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AN ENTIRE FUNCTION SHARING TWO VALUES WITH ITS LINEAR DIFFERENTIAL POLYNOMIAL

INDRAJIT LAHIRI

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Abstract

We consider the uniqueness of an entire function and a linear differential polynomial generated by it. One of our results improves a result of Li and Yang ['Value sharing of an entire function and its derivatives', *J. Math. Soc. Japan* **51**(4) (1999), 781–799].

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1. Introduction, definitions and results

Let f and g be two nonconstant meromorphic functions in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$, we say that f and g share the value a CM (counting multiplicities) or IM (ignoring multiplicities) if $f - a$ and $g - a$ have the same set of zeros counting multiplicities or ignoring multiplicities, respectively.

In 1976, Rubel and Yang [10] first considered the problem of uniqueness of an entire function f when it shares two values CM with its derivative f' and proved the following theorem.

Theorem A [10]. *Let f be a nonconstant entire function. If f and f' share two values a and b CM, then $f \equiv f'$.*

Considering $f(z) = e^{e^z} \int_0^z e^{-e^t}(1 - e^t)\,dt$ [12, page 386], one can easily verify that sharing of two values is essential.

In 1979, Mues and Steinmetz [9] improved Theorem A replacing CM shared values by IM shared values. In 1990, Yang [13] extended Theorem A to any kth-order derivative $f^{(k)}$ of the entire function f. In 2000, Li and Yang [8] improved the result of Yang [13] and settled a conjecture of Frank [2] (see also [12, page 394]) affirmatively. Their result can be stated as follows.

Theorem B [8]. *Let f be a nonconstant entire function, k a positive integer and a and b two distinct finite values. If f and $f^{(k)}$ share a and b IM, then $f \equiv f^{(k)}$.*

The natural extension of a derivative of an entire function f is a linear differential polynomial generated by f. In 1994, Gu [3] extended Theorem A to a linear differential polynomial. In order to state the result, we recall the definition of a small function: a meromorphic function $a = a(z)$ is called a small function of a meromorphic function f if $T(r, a) = S(r, f)$, where $S(r, f)$ stands for any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of finite linear measure.

Theorem C [3]. *Let f be a nonconstant entire function, a and b be distinct finite complex numbers and $L(f) = f^{(n)} + a_1 f^{(n-1)} + \cdots + a_n f$, where a_j $(j = 1, 2, \ldots, n)$ are small entire functions of f. If f and $L(f)$ share a and b CM and $a + b \neq 0$ or $a_n \not\equiv -1$, then $f \equiv L(f)$.*

The following theorem of Bernstein *et al.* [1] is an improvement of Theorem C.

Theorem D [1]. *Let f be a nonconstant entire function, a and b be distinct finite complex numbers and $L(f) = b_n f^{(n)} + b_{n-1} f^{(n-1)} + \cdots + b_1 f^{(1)} + b_0 f$, where the b_j $(j = 0, 1, 2, \ldots, n)$ are small meromorphic functions of f. If f and $L(f)$ share a and b CM, then $f \equiv L(f)$.*

In contrast to the derivative of an entire function, we see in the following examples that it is not possible in the case of a linear differential polynomial to replace any CM shared value by an IM shared value.

Example 1.1. Let $f = 1 + (e^z - 1)^2$ and $L(f) = \frac{1}{2} f^{(2)} - f^{(1)}$. Then f and $L(f)$ share 1 IM and 2 CM but $f \not\equiv L(f)$.

Example 1.2 [7]. Let $f = \frac{1}{2} e^z + \frac{1}{2} e^{-z}$ and $L(f) = f^{(2)} + f^{(1)}$. Then f and $L(f)$ share 1 and -1 IM but $f \not\equiv L(f)$.

Although one IM shared value and one CM shared value cannot ensure the equality of an entire function with a linear differential polynomial generated by it, Li and Yang [7] exhibited two possibilities in the following theorem.

