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# Remarks on endomorphisms and rational points

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Abstract

Let $X$ be an algebraic variety and let $f : X \dashrightarrow X$ be a rational self-map with a fixed point $q$, where everything is defined over a number field $K$. We make some general remarks concerning the possibility of using the behaviour of $f$ near $q$ to produce many rational points on $X$. As an application, we give a simplified proof of the potential density of rational points on the variety of lines of a cubic fourfold, originally proved by Claire Voisin and the first author in 2007.

## 1. Introduction

Let $X$ be an algebraic variety defined over a number field $K$. One says that the rational points are potentially dense in $X$, or that $X$ is potentially dense, if there is a finite extension $L$ of $K$, such that $X(L)$ is Zariski-dense.[1] For instance, unirational varieties are obviously potentially dense. A well-known conjecture of Lang affirms that a variety of general type cannot be potentially dense; more recently, the question of geometric characterization of potentially dense varieties has been raised by several mathematicians, for example, by Abramovich and Colliot-Thélène and especially by Campana [Cam04]. According to their points of view, one expects that the varieties with trivial canonical class should be potentially dense. This is well known for abelian varieties, but the simply connected case remains largely unsolved.

Bogomolov and Tschinkel [BT00] proved potential density of rational points for K3 surfaces admitting an elliptic pencil, or an infinite automorphism group. Hassett and Tschinkel [HT00] did this for certain symmetric powers of general K3 surfaces with a polarization of a suitable degree. The key observation of their work was that those symmetric powers are rationally fibered in abelian varieties over a projective space, and, as the elliptic K3 surfaces of [BT00], they admit a multisection with potentially dense rational points, which one can translate by suitable fiberwise rational self-maps to obtain the potential density of the ambient variety.

The K3 surfaces studied by Bogomolov and Tschinkel correspond to points lying on certain proper subvarieties in the moduli space of polarized K3. It is still unknown whether a K3 surface defined over a number field and 'general' in some sense, for instance, such that its Picard group is $\mathbb{Z}$, is potentially dense, and this question seems to be out of reach for the moment.

More recently, Amerik and Voisin [AV08] proved the potential density of the varieties of lines of certain sufficiently general cubic fourfolds $V$ defined over a number field. Such a variety of lines $X = \mathcal{F}(V)$ is an irreducible holomorphic symplectic fourfold, so it is a 'higher-dimensional

[1] As the referee observes, the question of potential density is relevant over an arbitrary field; most of our remarks from § 2 also work over any field of finite type over $\mathbb{Q}$, if one fixes a 'suitable' embedding of the field in $\mathbb{Q}_p$ for a suitable $p$.

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analogue' of a K3 surface. For generic $V$ (i.e. such that the corresponding point in the parameter space is outside of a countable union of proper subvarieties), $X$ has a cyclic Picard group. A theorem of Terasoma [Ter85] implies that this is also true for $X$ defined over a number field, provided that the parameter point is outside of a thin subset in the parameter space (where 'thin' is understood in the sense of Hilbert irreducibility, see [Ser92]).

So far these are the only examples of a simply connected variety defined over a number field, with trivial canonical class and cyclic Picard group, where the potential density is established. The starting idea is in some sense similar to that of [BT00]: as noticed in [Voi04], $X$ admits a rational self-map $f$ of degree 16. The map $f$ is defined as follows: for a general line $l \subset V$, there is a unique plane $P$ tangent to $V$ along $l$, and the map $f$ sends $l$ to the line $l'$ residual to $l$ in the intersection $P \cap V$.

Moreover, $X$ carries a two-parameter family $\Sigma_b, b \in B$, of surfaces birational to abelian surfaces; each $\Sigma_b$ parameterizes lines contained in a hyperplane section of $V$ with three double points. It is proved in [AV08] that under certain genericity conditions on the pair $(X, b)$, satisfied by many pairs defined over a number field, the union of the iterates $f^n(\Sigma_b), n \in \mathbb{N}$ is Zariski-dense in $X$. Since rational points are potentially dense on abelian varieties, this clearly implies that $X$ is potentially dense. The proof is done in two steps: first one shows that for many $(X, b)$ defined over a number field, the number of iterates $f^n(\Sigma_b)$ is infinite, so that these iterates are at least Zariski-dense in a divisor $D$. Already, at this first step, the proof is very involved, using, for instance, $\ell$-adic Abel–Jacobi invariants in the continuous étale cohomology. Using more geometry and some other $\ell$-adic Abel–Jacobi invariants, one excludes the case when the Zariski closure of $\bigcup_n f^n(\Sigma_b)$ is a divisor. As a consequence of the proof, the genericity conditions one has to impose on the pair $(X, b)$ are rather complicated, and it is not obvious whether one can check in practice if they are satisfied for a given $(X, b)$ and if a given $X$ is potentially dense.

The purpose of this article is to further investigate the connection between the existence of 'sufficiently non-trivial' rational self-maps and the potential density of rational points. In the first part we prove, among other facts, that if the differential of a rational self-map $f : X \dashrightarrow X$ at a non-degenerate fixed point $q \in X(\bar{\mathbb{Q}})$ has multiplicatively independent eigenvalues, then the rational points are potentially dense on $X$ (Corollary 2.7). More precisely, we show that, under this condition, one can find a point $x \in X(\bar{\mathbb{Q}})$ such that the set of its iterates is Zariski-dense. Note that this remains unknown for $X$ and $f$ as in [AV08], though the question has been raised in [AC08] where it is shown that the map $f : X \dashrightarrow X$ does not preserve any rational fibration and therefore the set of the iterates of a general complex point of $X$ is Zariski-dense. A related conjecture is stated in [Zha06], see Remark 2.12.

Unfortunately, it seems to be difficult to find interesting examples with multiplicatively independent eigenvalues of the differential at a fixed point. There are certainly plenty of such self-maps on rational varieties, but, since for those the potential density is obvious, we cannot consider their self-maps as being 'interesting'. In the case of [AV08], the eigenvalues of $Df_q$ at a fixed point $q$ are $1, 1, -2$ and $-2$ (Proposition 3.3). We do not know whether the multiplicative independence could hold for some fixed point of a power of $f$.

Nevertheless, even the independence of certain eigenvalues gives interesting new information. To illustrate this, we exploit this point of view in the second part, where a simplified proof of the potential density of the variety of lines of the cubic fourfold is given. In fact, we actually prove a somewhat stronger result. The first step of [AV08], that is, the existence of a surface $\Sigma \subset X$, defined over a number field, which is birationally abelian and not preperiodic under $f$, now becomes an immediate consequence of our general remarks. In particular, it holds for the variety of lines of an arbitrary cubic fourfold. On the second step, we do need some extra geometry since our eigenvalues are very far from the multiplicative independence, and, as we perform it, some genericity conditions do appear. However, it is sufficient for us to ask that $\mathrm{Pic}(X)=\mathbb{Z}$ in order to show that the iterates $f^n(\Sigma)$ are Zariski-dense. Let us remark that this condition can be verified, with an appropriate computer program, for an explicitly given $X$, in the same way as it is done by van Luijk [vL07] for quartic K3 surfaces.

The paper is organized as follows: in § 2, we consider a smooth projective variety $X$ with a rational self-map $f: X \dashrightarrow X$ which has a fixed point $q$, where $X$, $f$, $q$ are defined over a number field $K$. We show (Proposition 2.2) that for a suitable prime $\mathfrak{p}$, the point $q$ has an invariant $\mathfrak{p}$-adic neighbourhood. Looking at the behaviour of $f$ in this neighbourhood, we make a few observations about the global properties of $f$ under certain conditions on the eigenvalues of $Df_q$ (Corollaries 2.7 and 2.9). Our main tool is a linearization result (Theorem 4.1), due to Herman and Yoccoz in the case when $q$ is isolated. In § 4, we explain how to adapt their proof to the case of non-isolated fixed points, which we need for our main application; that is, the new proof of the potential density of the variety of lines on a cubic fourfold, explained in § 3 (our final result is Theorem 3.11). Finally, in § 5 we prove a version of Noether normalization lemma which we need in order to work in our invariant $\mathfrak{p}$-adic neighbourhood when $q$ is not isolated; this section is added following a referee's remark since we could not find a proof of this lemma in the literature.

As a final remark for this introduction, let us mention that the variety of lines on a cubic fourfold is the first in a series of examples of varieties with trivial canonical class admitting a rational self-map of high degree. Indeed, as observed by Voisin in [Voi04], one can look at the variety of $k$-dimensional projective subspaces contained in a cubic of dimension $n$; as soon as $n$ and $k$ are chosen in such a way that the canonical class of this variety is trivial, the same construction as above gives a rational self-map. It can be interesting to see what our method gives for these examples; however, this is not an easy question since their dimension is very high. More generally, it is a challenging problem to look for families of simply connected varieties with trivial canonical class, such that the general member admits a rational self-map of high degree. It is unclear when should such self-maps exist; on the other hand, so far there is only one negative result in this direction: a few months after this paper had been written, Chen [Che] proved that a generic K3 surface does not admit a rational self-map of degree greater than one.

Let us also mention that for varieties of lines of some special cubic fourfolds (roughly, of those admitting a hyperplane section with six double points), the potential density is shown in [HT] using the existence of an infinite-order birational *automorphism* on such varieties.

While finishing the writing of this paper, we have learned of a recent article [GT09], where a question of a different flavour is approached by similar methods.

## 2. Invariant neighbourhoods

Let $X$ be a smooth projective variety of dimension $n$ and let $f: X \dashrightarrow X$ be a rational self-map, both defined over a 'sufficiently large' number field $K$. We assume that $f$ has a fixed point $q \in X(K)$. This assumption is not restrictive if, for example, $f$ is a regular and polarized (that is, such that $f^*\mathcal{L}=\mathcal{L}^{\otimes k}$ for a certain ample line bundle $\mathcal{L}$ and an integer $k>1$) endomorphism: indeed, in this case the set of periodic points in $X(\bar{\mathbb{Q}})$ is even Zariski-dense [Fak03], so replacing $f$ by a power and taking a finite extension of $K$ if necessary, we find a fixed point.

We denote by $\mathcal{O}_K$ the ring of integers of the number field $K$; for a point $\mathfrak{p}$, i.e. an equivalence class of valuations of $K$, $K_{\mathfrak{p}}$ denotes the corresponding completion, $\mathcal{O}_{\mathfrak{p}}$ the ring of integers in $K_{\mathfrak{p}}$.

Our starting point is that, for any fixed point $q \in X(K)$ and a suitable prime ideal $\mathfrak{p} \subset \mathcal{O}_K$, we can find a '$\mathfrak{p}$-adic neighbourhood' $O_{\mathfrak{p},q}$ of $q$, which is a subset of $X(K_{\mathfrak{p}})$ containing $q$, on which $f$ is well defined and which is $f$-invariant.

More precisely, choose an affine neighbourhood $U \subset X$ of $q$, such that the restriction of $f$ to $U$ is regular. By Noether normalization lemma, there is a finite $K$-morphism $\pi = (x_1, \dots, x_n) : U \longrightarrow \mathbb{A}^n_K$ to the affine space. Recall that it can be obtained by embedding $U$ into an affine space and projecting from a suitable linear subspace at infinity; it is easy to see that this subspace can moreover be chosen in such a way that $\pi$ is étale at $q$. (In § 5, we prove a more general fact, Theorem 5.2, which should be well known but for which we could not find a proof in the literature.) We may suppose that $q$ maps to 0.

