \supset \mathfrak{p}^{n+1}$, $n \geqq 1$. *Then*

$$SC_2(\mathfrak{p}^n)/SC_2(\mathfrak{p}^{n+1}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2,$$

which is a group of order 8.

Proof. If $\mathfrak{p}^{n+1} \neq 0$, let $\overline{}$ be the homomorphism which factors by $\mathfrak{p}^{n+1}$. The kernel of $\overline{}$ restricted to $SC_2(\mathfrak{p}^n)$ is $SC_2(\mathfrak{p}^{n+1})$. Hence

$$SC_2(\mathfrak{p}^n)/SC_2(\mathfrak{p}^{n+1}) \cong \overline{SC_2(\mathfrak{p}^n)} = SC_2(\bar{\mathfrak{p}}^n) \cong \mathbf{Z}_2 \otimes \mathbf{Z}_2 \otimes \mathbf{Z}_2.$$

Among the eight elements of $SC_2(\mathfrak{p}^n)$, where $\mathfrak{p}^n \neq 0$ and $\mathfrak{p}^{n+1} = 0$, are the following important ones:

$$\phi_1 = \begin{pmatrix} 1 & \pi^n \\ \pi^n & 1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 1+\pi^n & \pi^n \\ 0 & 1+\pi^n \end{pmatrix}, \quad \phi_3 = \begin{pmatrix} 1+\pi^n & 0 \\ \pi^n & 1+\pi^n \end{pmatrix}.$$

LEMMA 3.1.4. *Let* $\mathfrak{p}^n \neq 0$ *and* $\mathfrak{p}^{n+1} = 0$, $n \geqq 1$. *Then*

$$[GL_2(\mathfrak{o}), SC_2(\mathfrak{p}^n)] = [SL_2(\mathfrak{o}), SC_2(\mathfrak{p}^n)] = \{1, \phi_1, \phi_2, \phi_3\},$$

which is a Klein group, and is normal in $GL_2(\mathfrak{o})$.

Proof. Let σ be a typical element of $SC_2(\mathfrak{p}^n)$ and let

$$\rho = \begin{pmatrix} \alpha & \mu \\ \nu & \beta \end{pmatrix}$$

be a typical element of $GL_2(\mathfrak{o})$. Then (ρ, σ) is a typical generator of $[GL_2(\mathfrak{o}), SC_2(\mathfrak{p}^n)]$ and

$$(\rho, \sigma) = \begin{pmatrix} 1+X & C \\ D & 1+X \end{pmatrix},$$

where $X = \alpha\nu c + \beta\mu d$, $C = (\alpha^2+1)c + \mu^2 d$, $D = \nu^2 c + (\beta^2+1)d$ which are in $\mathfrak{p}^n$.

If our σ has $o(\sigma) = 0$, then $c = 0 = d$, and (ρ, σ) is the identity. If $o(\sigma) = \mathfrak{p}^n$, at least one of c or d must generate $\mathfrak{p}^n$, that is, must be π^n. Examine the three cases, each time considering $X = 0$ and $X = \pi^n$. Note that adding the off-diagonal entries is at times useful. We find that the generators (ρ, σ) are the elements $1, \phi_1, \phi_2, \phi_3$, which form a Klein group.

The commutator of $\Gamma(-1, 1) \in SL_2(\mathfrak{o})$ and $E_{12}(\pi^n) \in SC_2(\mathfrak{p}^n)$ is ϕ_1, whose conjugate by $E_{12}(1)$ is ϕ_3, and $\phi_1\phi_3 = \phi_2$; hence

$$[SL_2(\mathfrak{o}), SC_2(\mathfrak{p}^n)] \geqq \{1, \phi_1, \phi_2, \phi_3\}.$$

3.2. *Groups of order* $\mathfrak{o}$. In classifying the normal subgroups of order $\mathfrak{o}$ we will first investigate the situation where $\mathfrak{p} \neq 0$ and $\mathfrak{p}^2 = 0$. Subsequently, the general solution will be readily obtained.

If $\mathfrak{p} \neq 0$ and $\mathfrak{p}^2 = 0$, then

$$\mathfrak{o} = \{0, \pi, 1, 1 + \pi\}, \qquad \mathfrak{p} = \{0, \pi\},$$

and thus each of the six types of elements of $GL_2(\mathfrak{o})$ can be formed in 16 ways. Hence, $GL_2(\mathfrak{o})$ has 96 elements and $SL_2(\mathfrak{o})$ has 48 elements (using the map det).