Theorem E [7]. *Let f be a nonconstant entire function and*

$$L(f) = b_{-1} + \sum_{j=0}^{n} b_j f^{(j)}, \tag{1.1}$$

where b_j $(j = -1, 0, 1, \ldots, n)$ are small meromorphic functions of f. Let a and b be two distinct finite values. If f and $L(f)$ share a CM and b IM, then either $f \equiv L(f)$ or f and $L(f)$ have the following forms: $f = b + (a - b)(e^{\alpha} - 1)^2$ and $L(f) = b + (a - b)(e^{\alpha} - 1)$, where α is an entire function.

For two meromorphic functions f and g, let us denote by $\overline{N}_E(r, a; f, g)$ the reduced counting function of those common a-points of f and g that have the same multiplicities. We put $\tau(a) = \liminf_{r\to\infty} \overline{N}_E(r, a; f, g)/\overline{N}(r, a; f)$ if $\overline{N}(r, a; f) \not\equiv 0$ and $\tau(a) = 1$ if $\overline{N}(r, a; f) \equiv 0$. Wang [11] improved Theorem E in the following manner.

THEOREM F [11]. *Let f be a nonconstant entire function and $L(f)$ be defined by (1.1). If f and $L(f)$ share two distinct finite values a and b IM and $\tau(a) > (n+2)/(n+3)$ for one of the shared values, say a, then the conclusion of Theorem E holds.*

Since $\tau(a) > 1 - 1/(n+3)$, we may suspect that f and $L(f)$ enjoy the advantage of sharing the value a CM in some sense, at least for large values of n.

If we look again at Theorem E, then we see that in the case of nonequality of f and $L(f)$, almost all the b-points of f and $L(f)$ are double and simple, respectively, whereas the a-points of f and $L(f)$ are almost all simple. In fact, we shall show that the simple a-points and b-points of f play a decisive role to ascertain the equality of f and $L(f)$. Also, we shall see that the simple a-points of f still play a crucial role even if the other value b is shared IM. To this end, we need the following idea of value sharing.

DEFINITION 1.3. Let f and g be meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $\overline{E}(a; f)$ the set of all distinct a-points of f.

Let $A \subset \mathbb{C}$ and k be a nonnegative integer or infinity. We denote by $E_k(a; f, A)$ the collection of those a-points of f that belong to A, where an a-point of f with multiplicity p is counted p times if $p \le k$ and $k+1$ times if $p \ge k+1$.

Also by $\overline{N}_A(r, a; f)$ we denote the reduced counting function of those a-points of f that lie in A. We now put $A = \overline{E}(a; f) \cap \overline{E}(a; g)$ and $B = \overline{E}(a; f) \Delta \overline{E}(a; g)$, where Δ denotes the symmetric difference of sets.

We shall say that f and g share the value a with weight k in the weak sense, written symbolically f, g share $(a, k)^*$, if $E_k(a; f, A) = E_k(a; g, A)$ and $\overline{N}_B(r, a; f) = S(r, f)$ and $\overline{N}_B(r, a; g) = S(r, g)$.

It is clear that if f, g share $(a, k)^*$, then f, g share $(a, p)^*$ for every integer p with $0 \le p < k$. Further, f, g share $(a, 0)^*$ if and only if f, g share the value a IM* and f, g share the value a CM* if f, g share $(a, \infty)^*$. For the definitions of IM* and CM*, we refer to [7]. We further note that the notion of weighted sharing in the weak sense coincides with that of weighted sharing (see [5, 6] for the definition) if $B = \emptyset$.

If $a = a(z)$ is a small function of f and g, then we shall say that f, g share $(a, k)^*$ if $f - a$ and $g - a$ share $(0, k)^*$.

We now state the results of the paper.

THEOREM 1.4. *Let f be a nonconstant entire function and $L(f)$ be defined by (1.1). Suppose that a and b are two distinct finite complex numbers. If f and $L(f)$ share $(a, 1)^*$ and $(b, 1)^*$, then $f \equiv L(f)$.*

By virtue of Examples 1.1 and 1.2, we see that the weight of the sharing of none of a and b can be reduced to zero. However, in such a case we can prove the following result, which improves Theorem E.