The $K$-algebra $\mathcal{O}(U)$ is integral over $K[x_1, \dots, x_n]$, i.e. it is generated over $K[x_1, \dots, x_n]$ by some regular functions $x_{n+1}, \dots, x_m$ integral over $K[x_1, \dots, x_n]$. The coordinate ring of $U$ is included into the local ring of $q$ and the latter is included into its completion: $\mathcal{O}(U) \subset \mathcal{O}_{U,q} \subset \widehat{\mathcal{O}}_{U,q} = K[[x_1, \dots, x_n]]$. In particular, $x_{n+1}, \dots, x_m$ become elements of $K[[x_1, \dots, x_n]]$. As $f^*$ defines an endomorphism of the ring $\mathcal{O}_{U,q}$ and of its completion, the functions $f^*x_1, \dots, f^*x_m$ also become power series in $x_i$ with coefficients in $K$.

We claim that the coefficients of the power series $x_{n+1}, \dots, x_m, f^*x_1, \dots, f^*x_m$ are $\mathfrak{p}$-integral for almost all primes $\mathfrak{p}$, that is,

$$x_{n+1}, \dots, x_m, f^*x_1, \dots, f^*x_m \in \mathcal{O}_K[1/N][[x_1, \dots, x_n]]$$

for some integer $N \geqslant 1$. This is a consequence of the following well-known result (a stronger version for $n = 1$ goes back to Eisenstein).

LEMMA 2.1. *Let $k$ be a field of characteristic zero and let $\phi \in k[[x_1, \dots x_n]]$ be a function algebraic over $k(x_1, \dots, x_n)$. Then $\phi \in A[[x_1, \dots x_n]]$, where $A$ is a finitely generated $\mathbb{Z}$-subalgebra of $k$.*

*Proof.* Let $G$ be a minimal polynomial of $\phi$ over $k[x_1, \dots, x_n]$, so $G(\phi) = 0$ and $G'(\phi) \neq 0$. Then $G'(\phi) \in \mathfrak{m}^s \smallsetminus \mathfrak{m}^{s+1}$ for some $s \geqslant 0$, where $\mathfrak{m}$ is the maximal ideal in $k[[x_1, \dots, x_n]]$.

Denote by $\phi_d$ the only polynomial of degree less than $d$ congruent to $\phi$ modulo $\mathfrak{m}^d$. For a formal series $\Phi$ in $x$ and an integer $m$ denote by $\Phi_{(m)}$ the homogeneous part of $\Phi$ of degree $m$. Clearly, $G'(\phi_d)_{(m)}$ is independent of $d$ for $d > m$.

We are going to show by induction on $d$ that the coefficients of the homogeneous component of $\phi$ of degree $d$ belong to the $\mathbb{Z}$-subalgebra of $k$ generated by coefficients of $G$ (as a polynomial in $n+1$ variables), by coefficients of $\phi_{s+1}$ and by the inverse of a certain (non-canonical) polynomial $D$ in coefficients of $G$ and in coefficients of $\phi_{s+1}$. To define $D$, choose a valuation $v$ of rank $n$ of the field $k(x_1, \dots, x_n)$, trivial on $k$, such that $v(x_i) > v(k(x_1, \dots, x_{i-1})^\times)$ for all $1 \leqslant i \leqslant n$ (equivalently, $0 < v(x_1^{m_1}) < \dots < v(x_n^{m_n})$ for all $m_1, \dots, m_n > 0$). Then $D$ is the coefficient of the monomial in $G'(\phi_{s+1})_{(s)}$ with the minimal valuation (such a monomial is unique, in fact two distinct monomials have distinct valuations).

For $d \leqslant s$ there is nothing to prove, so let $d > s$. Let $\Delta_d := \phi - \phi_d$, so $\Delta_d \in \mathfrak{m}^d$. Then $0 = G(\phi_d + \Delta_d) \equiv G(\phi_d) + G'(\phi_d)\Delta_d \pmod{\Delta_d^2}$, so in particular, $G(\phi_d)_{(d+s)} + G'(\phi_d)_{(s)}(\Delta_d)_{(d)} = 0$, or equivalently, $G(\phi_d)_{(d+s)} + G'(\phi_{s+1})_{(s)}\phi_{(d)} = 0$, and thus, $\phi_{(d)} = -G(\phi_d)_{(d+s)}/G'(\phi_{s+1})_{(s)}$.

The field of rational functions $k(x_1, \dots, x_n)$ is embedded into its completion with respect to $v$, and by our choice of $v$ this completion can be identified with the field of iterated Laurent series $k((x_1)) \cdots ((x_n))$. In particular, $(G'(\phi_{s+1})_{(s)})^{-1}$ becomes an iterated Laurent series, whose coefficients are polynomials over $\mathbb{Z}$ in the coefficients of $G'(\phi_{s+1})_{(s)}$ and in $D^{-1}$ (write $G'(\phi_{s+1})_{(s)}$ as a product of its minimal valuation monomial and a rational function, then write the inverse of the rational function as a geometric series). Then, by induction assumption, the coefficients of $\phi_{(d)}$ are polynomials over $\mathbb{Z}$ in coefficients of $G$, coefficients of $\phi_{s+1}$ and in $D^{-1}$. □

Therefore, for almost all primes $\mathfrak{p} \subset \mathcal{O}_K$, the coefficients of the power series $x_{n+1}, \ldots, x_m, f^*x_1, \ldots, f^*x_n$ are integral in $K_{\mathfrak{p}}$. Choose a $\mathfrak{p}$ satisfying this condition and such that, moreover, the irreducible polynomials $P_i(T) = P_i(x_1, \ldots, x_n; T) \in K[x_1, \ldots, x_n; T]$ which are minimal monic polynomials of $x_i$ for $n < i \leqslant m$, have $\mathfrak{p}$-integral coefficients and the elements $P_i'(0, \ldots, 0; x_i(q))$ are invertible in $\mathcal{O}_{\mathfrak{p}}$ for each $n < i \leqslant m$ (this last condition holds for almost all primes $\mathfrak{p}$ since the morphism $x$ is étale at $q$).

Define the system of $\mathfrak{p}$-adic neighbourhoods $O_{\mathfrak{p},q,s}$, $s \geqslant 1$, of the point $q$ as follows:

$$O_{\mathfrak{p},q,s} = \{t \in U(K_{\mathfrak{p}}) \mid x_i(t) \equiv x_i(q) \bmod \mathfrak{p}^s \text{ for all } 1 \leqslant i \leqslant m\}.$$

We set $O_{\mathfrak{p},q} := O_{\mathfrak{p},q,1}$.

PROPOSITION 2.2. (1) *For any $s \geqslant 1$, the functions $x_1, \ldots, x_n$ give a bijection between $O_{\mathfrak{p},q,s}$ and the $n$th cartesian power of $\mathfrak{p}^s$.*

(2) *The set $O_{\mathfrak{p},q}$ contains no indeterminacy points of $f$.*

(3) *One has $f(O_{\mathfrak{p},q,s}) \subseteq O_{\mathfrak{p},q,s}$ for any $s \geqslant 1$. Moreover, $f : O_{\mathfrak{p},q,s} \xrightarrow{\sim} O_{\mathfrak{p},q,s}$ is bijective if $\det Df_q$ is invertible in $\mathcal{O}_{\mathfrak{p}}$.*

(4) *The $\bar{\mathbb{Q}}$-points of $X$ are dense in $O_{\mathfrak{p},q,s}$ for any $s \geqslant 1$.*

*Proof.* These properties are clear from the definition and the inclusion of the elements $x_1, \ldots, x_m, f^*x_1, \ldots, f^*x_m$ into $\mathcal{O}_{\mathfrak{p}}[[x_1, \ldots, x_n]]$.

(1) The map $x$ from $O_{\mathfrak{p},q,s}$ to the $n$th cartesian power of $\mathfrak{p}^s$ is injective, since the coordinates $x_{n+1}, \ldots, x_m$ of a point $t$ are determined uniquely by the coordinates $x_1, \ldots, x_n$ and the condition $t \in O_{\mathfrak{p},q}$. Indeed, consider the equation $P_i(x_i) = P_i(x_1, \ldots, x_n; x_i) = 0$, where $P_i$ is the minimal monic polynomial of $x_i$ for $n < i \leqslant m$. For fixed values of $x_1, \ldots, x_n \in \mathfrak{p}$ this equation has precisely $\deg P_i$ solutions (with multiplicities) in $\bar{K}_{\mathfrak{p}}$. As $P_i'(x_i(q)) \in \mathcal{O}_{\mathfrak{p}}^{\times}$, $x_i(t) \equiv x_i(q) \pmod{\mathfrak{p}}$ is a simple root of $P_i \pmod{\mathfrak{p}}$, and thus, any root congruent to $x_i(q)$ modulo $\mathfrak{p}$ is not congruent to any other root modulo $\mathfrak{p}$.

It is surjective, since $x_{n+1}, \ldots, x_m$ are convergent series on $\mathfrak{p}^s$ with constant values modulo $\mathfrak{p}^s$.

(2) The functions $f^*x_i$, $1 \leqslant i \leqslant m$, are convergent series on $O_{\mathfrak{p},q}$.

(3) The functions $f^*x_i$ on the $n$th cartesian power of $\mathfrak{p}^s$, $1 \leqslant i \leqslant m$, are constant modulo $\mathfrak{p}^s$. This shows that $f(O_{\mathfrak{p},q,s}) \subseteq O_{\mathfrak{p},q,s}$. If $\det Df_q \neq 0$ then the inverse map $f^{-1}$ is well defined in a neighbourhood of 0. If $\det Df_q \in \mathcal{O}_{\mathfrak{p}}^{\times}$ then $f^{-1}$ is defined by series in $\mathcal{O}_{\mathfrak{p}}[[x_1, \ldots, x_n]]$, i.e. it is well defined on $O_{\mathfrak{p},q}$.

(4) The $\bar{\mathbb{Q}}$-points are dense in the $n$th cartesian power of $\mathfrak{p}^s$ and they lift uniquely to algebraic points of $O_{\mathfrak{p},q,s}$. □

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the tangent map $Df_q$. We assume from now on that $q$ is a non-degenerate fixed point of $f$, meaning, by definition, that $\lambda_i \neq 0$ for all $i$. Note that the $\lambda_i$ are algebraic numbers. Extending, if necessary, the field $K$, we may assume that $\lambda_i \in K$. The following is a consequence of the $p$-adic versions of several well-known results in dynamics and number theory.