In this situation, let $\overline{\quad}$ denote the homomorphism

$$\overline{\quad} : \mathfrak{o} \to \mathfrak{o}/\mathfrak{p} = \mathbf{F}_2,$$
$$\overline{\quad} : GL_2(\mathfrak{o}) \to GL_2(\mathfrak{o}/\mathfrak{p}) = GL_2(\mathbf{F}_2).$$

Then $\overline{DGL_2(\mathfrak{o})} = D\overline{GL_2(\mathfrak{o})} = DGL_2(\mathbf{F}_2)$, which has order 3 and consists of the identity e and of

$$d = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

since $GL_2(\mathbf{F}_2)$ is a representation of the symmetric group S_3. Hence the elements of $DGL_2(\mathfrak{o})$ are inverse images of these, and will be called of e-type, d-type, and f-type.

Are there conditions under which elements of d-type and f-type have order 3? For

$$\rho = \begin{pmatrix} 1 + x & 1 + z \\ 1 + w & y \end{pmatrix}$$

of d-type, a computation shows that ρ has order $3 \Leftrightarrow x + y + 2 = 0$ and $x + z + w = 0$. For

$$\sigma = \begin{pmatrix} x & 1 + z \\ 1 + w & 1 + y \end{pmatrix}$$

of f-type, a conjugation by $\Gamma(1, 1)$ reduces the problem to a d-type, and we find that σ of f-type has order $3 \Leftrightarrow x + y + 2 = 0$ and $y + z + w = 0$. We note that e-type elements have order 2 and fall in $SC_2(\mathfrak{p})$.

We give a description of $DGL_2(\mathfrak{o})$ in the form of a lemma.

LEMMA 3.2.1. *Let* $\mathfrak{p} \neq 0$ *and* $\mathfrak{p}^2 = 0$. *Then* $DGL_2(\mathfrak{o})$ *contains* $SC_2(\mathfrak{p})$, *has* 24 *elements, and has four* 3-*Sylow groups.*

Proof. We have that

$$[GL_2(\mathfrak{o}), SC_2(\mathfrak{p})] \leqq DGL_2(\mathfrak{o}) \cap SC_2(\mathfrak{p}) \leqq SC_2(\mathfrak{p});$$

hence, $DGL_2(\mathfrak{o}) \cap SC_2(\mathfrak{p})$ has order 4 or 8. However, $(\Gamma(1, 1), \Theta(1 + \pi, 1))$ is in $SC_2(\mathfrak{p})$ and not in $\{1, \phi_1, \phi_2, \phi_3\}$; see § 31. Hence, we must have that

$$DGL_2(\mathfrak{o}) \cap SC_2(\mathfrak{p}) = SC_2(\mathfrak{p}).$$

It follows that the restriction to $\mathrm{DGL}_2(\mathfrak{o})$ of the map $\overline{}$ above has kernel $\mathrm{SC}_2(\mathfrak{p})$, hence

$$\mathrm{DGL}_2(\mathfrak{o})/\mathrm{SC}_2(\mathfrak{p}) \cong \overline{\mathrm{DGL}_2(\mathfrak{o})} = \mathrm{DGL}_2(\mathbf{F}_2),$$

which has order 3, while $\mathrm{SC}_2(\mathfrak{p})$ has order 8. Hence $\mathrm{DGL}_2(\mathfrak{o})$ has order 24; its 3-Sylow groups have order 3 and there are either one or four. Let

$$\rho = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix};$$

then

$$\sigma = (\Gamma(1, 1), \rho) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \in \mathrm{DGL}_2(\mathfrak{o})$$

is an f-type element of order 3. However, σ_Θ, with $\Theta(1, 1 + \pi)$, falls in $\mathrm{DGL}_2(\mathfrak{o})$, has order 3, and does not belong to $\{1, \sigma, \sigma^2\}$; hence, it gives another 3-Sylow group. Thus, we have completed the proof.

We shall also need a description of $\mathrm{DSL}_2(\mathfrak{o})$. Denote by K_i, $i = 1, \ldots, 4$, the four 3-Sylow groups of $\mathrm{DGL}_2(\mathfrak{o})$ and recall the group of Lemma 3.1.4.

LEMMA 3.2.2. *Let* $\mathfrak{p} \neq 0$ *and* $\mathfrak{p}^2 = 0$. *Then* $\mathrm{DSL}_2(\mathfrak{o})$ *is a subgroup of order* 12 *of* $\mathrm{DGL}_2(\mathfrak{o})$.

Proof. Let K stand for $[\mathrm{GL}_2(\mathfrak{o}), \mathrm{SC}_2(\mathfrak{p})]$ and consider the set

$$H = K \cup K_1 \cup \ldots \cup K_4.$$

We claim that $H = K_1K$, which is a subgroup of $\mathrm{DGL}_2(\mathfrak{o})$, by (**7**, p. 24). Actually, K_1K has order $3 \times 4 = 12$ since $K_1 \cap K = \{1\}$. Also fix some generator σ of K_1.