THEOREM 1.5. *Let f be a nonconstant entire function and $L(f)$ be defined by (1.1). Suppose that a and b are two distinct finite complex numbers. If f and $L(f)$ share $(a, 1)^*$ and $(b, 0)^*$, then the conclusion of Theorem E holds.*

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As consequences of Theorems 1.4 and 1.5, respectively, we obtain the following corollaries.

COROLLARY 1.6. *Let f be a nonconstant entire function and $L(f)$ be defined by (1.1). Suppose that a and b are two distinct finite complex numbers. If f and $L(f)$ share a, b IM and f and $L(f)$ have the same set of simple a-points and b-points, then $f \equiv L(f)$.*

COROLLARY 1.7. *Let f be a nonconstant entire function and $L(f)$ be defined by (1.1). Suppose that a and b are two distinct finite complex numbers. If f and $L(f)$ share a, b IM and f and $L(f)$ have the same set of simple a-points, then the conclusion of Theorem E holds.*

Li and Yang [7] exhibited by an example that Theorem E is not valid for meromorphic functions. However, they proved the following extension of Theorem E.

THEOREM G [7]. *Let f be a nonconstant meromorphic function with $N(r, f) = S(r, f)$ and $L(f)$ be defined by (1.1). Let $a\,(\not\equiv \infty)$ and $b\,(\not\equiv \infty)$ be two distinct small functions of f. If f and $L(f)$ share a CM* and b IM*, then either $f \equiv L(f)$ or f and $L(f)$ have the following forms: $f = b + (a - b)(e^{\alpha} - 1)^2$ and $L(f) = b + (a - b)(e^{\alpha} - 1)$, where α is an entire function.*

It is possible to improve Theorems 1.4 and 1.5 along the lines of Theorem G.

For a meromorphic function f and $a \in \mathbb{C} \cup \{\infty\}$, we denote by $\overline{N}_{k)}(r, a; f)$ (respectively $\overline{N}_{(k}(r, a; f)$) the reduced counting function of a-points of f with multiplicities at most (at least) k. For standard definitions and notations of value distribution theory, we refer to [4] and [12].

2. Lemmas

In this section we present necessary lemmas. The first is a consequence of the second fundamental theorem.

LEMMA 2.1. *Let f and g be two meromorphic functions sharing $(a, 0)^*$, $(b, 0)^*$ and $(\infty, 0)^*$, where a and b are two distinct finite complex numbers. Then*

$$T(r, f) \leq 3T(r, g) + S(r, f) \quad \textit{and} \quad T(r, g) \leq 3T(r, f) + S(r, g).$$

Note. Lemma 2.1 implies that $S(r, f) = S(r, g)$.

The following lemma can be proved in a similar manner to [7, Lemma 5].

LEMMA 2.2. *Let f be a nonconstant entire function and $L(f)$ be defined by (1.1). Let a and b be two distinct finite complex numbers. If f and $L(f)$ share $(a, 0)^*$ and $(b, 0)^*$, then*

$$T(r, f) = \overline{N}(r, a; f) + \overline{N}(r, b; f) + S(r, f),$$

provided $f \not\equiv L(f)$.

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3. Proofs of the theorems

Proof of Theorem 1.4. Let $g = L(f)$ and

$$\phi = \frac{f'(f-g)}{(f-a)(f-b)}.$$

Since f and g share $(a,1)^*$, $(b,1)^*$ and $(\infty,0)^*$, by Lemma 2.1, $S(r,g) = S(r,f)$. We suppose that $f \not\equiv g$. Then, by the hypothesis, $N(r,\phi) = S(r,f)$. Since

$$\phi = \frac{1-b_0}{a-b}\left(\frac{af'}{f-a} - \frac{bf'}{f-b}\right) - \frac{b_{-1}}{a-b}\left(\frac{f'}{f-a} - \frac{f'}{f-b}\right) - \frac{f'}{f-a}\sum_{j=1}^{n}\frac{b_j f^{(j)}}{f-b},$$

from the lemma of the logarithmic derivative we see that $m(r,\phi) = S(r,f)$ and so $T(r,\phi) = S(r,f)$.