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Proposition 2.3. *Assume that $\lambda_1, \ldots, \lambda_n$ are multiplicatively independent. Then in some $\mathfrak{p}$-adic neighbourhood $O_{\mathfrak{p},q,s}$, the map $f$ is equivalent to its linear part $\Lambda$ (i.e. there exists a formally invertible $n$-tuple of formal power series $h = (h^{(1)}, \ldots, h^{(n)})$ in $n$ variables $(x_1, \ldots, x_n) = x$, convergent together with its formal inverse on a neighbourhood of zero, such that $h(\lambda_1 x_1, \ldots, \lambda_n x_n) = f(h(x_1, \ldots, x_n)))$.*

*Proof.* It is well known that in absence of relations

$$\lambda_1^{m_1} \cdots \lambda_n^{m_n} = \lambda_j, 1 \leqslant j \leqslant n, \qquad m = \sum m_i \geqslant 2, m_i \geqslant 0,$$

known as 'resonances', there is a unique formal linearization of $f$, obtained by formally solving the equation $f(h(x)) = h(\Lambda(x))$; the expressions $\lambda_1^{m_1} \cdots \lambda_n^{m_n} - \lambda_j$ appear in the denominators of the coefficients of $h$ (see for example [Arn88]). The problem is of course whether $h$ has non-zero radius of convergence, that is, whether the denominators are 'not too small'. By Siegel's theorem (see [HY83] for its $p$-adic version) this is the case as soon as the numbers $\lambda_i$ satisfy the diophantine condition

$$|\lambda_1^{m_1} \cdots \lambda_n^{m_n} - \lambda_j|_p > Cm^{-\alpha}$$

for some $C, \alpha > 0$. By [Yu90], this condition is always satisfied by the algebraic numbers. $\square$

When the fixed point $q$ is not isolated, the eigenvalues $\lambda_i$ are always resonant. However, as follows from the results proved in § 4, if 'all resonances come from the fixed subvariety', the linearization is still possible.

More precisely, extending $K$ if necessary, we may choose $q \in X(K)$ which is a smooth point of the fixed point locus of $f$. Let $F$ be the irreducible component of this locus containing $q$. Let $r = \dim F$. By a version of Noether's normalization lemma (Theorem 5.2 below), we may assume that our finite morphism $\pi = (x_1, \ldots, x_n) : U \longrightarrow \mathbb{A}^n_K$ which is étale at $q$, maps $F \cap U$ onto the coordinate plane $\{x_{r+1} = \cdots = x_n = 0\}$.

Proposition 2.4. *Let $q$ be a smooth point of the fixed point locus of $f$ as above. Suppose that the tangent map $Df_q$ is semisimple and that its eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfy the condition $\lambda_{r+1}^{m_{r+1}} \cdots \lambda_n^{m_n} \neq \lambda_i$ for all integer $m_{r+1}, \ldots, m_n \geqslant 0$ with $m_{r+1} + \cdots + m_n \geqslant 2$ and all $i$, $r < i \leqslant n$ (and $\lambda_1 = \cdots = \lambda_r = 1$).*

*Suppose moreover that the eigenvalues of $Df_q$ are constant in a neighbourhood of $q$ in $F$. Then, for each $q$ as above, the map $f$ can be linearized in some $\mathfrak{p}$-adic neighbourhood $O_{\mathfrak{p},q,s}$ of $q$, i.e. there exists a formally invertible $n$-tuple of formal power series $h = (h^{(1)}, \ldots, h^{(n)})$ in $n$ variables $(x_1, \ldots, x_n) = x$, convergent together with its formal inverse on a neighbourhood of zero, such that $h(\lambda_1 x_1, \ldots, \lambda_n x_n) = f(h(x_1, \ldots, x_n))$.*

*Proof.* Taking into account that the $\lambda_i$ are algebraic numbers and so the bad diophantine approximation condition

$$|\lambda_{r+1}^{m_{r+1}} \cdots \lambda_n^{m_n} - \lambda_j|_p > C\biggl(\sum_{r<i\leqslant n} m_i\biggr)^{-\alpha}$$

is automatically satisfied for some $C, \alpha > 0$, whenever $\sum_{r<i\leqslant n} m_i \geqslant 2$; this is just Theorem 4.1 below. $\square$

*Remark* 2.5. It might be worthwhile to mention explicitly that since the sum of the $m_i$ in the condition of this proposition must be at least two, the relation $\lambda_i = \lambda_j$ is *not* a resonance for $i, j$



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between $r+1$ and $n$, in other words, linearization is still possible when some (or even all) of the non-trivial eigenvalues are equal.

In order to apply these propositions to the study of iterated orbits of algebraic points, we need the following lemma.[2]

LEMMA 2.6. *Let $a_1, a_2, \dots$ be a sequence in $K_{\mathfrak{p}}^{\times}$ tending to 0. Let $b_1, b_2, \dots$ be a sequence of pairwise distinct elements in $\mathcal{O}_{\mathfrak{p}}^{\times}$. Then we have the following.*

(1) *There exists an $s \in \mathbb{N}$ such that $\sum_{i \geqslant 1} a_i b_i^s \neq 0$.*

(2) *If, moreover, $b_i/b_j$ is not a root of unity for any $i > j \geqslant 1$, then any infinite subset $S \subset \mathbb{N}$ contains an element $s$ such that $\sum_{i \geqslant 1} a_i b_i^s \neq 0$ (in other words, the set of $s \in \mathbb{N}$ such that $\sum_{i \geqslant 1} a_i b_i^s = 0$ is finite).*

*Proof.* Renumbering if necessary, we may suppose that $|a_1| = |a_2| = \cdots = |a_N| > |a_i|$ for any $i > N$. Suppose that $\sum_{i \geqslant 1} a_i b_i^s = 0$ for every $s \in S$, where $S$ is a subset of $\mathbb{N}$.

(1) First assume that $S = \mathbb{N}$. By our assumption, for any polynomial $P$, $\sum_{i \geqslant 1} a_i P(b_i) = 0$. By the triangle inequality, we shall get a contradiction as soon as we find a polynomial $P$ such that $|P(b_k)| < |P(b_1)|$ for $2 \leqslant k \leqslant N$ and $|P(b_k)| \leqslant |P(b_1)|$ for $k > N$. To construct such a $P$, choose an ideal $\mathfrak{q} = \mathfrak{p}^s$ such that the $b_i$ are pairwise distinct modulo $\mathfrak{q}$ for $i = 1, \dots, N$ and let

$$P(x) = \prod_{b \in \mathcal{O}_{\mathfrak{p}}/\mathfrak{q},\, b_1 \notin b} (x - \bar{b}),$$

where $\bar{b}$ denotes any representative of the class $b$. An easy check gives that $|P(x)| = |P(b_1)|$ when $x \equiv b_1 \pmod{\mathfrak{q}}$ and $|P(x)| < |P(b_1)|$ otherwise, so $P$ has the required properties.

(This polynomial $P$ has been indicated to us by A. Chambert-Loir.)

(2) Now let $S \subset \mathbb{N}$ be an arbitrary infinite set. Take an integer $M > 0$ such that $|b^M - 1| < |p|^{1/(p-1)}$ for all $b \in \mathcal{O}_{\mathfrak{p}}^{\times}$ (this is possible since $\mathcal{O}_{\mathfrak{p}}^{\times}$ is compact, equivalently, the residue rings are finite), then $c_i := \log b_i^M$ is defined for all $i$ and $\exp(c_i) = b_i^M$. Since no quotient $b_i/b_j$ is a root of unity, the $c_i$ are pairwise distinct. We claim that for a certain sequence $a'_1, a'_2, \dots$ in $K_{\mathfrak{p}}^{\times}$ tending to 0, we can write identities $\sum_{i \geqslant 1} a'_i c_i^m = 0$ for all $m \in \mathbb{N}$, so this case reduces to that of $S = \mathbb{N}$.

Indeed, consider $S$ as a subset of $\mathbb{Z}_p$; it has a limit point $s_0$. For a sequence of $m$-tuples $j_1 > j_2 > \cdots > j_m$ in $S$ such that $j_1 \equiv j_2 \equiv \cdots \equiv j_m \pmod{M}$ and $\lim j_1 = \lim j_2 = \cdots = \lim j_m = s_0$, and any analytic function $g$ on $\mathbb{Z}_p$, one has

$$\frac{g^{(m)}(s_0)}{m!} = \lim \sum_{l=1}^{m} \frac{g(j_l)}{\prod_{k \neq l} (j_l - j_k)}$$

(the Newton interpolation formula).

Pick a class modulo $M$ which contains a sequence in $S$ converging to $s_0$. There is an analytic function $g_i$ such that $g_i(k) = b_i^k$ when $k$ is in this class. By the formula above, we have

$$g_i^{(m)}(s_0) = \frac{c_i^m}{M^m} g_i(s_0) = m! \lim \sum_{l=1}^{m} \frac{g(j_l)}{\prod_{k \neq l} (j_l - j_k)},$$

[2] We thank the referee for pointing out that somewhat stronger assumptions from a previous version are not used in the proof.



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where all the $j_l$ are in the same class modulo $M$. This gives

$$\sum_{i=1}^{\infty} a_i g_i(s_0) c_i^m = M^m m! \lim \sum_{l=0}^{m} \frac{1}{\prod_{k \neq l}(j_l - j_k)} \sum_{i=1}^{\infty} a_i g_i(j_l) = 0$$

since the $j_l$ are in $S$. This holds for any $m \in \mathbb{N}$ and the $c_i$ are pairwise distinct, so that we are back to the case $S = \mathbb{N}$. □

The referees have pointed out the relation of this lemma with the $p$-adic proof of the Skolem–Mahler–Lech theorem, where the vanishing of finite sums $\sum_{i=1}^{k} a_i b_i^s$ for a certain fixed number $k$ is dealt with.

From now on, we assume that $\mathfrak{p}$ is chosen so that all $\lambda_i$ belong to $\mathcal{O}_{\mathfrak{p}}^{\times}$ (this is of course the case for almost all $\mathfrak{p}$).

The first part of the lemma (case $S = \mathbb{N}$) immediately implies the following corollary.

Corollary 2.7. *If $\lambda_1, \ldots, \lambda_n$ are multiplicatively independent, the rational points on $X$ are potentially dense. More precisely, there is a point $t \in X(\bar{\mathbb{Q}})$ with Zariski-dense iterated orbit $\{f^i(t) \mid i \in \mathbb{N}\}$.*

*Proof.* Since the algebraic points are dense in $O_{\mathfrak{p},q,s}$ and the union of the $y$-coordinate hyperplanes is a proper closed subset, we can find a point $t \in X(\bar{\mathbb{Q}})$ which is contained in $O_{\mathfrak{p},q,s}$, away from the coordinate hyperplanes in the local coordinates $(y_1, \ldots, y_n)$ linearizing $f$. We claim that the iterated orbit of such a point is Zariski-dense in $X$. Indeed, if not, there is a regular function $G$ on $U$ vanishing at $f^i(t)$ for all $i$; in the local linearizing coordinates on $O_{\mathfrak{p},q,s}$, $G$ becomes a convergent power series $G = \sum_I a_I y^I$. If $t = (t_1, \ldots, t_n)$, we get $\sum_I a_I t^I (\lambda^I)^i = 0$ for all $i \in \mathbb{N}$. Since the $\lambda_i$ are multiplicatively independent, the $\lambda^I$ are pairwise distinct, contradicting Lemma 2.6(1). □

Another useful version of this corollary is the following.