Clearly, $\sigma\phi_1 \notin K_1$, and using our earlier discussion of conditions for order 3, we find that $\sigma\phi_1$ has order 3, hence $\sigma\phi_1 \in K_2$, say; then $K_2 \leqq K_1K$. Similarly, using $\sigma\phi_1\phi_2$ and $\sigma\phi_1\phi_2\phi_3$, we deduce that $K_3 \leqq K_1K$ and $K_4 \leqq K_1K$. Hence $H \subseteq K_1K$; since both have 12 elements, our claim is proved.

Since $\bigcup_{i=1}^4 K_i$ exhausts the elements of order 3 in $\mathrm{DGL}_2(\mathfrak{o})$ and since $K \lhd \mathrm{GL}_2(\mathfrak{o})$, then $H \lhd \mathrm{GL}_2(\mathfrak{o})$. Then we can form $\mathrm{SL}_2(\mathfrak{o})/H$ which is a group of order 4, hence is commutative. Thus $H \geqq \mathrm{DSL}_2(\mathfrak{o})$.

Now, $\mathrm{DSL}_2(\mathfrak{o}) \geqq [\mathrm{SL}_2(\mathfrak{o}), \mathrm{SC}_2(\mathfrak{p})]$, which has order 4 (Lemma 3.1.4). The inclusion is strict since

$$o(\mathrm{DSL}_2(\mathfrak{o})) = \mathfrak{o} \supset \mathfrak{p};$$

hence by Lagrange's theorem, $\mathrm{DSL}_2(\mathfrak{o}) = H$.

PROPOSITION 3.2.3. *Let* $\mathfrak{p} \neq 0$ *and* $\mathfrak{p}^2 = 0$. *Let* $G \lhd \mathrm{GL}_2(\mathfrak{o})$, *with* $o(G) = \mathfrak{o}$. *Then either*

$$G \geqq \mathrm{DGL}_2(\mathfrak{o}) \quad \textit{or} \quad G \cap \mathrm{DGL}_2(\mathfrak{o}) = \mathrm{DSL}_2(\mathfrak{o}).$$

Proof. By Lemma 3.1.1 we may assume that $G \leqq \mathrm{DGL}_2(\mathfrak{o})$. Then the possible orders for G are

$$2, 3, 4, 6, 8, 12, 24.$$

Orders 2 and 4 are impossible since elements would be of e-type and in $\mathrm{SC}_2(\mathfrak{p})$. Orders 3 and 6 are impossible since $\mathrm{DGL}_2(\mathfrak{o})$ has four 3-Sylow groups of order 3. Finally, $\mathrm{SC}_2(\mathfrak{p})$ is the unique 2-Sylow group, of order 8.

The statement easily follows; in the case of order 12, look at $\mathrm{SL}_2(\mathfrak{o})/G$.

At this point we turn to the situation where $\mathfrak{p}^2$ is not necessarily 0. We prove the following lemma.

LEMMA 3.2.4. *Let* $G \lhd \mathrm{GL}_2(\mathfrak{o})$ *with* $o(G) = \mathfrak{o}$. *Then*

$$G \geqq \mathrm{SC}_2(\mathfrak{p}^2).$$

Proof. By Lemma 3.1.1 we may assume that $G \leqq \mathrm{DGL}_2(\mathfrak{o})$ and we can pick

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & b \end{pmatrix} \in G.$$

We claim that $b \in \mathfrak{u}$. Let $\overline{}$ denote the homomorphism which factors by $\mathfrak{p}^2$. Then

$$\bar{\sigma} \in \bar{G} \leqq \overline{\mathrm{DGL}_2(\mathfrak{o})} = \mathrm{DGL}_2(\bar{\mathfrak{o}}),$$

and $\bar{\mathfrak{o}}$ has $\bar{\mathfrak{p}} \neq \bar{0}$ and $\bar{\mathfrak{p}}^2 = \bar{0}$; hence $\bar{\sigma}$ is of f-type, and $\bar{b}$ and b are units.

Now, apply Formula 3.1.2 with $\mu = \pi$ and then Lemma 1.3.8.

THEOREM 3.2.5. *Let* $G \lhd \mathrm{GL}_2(\mathfrak{o})$ *with* $o(G) = \mathfrak{o}$. *Then either*

$$G \geqq \mathrm{DGL}_2(\mathfrak{o}) \quad \textit{or} \quad G \cap \mathrm{DGL}_2(\mathfrak{o}) = \mathrm{DSL}_2(\mathfrak{o}),$$

which has index 2 *and* 4 *in* $\mathrm{DGL}_2(\mathfrak{o})$ *and* $\mathrm{SL}_2(\mathfrak{o})$, *respectively. And we have that* $G \geqq \mathrm{SC}_2(\mathfrak{p})$ *only when* $G \geqq \mathrm{DGL}_2(\mathfrak{o})$.