Let z_0 be a zero of $f-a$ with multiplicity $p\,(\geq 2)$ and a zero of $g-a$ with multiplicity $q\,(\geq 2)$. Then z_0 is a zero of ϕ with multiplicity at least $\min\{p,q\} - 1 \geq 1$. Hence,

$$\overline{N}_{(2}(r,a;f \mid g = a, \geq 2) \leq N(r,0;\phi) = S(r,f),$$

where $\overline{N}_{(2}(r,a;f \mid g = a, \geq 2)$ denotes the reduced counting function of those multiple a-points of f which are also multiple a-points of g. Since f and g share $(a,1)^*$,

$$\overline{N}_{(2}(r,a;f) = \overline{N}_{(2}(r,a;f \mid g = a, \geq 2) + \overline{N}_{(2}(r,a;f \mid g = a, = 1) = S(r,f),$$

where $\overline{N}_{(2}(r,a;f \mid g = a, = 1)$ denotes the reduced counting function of multiple a-points of f which are also simple a-points of g. Similarly, $\overline{N}_{(2}(r,b;f) = S(r,f)$.

In view of Lemma 2.2, we consider the following cases.

Case I. $\overline{N}(r,a;f) \neq S(r,f)$. We put

$$\beta = \frac{g'}{g-b} - \frac{f'}{f-b}.$$

Since f and g share $(b,1)^*$,

$$N(r,\beta) = \overline{N}(r,\beta) \leq \overline{N}_{(2}(r,b;f) + S(r,f) = S(r,f).$$

Since $m(r,\beta) = S(r,f)$, we obtain $T(r,\beta) = S(r,f)$.

Now, from the definition of ϕ,

$$\phi\frac{f-a}{f'} = 1 - \frac{g-b}{f-b}. \tag{3.1}$$

Differentiating (3.1) and using (3.1) again,

$$(\phi+\beta)\frac{f'}{f-a} - \phi\frac{f''}{f'} + \phi' - \phi\beta = 0. \tag{3.2}$$

Since $\overline{N}(r,a;f) \neq S(r,f)$ and $\overline{N}_{(2}(r,a;f) = S(r,f)$, it follows from (3.2) that $\phi + \beta \equiv 0$ and so

$$\frac{f''}{f'} - \frac{\phi'}{\phi} + \frac{g'}{g-b} - \frac{f'}{f-b} = 0.$$

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Integration gives $\phi(f-b) = cf'(g-b)$, where c is a nonzero constant. Now, using the definition of ϕ,

$$f - g = c(f-a)(g-b). \tag{3.3}$$

From (3.3),

$$\frac{f-b}{g-b} = c\Big(f - \frac{ac-1}{c}\Big) \tag{3.4}$$

and

$$\frac{g-a}{f-a} = -c\Big(g - \frac{bc+1}{c}\Big). \tag{3.5}$$

Since f and g share $(a,1)^*$ and $(b,1)^*$, it follows from (3.4) and (3.5) that

$$\overline{N}\Big(r, \frac{ac-1}{c}; f\Big) = \overline{N}\Big(r, 0; \frac{f-b}{g-b}\Big) \le \overline{N}_{(2}(r, b; f) + S(r,f) = S(r,f)$$

and

$$\overline{N}\Big(r, \frac{bc+1}{c}; g\Big) = \overline{N}\Big(r, 0; \frac{g-a}{f-a}\Big) \le \overline{N}_{(2}(r, a; g) + S(r,g) = \overline{N}_{(2}(r, a; f) + S(r,g) = S(r,g)$$

and, by the second fundamental theorem,

$$T(r,f) = \overline{N}(r, a; f) + S(r,f) \tag{3.6}$$

and

$$T(r,g) = \overline{N}(r, b; g) + S(r,g) = \overline{N}(r, b; f) + S(r,g). \tag{3.7}$$

From (3.6) and (3.7) and Lemma 2.2, we find that $T(r,g) = S(r,g)$, which is a contradiction.