Corollary 2.8. *Let $n$ be a natural number, $\mathcal{T}$ be the $n$th cartesian power of $\mathcal{O}_{\mathfrak{p}}^{\times}$ and $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{T}$, where $\lambda_1, \ldots, \lambda_n$ are multiplicatively independent. Let $S \subset \mathbb{N}$ be an infinite subset. Then the set $\{\Lambda^i \mid i \in S\}$ is 'analytically dense' in $\mathcal{T}$, that is, for any non-zero Laurent series $\Phi = \sum_I a_I y^I$ convergent on $\mathcal{T}$, there is an $s \in S$ such that $\Phi(\Lambda^s) \neq 0$.*

*Proof.* Otherwise, after a renumbering of $a_I$ as $a_i$ and setting $b_i = \lambda^I$ for $i$ corresponding to $I$, we get $\sum_{i \geqslant 1} a_i b_i^s = 0$ for all $s \in S$. This contradicts Lemma 2.6. □

The case $S \neq \mathbb{N}$ yields the following remark which is useful in the study of the case when the fixed point $q$ is not isolated (as in the next section).

Corollary 2.9. *Under assumptions of Proposition 2.4, let $Y \subset U$ be such an irreducible subvariety that, possibly after a finite field extension, $Y(K_{\mathfrak{p}})$ meets a sufficiently small (i.e. where $f$ can be linearized) $\mathfrak{p}$-adic neighbourhood of $q$. Suppose that the multiplicative subgroup $H \subset \mathcal{O}_{\mathfrak{p}}^{\times}$ generated by the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $Df_q$ is torsion-free. Let $S \subset \mathbb{N}$ be an infinite subset. Then the Zariski closure of the union $\bigcup_{i \in S} f^i(Y)$ is independent of $S$, and, therefore, is irreducible.*

*Proof.* To check the independence of $S$, let us show that any regular function $G$ on $U$ vanishing on $\bigcup_{i \in S} f^i(Y)$ vanishes also on $f^i(Y)$ for any $i \geqslant 1$. The vanishing of $G$ on an irreducible subvariety $Z$ is equivalent to the vanishing of $G$ at a generic point of $Z$. Here, for a field extension $L/K$, a 'generic point' of $Z$ is an $L$-rational point in the complement to the union of all divisors on $Z$ defined over $K$ (i.e. a field embedding $K(Z) \hookrightarrow L$ over $K$).



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Let $y_1, \ldots, y_n$ be local coordinates at $q$ linearizing and diagonalizing $f$ on a neighbourhood $O_{\mathfrak{p},q,s}$ meeting $Y(K_{\mathfrak{p}})$. We claim that, possibly after a finite extension of $K$, $Y(K_{\mathfrak{p}})$ meets $O_{\mathfrak{p},q,s}$ at a generic point. Indeed, after a finite extension of $K$, one can find a point $u$ of $Y$ in $O_{\mathfrak{p},q,s}$ and then 'deform' $u$ to a generic point.[3]

As $f^i(Y)(K_{\mathfrak{p}}) \cap O_{\mathfrak{p},q,s}$ contains $f^i(Y(K_{\mathfrak{p}}) \cap O_{\mathfrak{p},q,s})$, each $f^i(Y)(K_{\mathfrak{p}})$ meets $O_{\mathfrak{p},q,s}$ at a generic point.

The function $G$ becomes a convergent power series $\sum_I a_I y^I$ on $O_{\mathfrak{p},q,s}$. For any point $t = (t_1, \ldots, t_n) \in Y(K_{\mathfrak{p}}) \cap O_{\mathfrak{p},q,s}$ the series $G$ vanishes at the points $f^i(t)$ for all $i \in S$ (that is, $\sum_I a_I \lambda^{iI} t^I = \sum_{I: t^I \neq 0} a_I \lambda^{iI} t^I = 0$ for all $i \in S$) and it remains to show that $G$ vanishes also at the points $f^i(t)$ *for all* $i \geqslant 1$.

The set of multi-indices $I \in \mathbb{Z}^n_{\geqslant 0}$ splits into the following equivalence classes: $I \sim I'$ if $\lambda^I = \lambda^{I'}$. By Lemma 2.6, the sum of $a_I t^I$ over each equivalence class is zero, and thus, $\sum_I a_I \lambda^{iI} t^I = 0$ for all $i \geqslant 1$.

At least one of the irreducible components of the Zariski closure of $\bigcup_{i\in S} f^i(Y)$ contains infinitely many of the $f^i(Y)$. Any such component contains $\bigcup_{i \geqslant 1} f^i(Y)$, i.e. the Zariski closure of $\bigcup_{i\in S} f^i(Y)$ is irreducible. □

Finally, the following is an obvious generalization of Corollary 2.7.

COROLLARY 2.10. *Under the assumptions of Proposition 2.4, let $t$ be a sufficiently general algebraic point of $U$ in a sufficiently small $\mathfrak{p}$-adic neighbourhood of $q$ and $H$ be the multiplicative group generated by $\lambda_1, \ldots, \lambda_n$. Then the dimension of the Zariski closure of the $f$-orbit $\{f^i(t) \mid i \in \mathbb{N}\}$ is greater than or equal to the rank $r$ of $H$. In particular, if the eigenvalues of the tangent map $Df_q$ are multiplicatively independent then the rational points on $X$ are potentially dense.*

(Here, as in the previous corollary, a sufficiently small neighbourhood is such a neighbourhood that $f$ is linearizable in it, and a sufficiently general algebraic point is just an algebraic point outside the union of the coordinate hyperplanes in the linearizing coordinates.)

*Proof.* Let $y_1, \ldots, y_n$ be local coordinates at $q$ linearizing and diagonalizing $f$ in a neighbourhood $O_{\mathfrak{p},q,s}$. Replacing $f$ by a power, we may assume that the eigenvalues $\lambda_i$ generate a torsion-free group. Take a point $t = (t_1, \ldots, t_n) \in O_{\mathfrak{p},q,s}$ away from the coordinate hyperplanes and consider the natural embedding of the $n$th cartesian power $\mathcal{T}$ of $\mathcal{O}_{\mathfrak{p}}^{\times}$:

$$t : \mathcal{T} \to O_{\mathfrak{p},q,s}, \quad (\mu_1, \ldots, \mu_n) \mapsto (\mu_1 t_1, \ldots, \mu_n t_n).$$

On the image of $\mathcal{T}$, we can find a monomial coordinate change such that in the new coordinates $y'_1, \ldots, y'_n$ one has

$$f(y'_1, \ldots, y'_n) = (\lambda'_1 y'_1, \ldots, \lambda'_r y'_r, y'_{r+1}, \ldots, y'_n),$$

where $\lambda'_1, \ldots, \lambda'_r \in H$ are multiplicatively independent (Euclid's algorithm). Now Corollary 2.8 (applied to the $r$th cartesian power of $\mathcal{O}_{\mathfrak{p}}^{\times}$ acting on $r$ first coordinates and to $\Lambda = (\lambda'_1, \ldots, \lambda'_r)$) asserts that the 'analytic closure' of the $f$-orbit of $t$ contains the torus $\{(y'_1, \ldots, y'_n) \in \mathcal{T} \mid y'_{r+1} = t_{r+1}, \ldots, y'_n = t_n\}$ (notice that since it is about Laurent series, we can allow monomial coordinate changes as above). □

[3] Choose some $x_{i_1}, \ldots, x_{i_{\dim Y}}$ whose restriction to $Y$ are algebraically independent. This defines a generically finite morphism $\gamma : Y \to \mathcal{A}_k^{\dim Y}$. Choose a sequence of (ordered) $\dim Y$-tuples of algebraically independent elements in $\mathfrak{p}^s$ tending to $\gamma(u)$. By Krasner's lemma, it lifts (possibly after a finite extension of $K_{\mathfrak{p}}$) to a sequence of unordered $\deg \gamma$-tuples of generic points of $Y$ tending to an unordered $\deg \gamma$-tuple containing $u$. This means that all the elements of the sequence, except for a finite number of them, contain a point of $O_{\mathfrak{p},q,s}$.

To end this section, let us mention that our considerations yield a simple and self-contained proof of the well-known result that rational points are potentially dense on abelian varieties (see for example [HT00, Proposition 3.1]). The following is a result of a discussion with A. Chambert-Loir.

PROPOSITION 2.11. *Let $A = A_1 \times \cdots \times A_n$ be a product of simple abelian varieties, let $k_1, \ldots, k_n$ be multiplicatively independent positive integers and set $f : A \to A$, $f = f_1 \times \cdots \times f_n$, where for each $i$, $f_i$ is the multiplication by $k_i$ on $A_i$. Then there exist points $x \in A(\bar{\mathbb{Q}})$ with Zariski-dense iterated orbit $\{f^m(x) \mid m \in \mathbb{N}\}$.*

*Proof.* Take a suitable number field $K$, $\mathfrak{p} \subset \mathcal{O}_K$ and for each $i$, a $\mathfrak{p}$-adic neighbourhood $O_i$ such that $f_i$ is linearized in $O_i$. Work in the linearizing coordinates. The same argument as in the above corollaries shows that for a point $(x_1, \ldots, x_n) \in O_1 \times \cdots \times O_n$, $x_i \neq 0$, the iterated orbit is analytically dense in $l_1 \times \cdots \times l_n$, where $l_i \subset O_i$ is the line through the origin generated by $x_i$. Induction by $n$ and the simplicity of the $A_i$ show that $l_1 \times \cdots \times l_n$ is not contained in a proper abelian subvariety of $A$. Since its Zariski closure is invariant by $f$, it must coincide with $A$. □

Since any abelian variety is isogeneous to a product of simple abelian varieties, this proposition proves that abelian varieties are potentially dense. Note that, unlike [HT00, Proposition 3.1], it does not show the existence of an algebraic point generating a Zariski-dense subgroup.

*Remark* 2.12. Let $f$ be a polarized endomorphism of a smooth projective variety $X$. Zhang conjectures in [Zha06] that one can find a point in $X(\bar{\mathbb{Q}})$ with a Zariski-dense iterated orbit. According to our Corollary 2.7, this conjecture is true provided that some power of $f$ has a non-degenerate fixed point with multiplicatively independent eigenvalues of the tangent map. Unfortunately not all polarized endomorphisms have this property: a simple example is the endomorphism of $\mathbb{P}^n$, $n > 1$, taking all homogeneous coordinates to the $m$th power, $m > 1$.

## 3. Variety of lines of the cubic fourfold

The difficulty in using the results of the previous section to prove potential density of rational points is that it can be hard to find an interesting example such that the eigenvalues of the tangent map at some fixed point are multiplicatively independent. For instance, if $f$ is an automorphism and $X$ is a projective K3 surface, or, more generally, an irreducible holomorphic symplectic variety, then the product of the eigenvalues is always a root of unity, as noticed for instance in [Bea83].

So even when a linearization in the neighbourhood of a fixed point is possible, the orbit of a general algebraic point may be contained in a relatively small analytic subvariety of the neighbourhood (of course this subvariety does not have to be algebraic, but it is unclear how to prove that it actually is not). Nevertheless, with some additional geometric information, one can still follow this approach to prove the potential density.

In the rest of this note, we illustrate this by giving a simplified proof of the potential density of the variety of lines of a cubic fourfold, which is the main result of [AV08]. The proof uses several ideas from [AV08], but we think that certain aspects become more transparent thanks to the introduction of the 'dynamical' point of view and the use of $\mathfrak{p}$-adic neighbourhoods. In particular, as it is already mentioned in the introduction, we obtain slightly stronger results.