Proof. By Lemma 3.1.1, reduce the problem to $G \leqq \mathrm{DGL}_2(\mathfrak{o})$. Consider the homomorphism $\overline{}$ which factors by $\mathfrak{p}^2$. When restricted to $\mathrm{SL}_2(\mathfrak{o})$, its kernel is $\mathrm{SC}_2(\mathfrak{p}^2)$. Then by Lemma 3.2.4, we see that the kernel of $\overline{}$ when restricted to $\mathrm{DGL}_2(\mathfrak{o})$, to $\mathrm{DSL}_2(\mathfrak{o})$, and to G, is also $\mathrm{SC}_2(\mathfrak{p}^2)$. Hence,

$$\mathrm{DGL}_2(\mathfrak{o})/\mathrm{SC}_2(\mathfrak{p}^2) \cong \overline{\mathrm{DGL}_2(\mathfrak{o})} = \mathrm{DGL}_2(\bar{\mathfrak{o}}),$$

say under the isomorphism ϕ;

$$\mathrm{DSL}_2(\mathfrak{o})/\mathrm{SC}_2(\mathfrak{p}^2) \cong \overline{\mathrm{DSL}_2(\mathfrak{o})} = \mathrm{DSL}_2(\bar{\mathfrak{o}}),$$

by restriction of the same ϕ;

$$G/\mathrm{SC}_2(\mathfrak{p}^2) \cong \bar{G} \leqq \mathrm{DGL}_2(\bar{\mathfrak{o}}),$$

again by restriction of ϕ.

By Proposition 3.2.3 (applied with $\bar{\mathfrak{o}}$) and because of this isomorphism ϕ, we deduce that $\mathrm{DSL}_2(\mathfrak{o})$ is the unique proper subgroup of $\mathrm{DGL}_2(\mathfrak{o})$ which is

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normal in $\mathrm{GL}_2(\mathfrak{o})$, which has order $\mathfrak{o}$, and whose quotient by $\mathrm{SC}_2(\mathfrak{p}^2)$ is isomorphic to $\mathrm{DSL}_2(\bar{\mathfrak{o}})$, under ϕ. By Proposition 3.2.3, we have that either $\bar{G} = \mathrm{DGL}_2(\bar{\mathfrak{o}})$ or $\bar{G} = \mathrm{DSL}_2(\bar{\mathfrak{o}})$.

Hence, either

$$G/\mathrm{SC}_2(\mathfrak{p}^2) = \mathrm{DGL}_2(\mathfrak{o})/\mathrm{SC}_2(\mathfrak{p}^2)$$

or

$$G/\mathrm{SC}_2(\mathfrak{p}^2) = \mathrm{DSL}_2(\mathfrak{o})/\mathrm{SC}_2(\mathfrak{p}^2),$$

so that G is as required. The remainder is quite easy.

This theorem has no converse, as shown by the following example.

Example 3.2.6. Let $\mathfrak{p} \neq 0$ and $\mathfrak{p}^2 = 0$. The element $\Theta(1 + \pi, 1)$ is not in $\mathrm{DSL}_2(\mathfrak{o})$ (look at $\det \Theta$) and has order 2; hence,

$$G = \mathrm{DSL}_2(\mathfrak{o}) \cup \Theta\mathrm{DSL}_2(\mathfrak{o})$$

is a subgroup of $\mathrm{GL}_2(\mathfrak{o})$ with

$$G \geqq \mathrm{DSL}_2(\mathfrak{o}), \qquad o(G) = \mathfrak{o}.$$

Now $(E_{12}(1), \Theta) = E_{12}(\pi)$, which has order 2. We check that

$$E_{12}(\pi) \notin \mathrm{DSL}_2(\mathfrak{o}),$$

which is described in Lemma 3.2.2; hence

$$E_{12}(1)\Theta E_{12}(-1) \notin G,$$

so that G is not even invariant under $\mathrm{SL}_2(\mathfrak{o})$.

If $\mathfrak{o}$ does not have $\mathfrak{p}^2 = 0$, again use $\Theta(1 + \pi, 1)$ and let G be the subgroup of $\mathrm{GL}_2(\mathfrak{o})$ generated by the set

$$\{\Theta\} \cup \mathrm{DSL}_2(\mathfrak{o}).$$

Then $G \geqq \mathrm{DSL}_2(\mathfrak{o})$, $o(G) = \mathfrak{o}$, but G is not invariant under $\mathrm{SL}_2(\mathfrak{o})$, as we can see by using the map $\overline{}$ which factors by $\mathfrak{p}^2$; indeed, $\bar{G}$ reduces to the group just considered above.