Case II. $\overline{N}(r, b; f) \neq S(r,f)$. We put

$$\gamma = \frac{g'}{g-a} - \frac{f'}{f-a}.$$

Since f and g share $(a,1)^*$,

$$N(r,\gamma) = \overline{N}(r,\gamma) \le \overline{N}_{(2}(r, a; f) + S(r,f) = S(r,f).$$

Also, $m(r,\gamma) = S(r,f)$ and so $T(r,\gamma) = S(r,f)$.

From the definition of ϕ,

$$\phi\frac{f-b}{f'} = 1 - \frac{g-a}{f-a}. \tag{3.8}$$

Differentiating (3.8) and using (3.8) again,

$$(\phi+\gamma)\frac{f'}{f-b} - \phi\frac{f''}{f'} + \phi' - \gamma\phi = 0. \tag{3.9}$$

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Since $\overline{N}(r, b; f) \neq S(r, f)$ and $\overline{N}_{(2}(r, b; f) = S(r, f)$, from (3.9) we get $\phi + \gamma \equiv 0$. So,

$$\frac{f''}{f'} - \frac{\phi'}{\phi} + \frac{g'}{g-a} - \frac{f'}{f-a} = 0.$$

Proceeding as in Case I,

$$\overline{N}\Big(r, \frac{ac+1}{c}; g\Big) = S(r, g) \quad \text{and} \quad \overline{N}\Big(r, \frac{bc-1}{c}; f\Big) = S(r, f).$$

By the second fundamental theorem, we have $T(r, f) = \overline{N}(r, b; f) + S(r, f)$ and $T(r, g) = \overline{N}(r, a; g) + S(r, g)$. Since $\overline{N}(r, a; g) = \overline{N}(r, a; f) + S(r, g)$, it follows from Lemma 2.2 that $T(r, g) = S(r, g)$, which is a contradiction. This proves the theorem. □

Proof of Theorem 1.5. Let $g = L(f)$ and define ϕ as in the proof of Theorem 1.4. Since f and g share $(a, 1)^*$, $(b, 0)^*$ and $(\infty, 0)^*$, by Lemma 2.1, $S(r, f) = S(r, g)$. Suppose that $f \not\equiv g$. By the hypothesis, $T(r, \phi) = S(r, f)$. Since f and g share $(a, 1)^*$, as in the proof of Theorem 1.4, $\overline{N}_{(2}(r, a; f) = S(r, f)$.

We first suppose that $\overline{N}(r, b; f) = S(r, f)$. Then, by Lemma 2.2, $\overline{N}(r, a; f) \neq S(r, f)$. Proceeding as the proof of Case I of Theorem 1.4,

$$T(r, g) = \overline{N}(r, b; g) + S(r, g) = \overline{N}(r, b; f) + S(r, g) = S(r, g),$$

which is a contradiction. Therefore, $\overline{N}(r, b; f) \neq S(r, f)$. Now, proceeding as the proof of Case II of Theorem 1.4, we obtain (3.9).

Suppose that $\phi + \gamma \equiv 0$. Then, from (3.9),

$$\frac{f''}{f'} - \frac{\phi'}{\phi} + \frac{g'}{g-a} - \frac{f'}{f-a} = 0. \tag{3.10}$$

Integrating (3.10) and using the definition of ϕ,

$$c_1(f - g) = (g - a)(f - b), \tag{3.11}$$

where c_1 is a nonzero constant. Let z_1 be a b-point of f with multiplicity p and a b-point of g with multiplicity q. From (3.11), it follows that $p \leq q$. By the Taylor expansion in some neighbourhood of z_1, we get $f(z) - b = \alpha_p(z - z_1)^p + O(z - z_1)^{p+1}$ and $g(z) - b = \beta_q(z - z_1)^q + O(z - z_1)^{q+1}$, where $\alpha_p\beta_q \neq 0$.