We recall the setting of [AV08] (the facts listed below are taken from [Ame09, Voi04]). Let $V$ be a smooth cubic in $\mathbb{P}^5$ and let $X \subset G(1,5)$ be the variety of lines on $V$.[4] This is a smooth fourfold with trivial canonical class, in fact an irreducible holomorphic symplectic fourfold: $H^{2,0}(X)$ is generated by a nowhere-vanishing form $\sigma$. For a general line $l \subset V$, there is a unique plane $P$ tangent to $X$ along $l$ (consider the Gauss map, it sends $l$ to a conic in the dual projective space). The map $f$ maps $l$ to the residual line $l'$. It multiplies the form $\sigma$ by $-2$; in particular, its degree is 16. If $V$ does not contain planes, the indeterminacy locus $S \subset X$ consists of points such that the image of the corresponding line by the Gauss map is a line (and the mapping is 2:1). This is a surface resolved by a single blow-up; for a general $V$ ('general' meaning 'outside of a proper subvariety in the parameter space') it is smooth and of general type. The universal family of lines, viewed as a correspondence between $V$ to $X$, induces the isomorphism between the primitive fourth cohomology of $V$ and the primitive second cohomology of $X$ (see [BD85]). This implies that for a generic $V$, and thus for a generic $X$ ('generic' meaning 'outside of a countable union of proper subvarieties in the parameter space'), the Picard group of $X$ is cyclic and so the Hodge structure on $H^2(X)^{\mathrm{prim}}$ is irreducible (thanks to $h^{2,0}(X) = 1$). The space of algebraic cycles of codimension two on $X$ is then generated by $H^2 = c_1^2(\mathcal{U}^*)$ and $\Delta = c_2(\mathcal{U}^*)$, where $\mathcal{U}$ is the restriction of $\mathcal{U}_{G(1,5)}$, the universal rank-two bundle on the Grassmannian $G(1,5)$. One deduces from Terasoma's theorem [Ter85] that this condition $\mathrm{Pic}(X) = \mathbb{Z}$ is also satisfied by a 'sufficiently general' $X$ defined over a number field, in fact even over $\mathbb{Q}$; 'sufficiently general' meaning 'outside of a thin subset in the parameter space' (thin in the sense of Hilbert irreducibility as in [Ser92]). One computes that the cohomology class of $S$ is $5(H^2 - \Delta)$ to conclude that $S$ is irreducible and non-isotropic with respect to $\sigma$.

### 3.1 Fixed points and linearization

The fixed point set $F$ of our rational self-map $f : X \dashrightarrow X$ is the set of points such that along the corresponding line $l$, there is a tritangent plane to $V$. Strictly speaking, this is the closure of the fixed point set, since some of such points are in the indeterminacy locus; but for simplicity we shall use the term 'fixed point set' as far as there is no danger of confusion.

PROPOSITION 3.1. *The fixed point set $F$ of $f$ is an isotropic surface, which is of general type if $X$ is general.*

*Proof.* It is clear from $f^*\sigma = -2\sigma$ that $F$ is isotropic. Since $\sigma$ is non-degenerate, the dimension of $F$ is at most two. Let $I \subset G(1,5) \times G(2,5)$ with projections $p_1, p_2$ be the incidence variety $\{(l,P) \mid l \subset P\}$ and let $\mathcal{F} \subset I \times \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^5}(3))$ denote the variety of triples $\{(l,P,V) \mid V \cap P = 3l\}$. This is a projective bundle over $I$, so $\mathcal{F}$ is smooth and thus its fiber $F'_V$ over a general $V \in \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^5}(3))$ is also smooth. This fiber clearly projects generically one-to-one on the corresponding $F = F_V$, since along a general line $l \subset V$ there is only one tangent plane, and *a fortiori* only one tritangent plane if any; so $F' = F'_V$ is a desingularization of $F$. Since $\dim(I) = 11$ and since intersecting the plane $P$ along the triple line $l$ imposes nine conditions on a cubic $V$, we conclude that $F'$ and $F$ are surfaces.

To compute the canonical class, remark that $F'$ is the zero locus of a section of a globally generated vector bundle on $I$. This vector bundle is the quotient of $p_2^* S^3 \mathcal{U}^*_{G(2,5)}$ (where $\mathcal{U}_{G(2,5)}$ denotes the tautological subbundle on $G(2,5)$) by a line subbundle $\mathcal{L}_3$ whose fiber at $(l,P)$ is the space of degree-three homogeneous polynomials on $P$ with zero locus $l$. One computes that

[4] Here and below, $G(k,n)$ denotes the Grassmannian of $k$-dimensional projective subspaces in $\mathbb{P}^n$.



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the class of $\mathcal{L}_3$ is three times the difference of the inverse images of the Plücker hyperplane classes on $G(2,5)$ and $G(1,5)$, and it follows that the canonical class of $F$ is $p_2^*(3c_1(\mathcal{U}^*))$, which is ample (we omit the details since an analogous computation is given in [Voi98], and a more detailed version of it in [Pac03]). $\square$

*Remark* 3.2. Since $F$ is isotropic and $S$ is not, $S$ cannot coincide with a component of $F$. In fact, a dimension count shows that $F \cap S$ is a curve.

Proposition 3.3. *Let $q$ be a smooth point of $F$ out of the indeterminacy locus of $f$. Then the tangent map $Df_q$ is diagonalized with eigenvalues* $1, 1, -2, -2$.

*Proof.* This follows from the fact that $f^*\sigma = -2\sigma$ and the fact that the map is the identity on the lagrangian plane $T_pF \subset T_pX$. Let $e_1, e_2, e_3, e_4$ be the Jordan basis with $e_1, e_2 \in T_pF$. There is no Jordan cell corresponding to the eigenvalue 1, since in this case $e_4$ would be an eigenvector with eigenvalue 4, but then $\sigma(e_1, e_4) = \sigma(e_2, e_4) = \sigma(e_3, e_4) = 0$, contradicting the fact that $\sigma$ is non-degenerate. By the same reason, the eigenvalues at $e_3$ and $e_4$ are both equal to $\pm 2$. Suppose that $Df_q$ is not diagonalized, so sends $e_3$ to $\pm 2e_3$ and $e_4$ to $e_3 \pm 2e_4$. In both cases $\sigma(e_3, e_4) = 0$. If $e_3$ goes to $2e_3$, we immediately see that $e_3 \in \mathrm{Ker}(\sigma)$, which is a contradiction. Finally, if $Df_q(e_3) = -2e_3$ and $Df_q(e_4) = e_3 - 2e_4$, we have

$$-2\sigma(e_1, e_4) = \sigma(e_1, e_3) - 2\sigma(e_1, e_4),$$

so that $\sigma(e_1, e_3) = 0$, but by the same reason $\sigma(e_2, e_3) = 0$, which is again a contradiction to non-degeneracy of $\sigma$. $\square$

Proposition 3.4. (1) *Let $q \in X(K)$ be a fixed point of $f$ as in Proposition 3.3 and let $O_{\mathfrak{p},q}$ be its $\mathfrak{p}$-adic neighbourhood for a suitable $\mathfrak{p}$, as in the previous section. Then $f$ is equivalent to its linear part in a sufficiently small subneighbourhood $O_{\mathfrak{p},q,s}$; that is, there exists a quadruple of power series $h = h_q$ in four variables $(t_1, t_2, t_3, t_4) = t$ such that $h(t_1, t_2, -2t_3, -2t_4) = f \circ h(t)$, convergent together with its inverse in some neighbourhood of zero.*

(2) *In the complex setting, the analogous statements are true. Moreover, the maps $h_{t_1,t_2}$, where $h_{t_1,t_2}(x, y) = h(t_1, t_2, x, y)$ extend to global meromorphic maps from $\mathbb{C}^2$ to $X$.*

*Proof.* (1) Since the couple of non-trivial eigenvalues $(-2, -2)$ is non-resonant, this is just the Proposition 2.4.

(2) In the complex case, the linearization is a variant of a classical result due to Poincaré [Poi28]. One writes the formal power series in the same way as in the $p$-adic setting, thanks to the absence of the resonances; but it is much easier to prove its convergence thanks to the fact that now $\lambda_3 = \lambda_4 = -2$ and $|-2| > 1$, and so the absolute values of the denominators which appear when one computes the formal power series are bounded from below (these denominators are in fact products of the factors of the form $\lambda_3^{m_3}\lambda_4^{m_4} - \lambda_i$ for $m_3 + m_4 \geqslant 2$, $m_3, m_4 \geqslant 0$). For the sake of brevity, we refer to [Ron08] which proves the analogue of Theorem 4.1 in the complex case and under a weaker diophantine condition on the eigenvalues (Rong assumes moreover that $|\lambda_i| = 1$, but as we have just indicated, in our case all estimates only become easier, going back to [Poi28]).

To extend the maps $h_{t_1,t_2}$ to $\mathbb{C}^2$, set

$$h_{t_1,t_2}(x) = f^r(h_{t_1,t_2}((-2)^{-r}x)),$$

where $(-2)^{-r}x$ is sufficiently close to zero; one checks that this is independent of the choices made. $\square$

We immediately get the following corollary (which follows from the results of [AV08], but for which there was no elementary proof).

Corollary 3.5. *There exist points in $X(\bar{\mathbb{Q}})$ which are not preperiodic for $f$.*

*Proof.* Indeed, $\bar{\mathbb{Q}}$-points are dense in $O_{\mathfrak{p},q}$. Take one in a suitable invariant subneighbourhood and use the linearization given by the proposition above. □

*Remark* 3.6. If $f$ were regular, this would follow from the theory of canonical heights; but this theory does not seem to work sufficiently well for polarized rational self-maps.

### 3.2 Non-preperiodicity of certain surfaces

The starting point of [AV08] was the observation that $X$ is covered by a two-parameter family $\Sigma_b, b \in B$ of birationally abelian surfaces, namely, surfaces parametrizing lines contained in a hyperplane section of $V$ with three double points. On a generic $X$, a generic such surface has cyclic Neron–Severi group [AV08]; moreover, for many $X$ defined over a number field, there are many $\Sigma_b$ defined over a number field with the same property, as shown by an argument similar to that of Terasoma [Ter85].

In [AV08], it is shown that the union of the iterates of a suitable $\Sigma_b$ defined over a number field and with cyclic Neron–Severi group is Zariski-dense in $X$. The first step is to prove non-preperiodicity, that is, the fact that the number of $f^k(\Sigma_b)$, $k \in \mathbb{N}$, is infinite. Already at this stage the proof is highly non-trivial, using the $\ell$-adic Abel–Jacobi invariant in the continuous étale cohomology.

In this subsection, we give an elementary proof of the non-preperiodicity of a suitable $\Sigma = \Sigma_b$ defined over a number field, which is based on Proposition 3.4. Moreover, this works without any assumptions on its Néron–Severi group, and also for any $X$, not only for a 'sufficiently general' one.