3.3. Inclusion theorems. We shall distinguish between two kinds of groups, one of which has bad behaviour.

Example 3.3.1. Suppose that $\mathfrak{p}^n \neq 0$ and $\mathfrak{p}^{n+1} = 0$ for some $n \geqq 1$. Let

$$\sigma = \begin{pmatrix} 1 & \pi^n \\ \pi^n & 1 + \pi^n \end{pmatrix}$$

and let G be the normal subgroup of $\mathrm{GL}_2(\mathfrak{o})$ generated by σ. Using $\mathfrak{p}^{n+1} = 0$, we can calculate that the conjugates of σ in $\mathrm{GL}_2(\mathfrak{o})$ turn out to be either σ or

$$\sigma_1 = \begin{pmatrix} 1 + \pi^n & \pi^n \\ \pi^n & 1 \end{pmatrix}.$$

Hence $G = \{1, \sigma, \sigma_1, \sigma\sigma_1 = \sigma_1\sigma\}$, which is a Klein group.

We see that $o(G) = \mathfrak{p}^n$ and the elements in G which have order $\mathfrak{p}^n$ have

$$(a - b)\mathfrak{o} = c\mathfrak{o} = d\mathfrak{o} = \mathfrak{p}^n.$$

Now, remove the assumption that $\mathfrak{p}^{n+1}$ is 0. Take σ as above and let H be the normal subgroup of $\mathrm{GL}_2(\mathfrak{o})$ generated by σ. Then $o(H) = \mathfrak{p}^n$ and the elements of H which have order $\mathfrak{p}^n$ satisfy

$$(a - b)\mathfrak{o} = c\mathfrak{o} = d\mathfrak{o} = \mathfrak{p}^n.$$

For, if we factor by $\mathfrak{p}^{n+1}$, then $\bar{H}$ is the normal subgroup of $\mathrm{GL}_2(\bar{\mathfrak{o}})$ generated by $\bar{\sigma}$. However, this is the situation discussed above; hence our statement follows.

Definition 3.3.2. Let $G \lhd \mathrm{GL}_2(\mathfrak{o})$ with $o(G) = \mathfrak{p}^n \neq 0$, $n \geqq 1$. Then G is called *of the second kind* if the elements of G which have order $\mathfrak{p}^n$ satisfy

$$(a - b)\mathfrak{o} = c\mathfrak{o} = d\mathfrak{o} = \mathfrak{p}^n.$$

The above example shows the existence of such groups for any given $n \geqq 1$, for which $\mathfrak{p}^n \neq 0$. We call *G of the first kind* if it is not of the second kind.

In a (normal) group of the first kind there is always an element with $d\mathfrak{o} = \mathfrak{p}^n$ and $a - b \in \mathfrak{p}^{n+1}$. For we can pick an element of order $\mathfrak{p}^n$ for which not all of $a - b, c, d$ generate $\mathfrak{p}^n$, and examining each case occurring we can transform elements by suitable conjugations and maneuvers.

PROPOSITION 3.3.3. *Let $G \lhd \mathrm{GL}_2(\mathfrak{o})$ with $o(G) = \mathfrak{p}^n$, where G is of the first kind. Then $G \cap \mathrm{SC}_2(\mathfrak{p}^n) \lhd \mathrm{GL}_2(\mathfrak{o})$ is of the first kind and*

$$o(G \cap \mathrm{SC}_2(\mathfrak{p}^n)) = \mathfrak{p}^n, \qquad G \cap \mathrm{SC}_2(\mathfrak{p}^n) > \mathrm{SC}_2(\mathfrak{p}^{n+1}).$$

Proof. Pick $\sigma \in G$ as described above. The element

$$(E_{12}(1), \sigma) \in G$$

shows that G is of the first kind and that $G \cap \mathrm{SC}_2(\mathfrak{p}^n)$ has order $\mathfrak{p}^n$. Secondly, applying Formula 3.1.2 with σ and $\mu = 1$ yields

$$\Xi(*, *, \eta\pi^n) \in G, \quad \text{where } \eta \in \mathfrak{u},$$

and Lemma 1.3.8 yields $\mathrm{SC}_2(\mathfrak{p}^{n+1}) \leqq G$.

PROPOSITION 3.3.4. *Let $G \lhd \mathrm{GL}_2(\mathfrak{o})$ with $o(G) = \mathfrak{p}^n$ and $\mathfrak{p}^{n+1} \neq 0$. Suppose that G is of the second kind. Then $G \cap \mathrm{SC}_2(\mathfrak{p}^{n+1}) \lhd \mathrm{GL}_2(\mathfrak{o})$ is of the first kind and*

$$o(G \cap \mathrm{SC}_2(\mathfrak{p}^{n+1})) = \mathfrak{p}^{n+1}, \qquad G \cap \mathrm{SC}_2(\mathfrak{p}^{n+1}) > \mathrm{SC}_2(\mathfrak{p}^{n+2}).$$

Proof. Pick $\sigma \in G$ with $o(\sigma) = \mathfrak{p}^n$. Then

$$(a - b)\mathfrak{o} = c\mathfrak{o} = d\mathfrak{o} = \mathfrak{p}^n.$$

Now examine $(E_{12}(\pi), \sigma)$, and secondly, make applications of Formula 3.1.2 (with $\mu = \pi$) and Lemma 1.3.8.