We suppose that $p < q$. Then, in some neighbourhood of z_1,

$$\frac{f(z) - g(z)}{f(z) - b} = \frac{\alpha_p + O(z - z_1)}{\alpha_p + O(z - z_1)}.$$

Therefore, putting $z = z_1$ in (3.11), we get $c_1 = b - a$ and so again from (3.11) we obtain $(f - a)(g - b) \equiv 0$, which is a contradiction. Therefore, $p = q$ and so f and g share $(b, \infty)^*$. Then, by Theorem 1.4, $f \equiv g$, which is a contradiction.

Hence, $\phi + \gamma \not\equiv 0$. So, from (3.9),

$$\overline{N}_{1)}(r, b; f) \leq N(r, 0; \phi + \gamma) + S(r, f) = S(r, f).$$

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Let z_2 be a b-point of f with multiplicity greater than or equal to $n+2$. If z_2 is a b-point of g, then, from (1.1) and the hypothesis, $b = b_{-1}(z_2) + bb_0(z_2)$. If $b \not\equiv b_{-1}(z) + bb_0(z)$, then

$$\overline{N}_{(n+2}(r, b; f) \le N(r, b; b_{-1} + bb_0) + S(r, f) = S(r, f).$$

If $b \equiv b_{-1}(z) + bb_0(z)$, then, from (1.1), $g - f = (b_0 - 1)(f - b) + \sum_{j=1}^{n} b_j f^{(j)}$. Hence, if z_2 is not a pole of any one of b_j $(j = 0, 1, 2, \ldots, n)$, then z_2 is a multiple zero of $g - f$ and so is a zero of ϕ. Therefore, $\overline{N}_{(n+2}(r, b; f) \le N(r, 0; \phi) + \sum_{j=0}^{n} N(r, \infty; b_j) = S(r, f)$. Hence, in any case, $\overline{N}_{(n+2}(r, b; f) = S(r, f)$.

Next let z_3 be a b-point of f with multiplicity p $(2 \le p \le n+1)$. If z_3 is not a pole of $\phi' - \phi\gamma$, then we see from (3.9) that $\phi(z_3) + p\gamma(z_3) = 0$.

We suppose that $\phi(z) + p\gamma(z) \not\equiv 0$ for any $p \in \{2, 3, \ldots, n+1\}$. Then, from above,

$$\overline{N}_{n+1)}(r, b; f) - \overline{N}_{1)}(r, b; f) \le \sum_{p=2}^{n+1} N(r, 0; \phi + p\gamma) + N(r, \infty; \phi' - \phi\gamma) = S(r, f)$$

and so $\overline{N}_{n+1)}(r, b; f) = S(r, f)$. Therefore,

$$\overline{N}(r, b; f) = \overline{N}_{n+1)}(r, b; f) + \overline{N}_{(n+2}(r, b; f) = S(r, f),$$

which is a contradiction. Therefore, there exists a $p \in \{2, 3, \ldots, n+1\}$ such that $\phi(z) + p\gamma(z) \equiv 0$. Then, from (3.9),

$$\Big(1 - \frac{1}{p}\Big)\frac{f'}{f - b} - \frac{f''}{f'} + \frac{\phi'}{\phi} - \frac{g'}{g - a} + \frac{f'}{f - a} = 0.$$

Integrating and using the definition of ϕ,

$$(f - g)^p = c_2 (f - b)(g - a)^p, \tag{3.12}$$

where c_2 is a nonzero constant. Suppose that $\overline{N}(r, a; f) = S(r, f)$. Since f and g share $(a, 1)^*$, we have $\overline{N}(r, a; g) = S(r, f) = S(r, g)$. So, f and g share the value a CM*. Then, by Theorem G, there exists an entire function α such that $f = b + (a - b)(e^\alpha - 1)^2$. Hence, $f - a = (a - b)e^\alpha(e^\alpha - 2)$ and so

$$\overline{N}(r, a; f) = \overline{N}(r, 2; e^\alpha) + S(r, e^\alpha) = T(r, e^\alpha) + S(r, e^\alpha) = \tfrac{1}{2}T(r, f) + S(r, f),$$

which is a contradiction. Therefore, $\overline{N}(r, a; f) \ne S(r, f)$.