Lemma 3.7. *Let $\Sigma \subset X$ be a surface parameterizing lines contained in a hyperplane section of $V$ with three double points and no other singularities, as above. Then $\Sigma$ cannot be invariant by $f$.*

*Proof.* The surface $\Sigma$ is the variety of lines contained in the intersection $Y = V \cap H$, where $H$ is a hyperplane in $\mathbb{P}^5$ tangent to $V$ at exactly three points. For a general line $l$ corresponding to a point of $\Sigma$, there is a unique plane $P$ tangent to $V$ along $l$, and the map $f$ sends $l$ to the residual line $l'$. If $\Sigma$ is invariant, $l'$ and therefore $P$ lie in $H$, and $P$ is tangent to $Y$ along $l$. However, this means that $l$ is 'of the second type' on $Y$ in the sense of Clemens–Griffiths (i.e. the Gauss map of $Y \subset H = \mathbb{P}^4$ maps $l$ to a line in $(\mathbb{P}^4)^*$ as a double covering, or equivalently, the normal bundle of $l$ in $\mathcal{U}$ is $\mathcal{O}_l(-1) \oplus \mathcal{O}_l(1)$), see [CG72, 6.6, 6.19]. At the same time it is well known that a general line on a cubic threefold with double points is 'of the first type' (i.e. has trivial normal bundle, or equivalently is mapped bijectively onto a conic by the Gauss map; see [CG72, Beginning of § 7] for dimension estimates in the case of a cubic $n$-fold), which is a contradiction. □

Passing to the $p$-adic setting and taking a $\Sigma$ meeting a small neighbourhood of a general fixed point $q$ of $f$, we see by Corollary 2.9 that the Zariski closure of $\cup_k f^k(\Sigma)$ is irreducible. Since $\Sigma$ cannot be $f$-invariant by the lemma above, this means that $\Sigma$ is not preperiodic and so the Zariski closure $D$ of $\cup_k f^k(\Sigma)$ is at least a divisor.

Coming back to the complex setting and taking a $\Sigma$ passing close to $q$ in both $p$-adic and complex topologies, let us make a few remarks on the geometry of $D$.



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In a neighbourhood of our fixed point $q$, the intersections of $D$ with the images of $h_{t_1,t_2}$ are $f$-invariant analytic subsets. From the structure of $f$ as in Proposition 3.4 we deduce that such a subset is either the whole image of $h_{t_1,t_2}$, or a finite union of 'lines through the origin' (that is, images of such lines by $h_{t_1,t_2}$). If the last case holds generically, $D$ must contain $F$ by dimension reasons. If the generic case is the first one, $D$ might only have a curve in common with $F$.

To sum up, we have the following theorem.

Theorem 3.8. *Let $X$ be the variety of lines of a smooth cubic fourfold $V$, and let $q$ be a smooth point of $F$ outside of the indeterminacy locus of $f$, as above. For any $\Sigma$ as in Lemma 3.7 meeting a sufficiently small neighbourhood of $q$, the Zariski closure $D$ of $\cup_k f^k(\Sigma)$ is of dimension at least three. If it is of dimension three, this is an irreducible divisor which either contains the surface of fixed points $F$, or has a curve in common with $F$. In this last case, $D$ contains corresponding 'leaves' (images of $\mathbb{C}^2$ from Proposition 3.4) through the points of this curve.*

### 3.3 Potential density

In this subsection, we exclude the case when $D$ is a divisor, under the assumption that $\mathrm{Pic}(X)=\mathbb{Z}$. So let us assume, by contradiction, that $D$ is a divisor (irreducible by Theorem 3.8).

Let $\mu:\tilde{D}\to D\subset X$ denote a desingularization of $D$; $\tilde{D}$ is equipped with a rational self-map $\tilde{f}$ satisfying $\mu\tilde{f}=f\mu$.

Our proof is a case-by-case analysis on the Kodaira dimension of $D$ (meaning the Kodaira dimension of any desingularization). To start with, this Kodaira dimension cannot be maximal because of the presence of a self-map of infinite order (that $f$ cannot be of finite order on $D$ is immediate, for instance, from $f^*\sigma=-2\sigma$; the fact that any dominant rational self-map of a variety of general type is of finite order is well known, see for example [Kob98, Theorem 7.6.1]). In [AV08], we already have simple geometric arguments ruling out the cases of $\kappa(D)=-\infty$ and $\kappa(D)=0$. Though the main part of the discussion in [AV08] is quite different from ours, these two cases also occur there at a certain point and are just sketched below for the convenience of the reader.

*Case* $\kappa(D)=-\infty$. This is especially simple since then the holomorphic 2-form would be coming from the rational quotient of $D$, but $\Sigma$, being non-uniruled, must dominate the rational quotient and thus $\Sigma$ would not be isotropic [AV08, p. 400].

*Case* $\kappa(D)=0$. This is much less obvious and uses the fact that $\mathrm{Pic}(X)=\mathbb{Z}$ or, equivalently, that the Hodge structure $H^2(X)^{\mathrm{prim}}$ is irreducible of rank 22. Namely, an argument using Minimal Model theory and the existence of an holomorphic 2-form on $D$ gives that $D$ must be rationally dominated by an abelian threefold or by a product of a K3 surface with an elliptic curve. However, the second transcendental Betti number of those varieties cannot exceed 21, which contradicts the fact that $\tilde{D}$ carries an irreducible Hodge substructure of rank 22; see [AV08, Proposition 3.4] for details.

We now treat the Cases $\kappa(D)=2$ and $\kappa(D)=1$.

*Case* $\kappa(D)=2$. We need the following lemma.

Lemma 3.9. *On an $X$ with cyclic Picard group, the points of period 3 with respect to $f$ form a curve.*

*Proof.* Let $l_1$ be (a line corresponding to) such a point, $l_2=f(l_1)$, $l_3=f^2(l_1)$, so that $f(l_3)=l_1$. There are thus planes $P_1,P_2,P_3$, such that $P_1$ is tangent to $V$ along $l_2$ and contains $l_3$, etc.

Clearly, the planes $P_1, P_2, P_3$ are pairwise distinct. The span of the planes $P_j$ is a projective 3-space $Q$. Let us denote the two-dimensional cubic, intersection of $V$ and $Q$, by $W$. We can choose the coordinates $(x:y:z:t)$ on $Q$ such that $l_1$ is given by $y=z=0$, etc. Then the intersection of $W$ and $P_1$ is given by the equation $z^2y=0$, etc. The only other monomial from the equation of $W$, up to a constant, can be $xyz$, since it has to be divisible by the three coordinates. Therefore $W$ is a cone (with vertex at 0) over the cubic given by the equation

$$ax^2y + by^2z + cy^2z + dxyz = 0$$

in the plane at infinity. Now a standard dimension count [Ame09] shows that a general cubic admits a one-parameter family of two-dimensional linear sections which are cones, and if some special cubic admits a two-parameter family of such linear sections, then some of them would degenerate into the union of a plane and a quadratic cone, contradicting $\mathrm{Pic}(X) = \mathbb{Z}$ (recall that by [BD85], the primitive classes of algebraic 2-cycles on $V$ correspond to primitive divisor classes on $X$). Each cone on $V$ gives rise to a plane cubic on $X$. This cubic is invariant under $f$, and $f$ acts by multiplication by $-2$ for a suitable choice of zero point on the cubic. Indeed, by construction of $f$, for $x$ on such a cubic, $f(x)$ is just the second intersection point of the cubic and its tangent line at $x$. The points of period 3 with respect to $f$ lie on such cubics and are their points of 9-torsion. □

*Remark* 3.10. In fact the lemma says slightly more: it applies to the indeterminacy points which are '3-periodic in the generalized sense', that is, points appearing if one replaces the condition '$f(l_1) = l_2, f(l_2) = l_3, f(l_3) = l_1$' by '$l_2 \in f(l_1), l_3 \in f(l_2), l_1 \in f(l_3)$'; here by $f(l_1)$ we mean the rational curve which is the image of $l_1$ by the correspondence which is the graph of $f$ (equivalently, $l_2 \in f(l_1)$ says that for some plane $P_3$ tangent to $V$ along $l_1$, the residual line in $P_3 \cap V$ is $l_2$).

By blowing-up $\tilde{D}$, we may assume that the Iitaka fibration $\tilde{D} \to B$ is regular. Its general fiber is an elliptic curve. By [NZ07, Theorem A], the rational self-map $\tilde{f}$ descends to $B$ and induces a transformation of finite order, so the elliptic curves are invariant by a power of $\tilde{f}$. From Proposition 3.4, we obtain that they are in fact invariant by $\tilde{f}$ itself: indeed, locally in a neighbourhood of our fixed point $q$, the curves invariant by $f$ are the same as the curves invariant by its power. On each elliptic curve, there is a finite (non-zero) number of points of period three, since $\tilde{f}$ acts as multiplication by $-2$. We have two possibilities.

(1) These are mapped to points of period three (in the generalized sense as in the Remark 3.10) on $X$ (or the surface formed by those points of period three on $\tilde{D}$ is contracted to any other curve on $X$). Then we claim that any preimage of our surface by an iteration of $\tilde{f}$ is contracted as well, and this gives a contradiction since there are infinitely many of such preimages.

Indeed, the claim would be completely obvious from the equality $\mu \circ \tilde{f} = f \circ \mu$ (recall that $\mu$ denotes the map from $\tilde{D}$ to $X$) if $f$ were regular. Since $f$ is only rational, we have to consider specially the case when either the surface $F_3$ of period-three points on $\tilde{D}$ or any of its preimages is mapped into the indeterminacy locus $S$. However, since the map $f$ is defined along the image of a general elliptic curve of $\tilde{D}$ (or at least each of the branches of this image at a singular point), one sees that also in this case $F_3$ must be contracted to a set of points described by Remark 3.10; moreover, we know from [Ame09] that the resolution of indeterminacy of $f$ obtained by blowing-up $S$ does not contract surfaces, and so the preimages of $F$ must be contracted by $\mu$ as well even if some of them are mapped to $S$.

(2) This surface dominates a component of the surface of fixed points of $f$. In this case, several points of period three must collapse to the same fixed point $p$. However, then the



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resulting branches of each elliptic curve near the generic fixed point are interchanged by $f$, which contradicts the local description of $f$ in Proposition 3.4.

This rules out the possibility $\kappa(D)=2$.

Finally, let us consider the following.

*Case* $\kappa(D)=1$. The Iitaka fibration $\tilde{D}\to C$ maps $\tilde{D}$ to a curve $C$ and the general fiber $U$ is of Kodaira dimension zero. As before, by [NZ07] $\tilde{f}$ induces a finite order automorphism on $C$, and one deduces from Proposition 3.4 (applied to $f$) that this is in fact the identity. We have two possible subcases.

*Subcase* 1*: $U$ is not isotropic with respect to the holomorphic 2-form $\sigma$.* We use the idea from [AV08] as in the case $\kappa(D)=0$. Namely, since $X$ is generic, the Hodge structure $H^2_{\mathrm{prim}}(X,\mathbb{Q})$ is simple. As the pull-back of $\sigma$ to $\tilde{D}$ is non-zero, $H^2(\tilde{D},\mathbb{Q})$ carries a simple Hodge substructure of rank 22. Since $U$ is non-isotropic, the same is true for $U$, but a surface of Kodaira dimension zero never satisfies this property.

*Subcase* 2*: $U$ is isotropic with respect to $\sigma$.* The kernel of the pull-back $\sigma_D$ of $\sigma$ to $\tilde{D}$ gives a locally free subsheaf of rank one in the tangent bundle $T_{\tilde{D}}$, which is in fact a subsheaf of $T_U$ since $U$ is isotropic. There is thus a foliation in curves on $U$, and this foliation has infinitely many algebraic leaves (these are intersections of $U$ with the iterates of our original surface $\Sigma$). By Jouanolou's theorem [Jou78], this is a fibration. In other words, $D$ is (rationally) fibered over a surface $T$ in integral curves of the kernel of $\sigma_D$, and $U$ projects to a curve; as $U$ varies, such curves cover $T$. These cannot be rational curves since the surface $T$ is not uniruled (indeed, the form $\sigma_D$ must be a lift of a holomorphic 2-form on $T$). Therefore these are elliptic curves, and, since $\kappa(U)=0$, so are the fibers of $\xi: D \dashrightarrow T$.