The following result may have some interest in number theory.

Remark 3.3.5. *Suppose that* $\mathfrak{p} = 2\mathfrak{o}$. *Let* $G \lhd \mathrm{GL}_2(\mathfrak{o})$ *with* $o(G) = \mathfrak{p}^n$, $n \geqq 2$, *where* G *is of the second kind. Then* $G \geqq \mathrm{SC}_2(\mathfrak{p}^{n+1})$.

Proof. Pick σ of order $\mathfrak{p}^n$ in G. Note that

$$a + b = a - b + 2b$$

generates $\mathfrak{p}$. Now apply Formula 3.1.2 with $\mu = 1$, and then Lemma 1.3.7 (i).

For the sake of completion we try to find some conditions under which a normal subgroup G of order $\mathfrak{p}^n$ actually contains $\mathrm{SC}_2(\mathfrak{p}^n)$. *A necessary condition is that* G *be of the first kind.*

An element $\sigma \in \mathrm{GL}_2(\mathfrak{o})$, of order $\mathfrak{p}^n$, is one of the following types:

$$\begin{array}{llll}
T_{n1}: & (a-b)\mathfrak{o} = \mathfrak{p}^n, & c \in \mathfrak{p}^{n+1}, & d \in \mathfrak{p}^{n+1}. \\
T_{n2}: & (a-b)\mathfrak{o} = \mathfrak{p}^n, & c\mathfrak{o} = \mathfrak{p}^n, & d \in \mathfrak{p}^{n+1}. \\
T_{n3}: & (a-b)\mathfrak{o} = \mathfrak{p}^n, & c \in \mathfrak{p}^{n+1}, & d\mathfrak{o} = \mathfrak{p}^n. \\
T_{n4}: & a-b \in \mathfrak{p}^{n+1}, & c\mathfrak{o} = \mathfrak{p}^n, & d\mathfrak{o} = \mathfrak{p}^n. \\
T_{n5}: & a-b \in \mathfrak{p}^{n+1}, & c\mathfrak{o} = \mathfrak{p}^n, & d \in \mathfrak{p}^{n+1}. \\
T_{n6}: & a-b \in \mathfrak{p}^{n+1}, & c \in \mathfrak{p}^{n+1}, & d\mathfrak{o} = \mathfrak{p}^n. \\
T_{n7}: & (a-b)\mathfrak{o} = \mathfrak{p}^n, & c\mathfrak{o} = \mathfrak{p}^n, & d\mathfrak{o} = \mathfrak{p}^n.
\end{array}$$

When we say that "G has T_{ni}" or "T_{ni} exists in G", we shall mean that the group G has elements of type T_{ni}.

LEMMA 3.3.6. *Let* $G \lhd \mathrm{GL}_2(\mathfrak{o})$. *Then:*

(i) G *has* $T_{n1} \Leftrightarrow G$ *has* $T_{n2} \Leftrightarrow G$ *has* T_{n3};

(ii) G *has* $T_{n4} \Leftrightarrow G$ *has* $T_{n5} \Leftrightarrow G$ *has* T_{n6}.

Proof. Trivial, using suitable conjugations.

PROPOSITION 3.3.7. *Let* $G \lhd \mathrm{GL}_2(\mathfrak{o})$ *with* $o(G) = \mathfrak{p}^n$, *and suppose that* G *is of the first kind.*

(i) *A sufficient condition that* $G \geqq \mathrm{SC}_2(\mathfrak{p}^n)$ *is that* T_{n1} *exist in* G, *which is equivalent to the existence of* T_{n1}, T_{n2}, T_{n3} *in* G. *The condition is not necessary.*

(ii) *A sufficient condition that* $G \geqq \mathrm{SC}_2(\mathfrak{p}^n)$ *is that* T_{n4} *and* T_{n7} *exist in* G, *which is equivalent to the existence of* T_{n4}, T_{n5}, T_{n6}, *and* T_{n7} *in* G. *The condition is not necessary.*

Proof. Let $\sigma \in G$ be of type T_{n1}. By Proposition 3.3.3, we have that

$$\tau = E_{21}(-a^{-1}d) \in \mathrm{SC}_2(\mathfrak{p}^{n+1}) \leqq G;$$

hence, $\tau\sigma = \Xi(a, b - a^{-1}cd, *) \in G$, and hence $G \geqq \mathrm{SC}_2(\mathfrak{p}^n)$ by Lemma 1.3.7 (i).