Let z_4 be an a-point of f and g with respective multiplicities q and s. From (3.12), we see that $s \le q$. We suppose that $s < q$. From (3.12), $c_2 = (-1)^p/(a - b)$. So, again from (3.12),

$$f = b + (-1)^p (a - b)(h - 1)^p \tag{3.13}$$

and

$$g = b + \frac{(a - b)(h - 1)}{h}[(-1)^p (h - 1)^{p-1} + 1],$$

where $h = (f - a)/(g - a)$. Since f is entire, from (3.13), we see that h is also entire. Also, (3.13) implies that

$$pT(r, h) = T(r, f) + S(r, f).$$

Further, we see that $\overline{N}(r, 0; h) \leq \overline{N}_{(2}(r, a; f) + S(r, f) = S(r, f) = S(r, h)$. Therefore, by the second fundamental theorem, $\overline{N}(r, d; h) \neq S(r, f)$ for a complex number d $(\neq 0, \infty)$ with $(-1)^p(d-1)^{p-1} + 1 = 0$. Since f and g share $(b, 0)^*$, we must have $p = 2$. Hence, $f - a = (a - b)h(h - 2)$ and $g - a = (a - b)(h - 2)$. Since z_4 is a common zero of $f - a$ and $g - a$, we have $s = q$, which is a contradiction to the supposition. Therefore, f and g share $(a, \infty)^*$. Now we achieve the result by Theorem G. This proves the theorem. □

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References

[1] C. A. Bernstein, D. C. Chang and B. Q. Li, 'On uniqueness of entire functions in $\mathbb{C}^n$ and their partial differential polynomials', *Forum Math.* **8** (1996), 379–396.
[2] G. Frank, 'Lecture notes on sharing values of entire and meromorphic functions', in: *Workshops in Complex Analysis,* Tianjin, China, 1991.
[3] Y. X. Gu, 'Uniqueness of an entire function and its differential polynomial', *Acta Math. Sin.* **37**(6) (1994), 791–798.
[4] W. K. Hayman, *Meromorphic Functions* (Clarendon Press, Oxford, 1964).
[5] I. Lahiri, 'Weighted sharing and uniqueness of meromorphic functions', *Nagoya Math. J.* **161** (2001), 193–206.
[6] I. Lahiri, 'Weighted value sharing and uniqueness of meromorphic functions', *Complex Var. Theory Appl.* **46**(3) (2001), 214–253.
[7] P. Li and C. C. Yang, 'Value sharing of an entire function and its derivatives', *J. Math. Soc. Japan* **51**(4) (1999), 781–799.
[8] P. Li and C. C. Yang, 'When an entire function and its linear differential polynomial share two values', *Illinois J. Math.* **44**(2) (2000), 349–362.
[9] E. Mues and N. Steinmetz, 'Meromorphe Funktionen die mit ihrer Ableitung Werte teilen', *Manuscripta Math.* **29** (1979), 195–206.
[10] L. A. Rubel and C. C. Yang, 'Values shared by an entire function and its derivatives', in: *Proc. Conf. Complex Analysis,* University of Kentucky, Lexington, KY, 1976, Lecture Notes in Mathematics, 599 (Springer, Berlin, 1977), 101–103.
[11] J. Wang, 'Entire functions that share two values with their derivatives', *Indian J. Pure Appl. Math.* **34**(3) (2003), 371–383.
[12] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions* (Science Press/Kluwer Academic, Dordrecht, 2003).
[13] L. Z. Yang, 'Entire functions that share finite values with their derivatives', *Bull. Aust. Math. Soc.* **41** (1990), 337–342.

INDRAJIT LAHIRI, Department of Mathematics, University of Kalyani,
West Bengal 741235, India
e-mail: ilahiri@hotmail.com