Recall from Theorem 3.8 that either $D$ contains $F$, or it contains a curve on $F$; and in this last case, locally near generic such point, $D$ is a fibration in (isotropic) two-dimensional disks over a curve; in particular, such a point is a smooth point of $D$. If $D$ contains $F$, we get a contradiction with Proposition 3.1: indeed, $F$ must be dominated by a union of fibers of $\xi$, but $F$ is of general type and the fibers are elliptic. If $D$ contains a curve on $F$, then we look at the 'leaf' (image of $\mathbb{C}^2$ from the Proposition 3.4) at a general point $q$ of this curve. If the closure of such a leaf does not coincide with the image of $U$, their intersection is an invariant curve, that is, the image of a line through the origin. Since $U$ and the leaf are both isotropic, this must be an integral curve of the kernel of the restriction of $\sigma$ to $D$. However, $U$ varies in a family, and this implies that the restriction of $\sigma$ to $D$ is zero at $q$, a contradiction since $\sigma$ is non-degenerate.

Finally, if the image of $U$ and the closure of a general leaf coincide, then $U$ is mapped onto itself by $f|_U$ and at some point of $U$ the tangent map is just the multiplication by $-2$. We recall that by definition $\kappa(U)=0$ and so either the geometric genus of $U$ or its $m$th plurigenus for some $m>0$ is equal to 1. The rational self-map $f$ preserves the spaces of pluricanonical forms, so it must multiply some non-zero pluricanonical form by a scalar. However, because of what we know about the tangent map, this scalar can only be equal to 4, and so the degree of $f|_U$ is 16, contradicting the calculations of [Ame09] as in [AV08, Proof of Proposition 2.3].

We thus come to a conclusion that $D$ cannot be a divisor, so $D=X$. Since rational points are potentially dense on $\Sigma$ and the iterates of $\Sigma$ are Zariski-dense in $D$, $X$ is potentially dense. To sum up, we have proved the following result.



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THEOREM 3.11. *Let $X = \mathcal{F}(V)$ be the variety of lines of a cubic fourfold $V$, defined over a number field. If $\mathrm{Pic}(X) = \mathbb{Z}$ (or, equivalently, if the group of algebraic cycles in the middle cohomology of $V$ is cyclic), then $X$ is potentially dense.*

*Remark* 3.12. Here, unlike in the proof of non-preperiodicity from the previous subsection, we do need a genericity condition, that is, $\mathrm{Pic}(X) = \mathbb{Z}$. It would be interesting to check whether one can modify the argument to get rid of this assumption.

## 4. A version of Siegel's theorem

In this section we explain how to modify the proof of Siegel's theorem on linearization of $p$-adic diffeomorphisms given in [HY83, Theorem 1, § 4, page 423] in order to adapt it to the situation where the fixed point is not isolated.

Let $k$ be a complete non-archimedean non-discrete field with the absolute value $|\,|_k : k^\times \to \mathbb{R}^\times_{>0}$. For any $\rho \in \mathbb{R}^\times_{>0}$, the set $\mathcal{B} = \mathcal{B}_\rho := \{a \in k \mid |a|_k < \rho\}$ is called the open disc of radius $\rho$ over $k$; for any $s \in \mathbb{N}$ an $s$-dimensional open polydisc is the $s$th cartesian power $\mathcal{B}^s = \mathcal{B}^s_\rho$ of a disc.

An *analytic diffeomorphism* of $\mathcal{B}^s$ is a self-bijection given, together with its inverse, by an $s$-tuple of convergent formal power series in $s$ variables over $k$.

We fix a pair of non-negative integers $r < n$. Denote by $p_1 : \mathcal{B}^n = \mathcal{B}^r \times \mathcal{B}^{n-r} \to \mathcal{B}^r$ and $p_2 : \mathcal{B}^n \to \mathcal{B}^{n-r}$ the natural projections.

THEOREM 4.1. *Let $f$ be an analytic diffeomorphism of a polydisc $\mathcal{B}^n$ over $k$. Assume that the following conditions hold.*

(i) *The fixed-point-set $F$ of $f$ is $\mathcal{B}^r \times \{(0)\} \subset \mathcal{B}^r \times \mathcal{B}^{n-r} = \mathcal{B}^n$.*

(ii) *The tangent map of $f$ is diagonalizable over $k$ at all points $q \in F$, only $r$ of its eigenvalues are equal to 1, and that its eigenvalues $\lambda_{r+1}, \ldots, \lambda_n$, distinct from 1, are constant and satisfy the following bad diophantine approximation property:*

$$|\lambda_{r+1}^{i_{r+1}} \cdots \lambda_n^{i_n} - \lambda_j|_k \geqslant C(i_{r+1} + \cdots + i_n)^{-\beta} \quad \text{for some } \beta, C > 0, \tag{1}$$

*any $r < j \leqslant n$ and $(i_{r+1}, \ldots, i_n) \in \mathbb{Z}^{n-r}_{\geqslant 0}$ such that $i_{r+1} + \cdots + i_n \geqslant 2$.*

*Then, for any $q \in F$, there exist a neighbourhood $\mathcal{V}_q$ of $q$ and an analytic diffeomorphism $h$ of $\mathcal{V}_q$, identical on the intersection with $F$ and such that $h^{-1}(f(h(x))) = \Lambda(x)$ for any $x \in \mathcal{V}_q$, where $\Lambda(x_1, \ldots, x_n) = (x_1, \ldots, x_r, \lambda_{r+1}x_{r+1}, \ldots, \lambda_n x_n)$.*

The desired diffeomorphism $h$ is a solution of the equation $\Psi(h) = 0$, where $\Psi(h) := f \circ h - h \circ \Lambda$ is a non-linear operator in an appropriate space.

### 4.1 Plan of the proof

Corollary 4.3 allows us to choose a diffeomorphism $h_0$ of the whole polydisc such that $f$, conjugated by $h_0$, coincides with $\Lambda$ on 'the first infinitesimal neighbourhood of $F$'.[5] Next, it is easy to see that in the non-resonant case there is a unique formal solution $h$ of the equation $\Psi(h) = 0$ coincident with $h_0$ on 'the first infinitesimal neighbourhood of $F$'. The remaining task is to show the convergence of the formal solution. For that, one follows the same procedure as in [HY83], but instead of working in the spaces $A^2_\rho(k^n)$ and $B^2_\rho(k^n)$ of *loc. cit.* we work in smaller

[5] Our modest purposes do not require the use of scheme-like structures, a drawback of avoiding such a language being that the infinitesimal neighbourhoods are not formally defined.



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(when $r > 0$) spaces $A_\rho^{(r)}(k^n)$ and $B_\rho^{(r)}(k^n)$, cf. below. Instead of showing the convergence of the formal solution, one constructs $h$ as the limit (in an appropriate non-archimedean Banach space) of the sequence of approximations $h_0, h_1, h_2, \ldots$, obtained by the Newton's method. In fact, $h_i$ is a diffeomorphism of the open $\rho_i$-neighbourhood (possibly onto a neighbourhood of different radius) of a fixed point of $f$ for each $i \geqslant 1$, where $\rho_0 > \rho_1 > \rho_2 > \cdots$ is a decreasing sequence of radii with a positive lower bound.

### 4.2 'The first infinitesimal neighbourhood of *F*'

Let $y_j : \mathcal{B}^n \to \mathcal{B}^1$ be the projection to the $j$th factor, $1 \leqslant j \leqslant n$.

Lemma 4.2. *Let $f$ be such an analytic diffeomorphism of a polydisc $\mathcal{B}^n$ over $k$ that the fixed point set $F$ of $f$ is $\mathcal{B}^r \times \{(0)\} \subset \mathcal{B}^r \times \mathcal{B}^{n-r} = \mathcal{B}^n$. Assume that only $r$ eigenvalues of the tangent map of $f$ at any $q \in F$ are equal to 1. Then there exists an analytic diffeomorphism $h$ of $\mathcal{B}^n$ identical on $F$ and such that the power series expression of $p_1 \circ h^{-1} \circ f \circ h$ is $p_1$ modulo terms quadratic in $y_{r+1}, \ldots, y_n$.*

*Proof.* Modulo terms quadratic in $p_2$ (that is, in $y_{r+1}, \ldots, y_n$), the formal series expansion of $f$ is $\mathrm{id} + \sum_{i=r+1}^n a_i(p_1)y_i =: \Xi$ for some analytic $a_i = (a_i', a_i'') : \mathcal{B}^r \to \mathcal{B}^n = \mathcal{B}^r \times \mathcal{B}^{n-r}$. We rewrite $\Xi = (p_1 + a'p_2; p_2 + a''p_2)$, where $a' = a'(p_1)$ is the matrix with columns $a_i'(p_1)$ for $r < i \leqslant n$ and $a'' = a''(p_1)$ is the matrix with columns $a_i''(p_1)$ for $r < i \leqslant n$ (so $a''$ is an $(n-r) \times (n-r)$-matrix, invertible since $f$ is a diffeomorphism). Let $h = (p_1 + a'(a'')^{-1}p_2; p_2)$. Then, modulo terms quadratic in $p_2$, $h^{-1} \equiv (p_1 - a'(a'')^{-1}p_2; p_2)$, $f \circ h \equiv (p_1 + a'(a'')^{-1}p_2 + a'p_2; p_2 + a''p_2)$, and finally, $h^{-1} \circ f \circ h \equiv (p_1; p_2 + a''p_2)$. □

Corollary 4.3. *In the setting of Lemma 4.2, assume the following conditions hold.*

(i) *The tangent maps of $f$ at all points of $F$ are semisimple.*

(ii) *Their eigenvalues $\lambda_1, \ldots, \lambda_n$ do not vary.*

*Then there exists an analytic diffeomorphism $h$ of $\mathcal{B}^n$, identical on $F$, and such that the power series expression of $h^{-1} \circ f \circ h$ is $(\lambda_1 y_1, \ldots, \lambda_n y_n)$ modulo terms quadratic in $y_{r+1}, \ldots, y_n$.*

*Proof.* By Lemma 4.2, we may assume that $p_1 \circ f \equiv p_1$ (where $\equiv$ is modulo terms quadratic in $y_{r+1}, \ldots, y_n$). Then we look for a diffeomorphism $h$ such that $h^{-1} \circ f \circ h \equiv (\lambda_1 y_1, \ldots, \lambda_n y_n)$, $p_1 \circ h = p_1$ and $h$ is linear on the fibers of $p_1$.

In the setting of the proof of Lemma 4.2, this is equivalent to looking for an $h$ which is:

(i) identical on $y_1, \ldots, y_r$;

(ii) linear in the coordinates $y_{r+1}, \ldots, y_n$ with functions of $y_1, \ldots, y_r$ as coefficients; and

(iii) making diagonal the matrix $a''$: $h^{-1} \circ \Xi \circ h \equiv (\lambda_1 y_1, \ldots, \lambda_n y_n)$.