Now let $\sigma, \rho \in G$ have type T_{n4} and T_{n7}, respectively. Verify that $\sigma\rho \in G$ has type T_{n1} and apply the above step.

To show that the conditions of (i) and (ii) are not necessary, we consider $\mathrm{SC}_2(\mathfrak{p}^n)$ itself. Its elements can never satisfy the condition $(a - b)\mathfrak{o} = \mathfrak{p}^n$, hence $\mathrm{SC}_2(\mathfrak{p}^n)$ has neither T_{n1} nor T_{n7}.

One can use Propositions 3.3.4 and 3.3.7 to obtain conditions under which G of the second kind contains $\mathrm{SC}_2(\mathfrak{p}^{n+1})$.

3.4. CLASSIFICATION THEOREMS. A few remarks are in order; then we will be ready for our theorems.

Suppose that $\mathfrak{p}^n \neq 0$ and $\mathfrak{p}^{n+1} = 0$. Lemma 3.1.3 gave us a description of $\mathrm{SC}_2(\mathfrak{p}^n)$. What subgroups of order $\mathfrak{p}^n$ of $\mathrm{SC}_2(\mathfrak{p}^n)$ are normal in $\mathrm{GL}_2(\mathfrak{o})$? Those of two elements are not normal in $\mathrm{GL}_2(\mathfrak{o})$ as we can easily verify with conjugations by $\Gamma(1, 1)$ or $E_{12}(1)$. Those of four elements are Klein groups, determined by pairs of non-identity elements. We can list seven, and six of them are easily found to be not normal. The only survivor is

$$[\mathrm{GL}_2(\mathfrak{o}), \mathrm{SC}_2(\mathfrak{p}^n)] = \{1, \phi_1, \phi_2, \phi_3\}.$$

When $\mathfrak{p}^{n+1} \neq 0$, the homomorphism $\overline{}$ which factors by $\mathfrak{p}^{n+1}$ provides us with

$$\overline{[\mathrm{GL}_2(\mathfrak{o}), \mathrm{SC}_2(\mathfrak{p}^n)]} = [\mathrm{GL}_2(\bar{\mathfrak{o}}), \mathrm{SC}_2(\bar{\mathfrak{p}}^n)],$$

and we can deduce that $[\mathrm{GL}_2(\mathfrak{o}), \mathrm{SC}_2(\mathfrak{p}^n)]$ has order $\mathfrak{p}^n$ and is of the first kind.

THEOREM 3.4.1. *Let $G \lhd \mathrm{GL}_2(\mathfrak{o})$ with $o(G) = \mathfrak{p}^n$, where G is of the first kind. Then either*

$$G \geqq \mathrm{SC}_2(\mathfrak{p}^n) \quad or \quad G \cap \mathrm{SC}_2(\mathfrak{p}^n) = [\mathrm{GL}_2(\mathfrak{o}), \mathrm{SC}_2(\mathfrak{p}^n)],$$

which is of index 2 in $\mathrm{SC}_2(\mathfrak{p}^n)$.

Proof. If $\mathfrak{p}^n \neq 0$ and $\mathfrak{p}^{n+1} = 0$, Proposition 3.3.3 and the above remarks complete the result.

Now the general case. By Proposition 3.3.3, we may assume that $G \leqq \mathrm{SC}_2(\mathfrak{p}^n)$, and we also obtain that the restrictions to $\mathrm{SL}_2(\mathfrak{o})$, to $[\mathrm{GL}_2(\mathfrak{o}), \mathrm{SC}_2(\mathfrak{p}^n)]$, and to G, of the homomorphism $\overline{}$ which factors by $\mathfrak{p}^{n+1}$, all have kernel $\mathrm{SC}_2(\mathfrak{p}^{n+1})$. In $\bar{\mathfrak{o}}$ we have $\bar{\mathfrak{p}}^{n+1} = \bar{0}$; hence, we use the above result and arguments as in the proof of Theorem 3.2.5 to complete our proof.

THEOREM 3.4.2. *Let $G \lhd \mathrm{GL}_2(\mathfrak{o})$ with $o(G) = \mathfrak{p}^n$, where G is of the second kind. Suppose that $\mathfrak{p}^{n+1} \neq 0$. Then either*

$$G \geqq \mathrm{SC}_2(\mathfrak{p}^{n+1}) \quad or \quad G \cap \mathrm{SC}_2(\mathfrak{p}^{n+1}) = [\mathrm{GL}_2(\mathfrak{o}), \mathrm{SC}_2(\mathfrak{p}^{n+1})],$$

which is of index 2 in $\mathrm{SC}_2(\mathfrak{p}^{n+1})$.

Proof. Apply Theorem 3.4.1 to $G \cap \mathrm{SC}_2(\mathfrak{p}^{n+1})$, which is described by Proposition 3.3.4.

Our two theorems have no converse. This is easy to see for Theorem 3.4.2. Relative to Theorem 3.4.1 we offer the following example.

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Example 3.4.3. Take $\Theta(1 + \pi^n, 1)$ and let G be the subgroup of $\mathrm{GL}_2(\mathfrak{o})$ generated by the set

$$\{\Theta\} \cup [\mathrm{GL}_2(\mathfrak{o}), \mathrm{SC}_2(\mathfrak{p}^n)].$$

Then $G \geqq [\mathrm{GL}_2(\mathfrak{o}), \mathrm{SC}_2(\mathfrak{p}^n)]$. Proceeding as in Example 3.2.6, first considering $\mathfrak{p}^n \neq 0$ and $\mathfrak{p}^{n+1} = 0$, we easily find that $o(G) = \mathfrak{p}^n$ and that G is not invariant under $\mathrm{SL}_2(\mathfrak{o})$.

For a normal subgroup G of the second kind, with $o(G) = \mathfrak{p}^n$ and where $\mathfrak{p}^{n+1} = 0$, there is nothing interesting, as $\mathrm{SC}_2(\mathfrak{p}^{n+1}) = \{1\}$, and there is no possibility for a converse. We may mention that $G \cap \mathrm{SC}_2(\mathfrak{p}^n)$ consists of the identity and of the radiation $\mathrm{diag}(1 + \pi^n)$, which is no contribution.

The core of our efforts to measure the eccentricity of the situation where $N\mathfrak{p} = 2$ can be stated as our Second Main Theorem.

Second Main Theorem 3.4.4. *Let $\mathfrak{o}$ be a local ring with principal maximal ideal $\mathfrak{p}$ and with $N\mathfrak{p} = 2$. Suppose that we are given $G \lhd \mathrm{GL}_2(\mathfrak{o})$ with $o(G) = \mathfrak{p}^n$, where $n = 0, 1, 2, \ldots$. Then $G \geqq \mathrm{SC}_2(\mathfrak{p}^{n+2})$.*

Note that we have some better estimates. With G as in the theorem we find, in particular, that when $n = 0$, $G \geqq \mathrm{SC}_2(\mathfrak{p}) \Leftrightarrow G \geqq \mathrm{DGL}_2(\mathfrak{o})$, and when $n \geqq 1$, $G \geqq \mathrm{SC}_2(\mathfrak{p}^{n+1})$ if G is of the first kind. Still sharper estimates are given by mixed commutator groups.

Summary. When $N\mathfrak{p} = 3$, normal subgroups of order $\mathfrak{o}$ are "almost" classified by $\mathrm{DSL}_2(\mathfrak{o})$ which is of index 3 in $\mathrm{SL}_2(\mathfrak{o})$. That is, we obtain a theorem but it has no converse, as we find counterexamples. Normal subgroups of order $\mathfrak{a} \subseteq \mathfrak{p}$ are classified by $\mathrm{SC}_2(\mathfrak{a})$. For these results it is enough to assume only invariance under $\mathrm{SL}_2(\mathfrak{o})$.

For $N\mathfrak{p} \geqq 4$, invariance under $\mathrm{SL}_2(\mathfrak{o})$ is sufficient to classify normal subgroups of order $\mathfrak{o}$ (by means of $\mathrm{SL}_2(\mathfrak{o})$ as usual). But when the order $\mathfrak{a} \subseteq \mathfrak{p}$, invariance under $\mathrm{SL}_2(\mathfrak{o})$ is not enough and we need additional conditions on the ring (Property T) to achieve a classification by $\mathrm{SC}_2(\mathfrak{a})$; however, normality in $\mathrm{GL}_2(\mathfrak{o})$ yields the classification by $\mathrm{SC}_2(\mathfrak{a})$, and there is no need for special conditions on the ring.

When $N\mathfrak{p} = 2$, with our particular local ring, normal subgroups of order $\mathfrak{o}$ are "almost" classified by $\mathrm{DSL}_2(\mathfrak{o})$, of index 4 in $\mathrm{SL}_2(\mathfrak{o})$. Normal subgroups of order $\mathfrak{p}^n$ are separated into two kinds. Those of the first kind are "almost" classified by $[\mathrm{GL}_2(\mathfrak{o}), \mathrm{SC}_2(\mathfrak{p}^n)]$, of index 2 in $\mathrm{SC}_2(\mathfrak{p}^n)$. Those of the second kind have a still weaker "pseudo-classification".

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