(Recall that $f \equiv \Xi = (p_1; (1 + a'')p_2)$.)

After a $k$-linear change of $p_2$-coordinates we may assume that $a''$ is diagonal at some point $q \in F$, so if the vectors $\{e_1, \ldots, e_n\}$ correspond to the coordinates $y_1, \ldots, y_n$ then $\{e_{r+1}, \ldots, e_n\}$ are eigenvectors of $a''(q)$. Let $\pi_i := \prod_{j:\lambda_j \neq \lambda_i} (\lambda_i - \lambda_j)^{-1}(a'' - \lambda_j + 1)$ be a projector onto the $\lambda_i$-eigenspace of $a''$. Then $\{\pi_i e_i\}_{r<i\leqslant n}$ is a system of eigenvectors of $a''$ and its reduction modulo the maximal ideal in $k[[x_1, \ldots, x_r]]$ is the eigenbasis $\{e_{r+1}, \ldots, e_n\}$ of $a''(q)$. This means that $\pi_1 e_1, \ldots, \pi_n e_n$ generate the tangent bundle, cf. [AM69, Proposition 2.8]. We set $h(p_1, p_2) := \sum_{i=1}^r y_i e_i + \sum_{i=r+1}^n y_i \pi_i e_i =: (p_1; A(p_1)p_2)$

for a matrix $A(p_1)$. Then $h^{-1}(p_1,p_2)=(p_1;A(p_1)^{-1}p_2)$, $f(h(p_1,p_2))\equiv(p_1;(1+a'')Ap_2)$ and $h^{-1}(f(h(p_1,p_2)))\equiv(p_1;A^{-1}(1+a'')Ap_2)$. One has $A^{-1}a''Ae_i=A^{-1}a''\pi_ie_i=(\lambda_i-1)A^{-1}\pi_ie_i=(\lambda_i-1)e_i$, and thus, $h^{-1}\circ f\circ h\equiv(p_1;\Lambda'p_2)$, where $\Lambda'$ is the diagonal matrix with entries $\lambda_{r+1},\dots,\lambda_n$. □

### 4.3 Newton's method

From now on we assume that $f$ is in the form attained in Corollary 4.3, and that $y_1,\dots,y_n$ are the corresponding coordinates.

Let $\Lambda$ be the linear part of $f$, i.e. an $n\times n$ diagonal matrix with entries $\lambda_1,\dots,\lambda_n$, the first $r$ ones being equal to 1.

For any real $\rho>0$ define the spaces:

- $A_\rho(k^n):=\{\phi=\sum_I a_Ix^I\in k[[x_1,\dots,x_n]]\mid \sup|a_I|_k\rho^{|I|}=:\|\phi\|_\rho<\infty\}$ of the formal series convergent and bounded on $\mathcal{B}^n_\rho$ (this is a non-archimedean Banach algebra; notation coincides with that of [HY83]);
- $A^{(r)}_\rho(k^n)$ consisting of all $\phi\in A_\rho(k^n)$ such that $\phi(x_1,\dots,x_r,0,\dots,0)=0$ and $(\partial\phi/\partial x_i)(x_1,\dots,x_r,0,\dots,0)=0$ for all $r<i\leqslant n$ (this is the ideal in $A_\rho(k^n)$ generated by $x_ix_j$ for all $r<i\leqslant j\leqslant n$; $A^{(0)}_\rho(k^n)$ is denoted in [HY83] by $A^2_\rho(k^n)$);
- $B^{(r)}_\rho(k^n):=\{\phi\in A^{(r)}_\rho(k^n)\mid\phi\circ\Lambda\in A^{(r)}_\rho(k^n)\}$, so, in particular, $B^{(r)}_\rho(k^n)=A^{(r)}_\rho(k^n)$ if $|\lambda_1|_k=\cdots=|\lambda_n|_k=1$.

We consider $f$ as an $n$-tuple of formal series convergent and bounded on $\mathcal{B}^n_\rho$ for an appropriate $\rho$. In fact, $f-\Lambda\in(A^{(r)}_\rho(k^n))^n$, since $f$ is in the form attained in Corollary 4.3. Replacing if necessary $f$ by $uf(u^{-1}x)$ ($u\in k$, $|u|_k\gg1$), we may assume that $f$ is convergent and bounded on $\mathcal{B}^n_1$ and $\|f-\Lambda\|_1$ is as small as we want. (Given a non-archimedean Banach space with norm $\|\ \|$, we endow its arbitrary finite cartesian power with the 'max norm': $\|(u_1,\dots,u_s)\|:=\max(\|u_1\|,\dots,\|u_s\|)$. Examples are $(A_\rho(k^n))^n$ and $\mathrm{End}(k^n)\cong k^{n^2}$.)

We assume that $\lambda_{r+1},\dots,\lambda_n$ satisfy the bad diophantine approximation property (1). The set $x+(A^{(r)}_\rho(k^n))^n$ is a group (with respect to the composition) acting on $A^{(r)}_\rho(k^n)$. We are looking for a root of the equation $\Psi(h)=0$ in $x+(A^{(r)}_{1/2}(k^n))^n$, which is of the form $h(x)=\lim_{i\to\infty}h_i(x)$, where $h_i(x)-x\in(A^{(r)}_{\rho_i}(k^n))^n\subset(A^{(r)}_{1/2}(k^n))^n$ with $\rho_i=1/2+2^{-i-1}$ for all integer $i\geqslant0$.

The sequence $h_0,h_1,h_2,\dots$ is constructed inductively, following the Newton's method: let $L=L_\rho:(B^{(r)}_\rho(k^n))^n\to(A^{(r)}_\rho(k^n))^n$ be the linear operator, defined by $Lw=w\circ\Lambda-\Lambda\circ w$; set $h_0=\mathrm{id}$ and define $h_{i+1}$ so that

$$\Psi(h_i)=(Dh_i\circ\Lambda)(LE_i),\tag{2}$$

where $E_i:=(Dh_i)^{-1}\cdot(h_{i+1}-h_i)$. To make this work, we have to invert $L$.

The injectivity of $L$ is evident: $\phi=\sum_Ia_Ix^I\mapsto(\sum_I(\lambda^I-\lambda_i)a^{(i)}_Ix^I)_{1\leqslant i\leqslant n}$, where $\lambda^I=\lambda_1^{i_1}\cdots\lambda_n^{i_n}$. Moreover, it is also evident that for any $g\in(A^{(r)}_\rho(k^n))^n$ there exists a unique $n$-tuple of formal series $w$ such that $Lw=g$, $w(p_1;0)=0$ and $(\partial w/\partial x_i)(p_1;0)=0$ for all $r<i\leqslant n$. According to Lemma 4.4, $L_\rho$ is 'almost surjective'. In particular, the formal solution $E_i$ of (2) belongs, in fact, to $(B^{(r)}_{\rho_i-\delta}(k^n))^n$ for any positive $\delta$.

It follows from the continuity of $\Psi$ (see [HY83, Lemma 14]) and the identity

$$\begin{aligned}\Psi(h+\varepsilon) &= f\circ(h+\varepsilon) - f\circ h + \Psi(h) - \varepsilon\circ\Lambda \\ &= [f\circ(h+\varepsilon) - f\circ h + D\Psi(h)\cdot E - Df\circ h\cdot\varepsilon] + \Psi(h) - (Dh\circ\Lambda)(LE),\end{aligned} \tag{3}$$

where $E := (Dh)^{-1}\cdot\varepsilon$, that if the sequence $(h_i)_{i\geqslant 0}$ converges to $h_\infty$ then $\Psi(h_\infty) = 0$.

It remains to show that $h_{i+1} - h_i \in (A_{1/2}(k^n))^n$ and $\|h_{i+1} - h_i\|_{1/2}$ tends to 0.

For that we need the following version of [HY83, Lemma 15] (with the same proof).

Lemma 4.4. *For any $\delta > 0$ and any $g \in (A_\rho^{(r)}(k^n))^n$ the solution $w$ of the linear equation $Lw = g$ belongs to $(B_{\rho-\delta}^{(r)}(k^n))^n$ and satisfies $\|w\|_{\rho-\delta} \leqslant C_1(\|g\|_\rho/\delta^\beta)\rho^\beta$, $\|Dw\|_{\rho-\delta} \leqslant C_1(\|g\|_\rho/\delta^\beta)$ $(\rho^\beta/(\rho-\delta))$, $\|Dw\circ\Lambda\|_{\rho-\delta} \leqslant C_1(\|g\|_\rho/\delta^\beta)(\rho^\beta/(\rho-\delta))$, where $C_1$ is a constant depending only on $C$, $\beta$ (from the bad diophantine approximation condition (1)) and $\|\Lambda\|$. (We put on $(B_\rho^{(r)}(k^n))^n$ the max norm: $\|\phi\|_\rho = \max(\|\phi\|_\rho, \|\phi\circ\Lambda\|_\rho)$.)*

From Lemma 4.4 we get

$$\|E_i\|_{\rho_{i+1}} \leqslant C_2\frac{\|\Psi(h_i)\|_{\rho_i}}{(\rho_i-\rho_{i+1})^\beta}\|(Dh_i\circ\Lambda)^{-1}\|_{\rho_i} \leqslant C_2\frac{\|\Psi(h_i)\|_{\rho_i}}{(\rho_i-\rho_{i+1})^\beta}, \tag{4}$$

$\|DE_i\|_{\rho_{i+1}} \leqslant C_2(\|\Psi(h_i)\|_{\rho_i}/(\rho_i-\rho_{i+1})^\beta)$, $\|DE_i\circ\Lambda\|_{\rho_{i+1}} \leqslant C_2(\|\Psi(h_i)\|_{\rho_i}/(\rho_i-\rho_{i+1})^\beta)$, where we have used the estimate $\|Dh_i\circ\Lambda - \mathrm{id}\| < 1/2$ and the following lemma (which is evident from the identity $(1-x)^{-1} = \sum_{i\geqslant 0} x^i$ for $\|x\| < 1$).

Let $A_\rho(k^n, \mathrm{End}(k^n))$ be the non-archimedean Banach algebra of the formal series (with coefficients in $\mathrm{End}(k^n)$) that are convergent and bounded on $\mathcal{B}_\rho^n$.

Lemma 4.5 [HY83, Lemma 13]. *If $\varphi \in A_\rho(k^n, \mathrm{End}(k^n))$ and $\|\varphi\|_\rho < 1$ then $1+\varphi$ is invertible in $A_\rho(k^n, \mathrm{End}(k^n))$ and $\|(1+\varphi)^{-1} - 1\|_\rho = \|\varphi\|_\rho$.*

It follows from the estimates $\|Dh_i\|_{\rho_i} \leqslant 1$, $\|Dh_i^{-1}\|_{\rho_i} \leqslant 1$, $\|Dh_i\circ\Lambda\|_{\rho_i} \leqslant 1$, $\|(Dh_i\circ\Lambda)^{-1}\|_{\rho_i} \leqslant 1$ that $h_i - h_{i+1} \in (B_{\rho_{i+1}}^{(r)}(k^n))^n$ and satisfies the same estimates as $E_i$:

$$\|h_{i+1} - h_i\|_{\rho_{i+1}} \leqslant C_2\frac{\|\Psi(h_i)\|_{\rho_i}}{(\rho_i-\rho_{i+1})^\beta}. \tag{5}$$

Using Cauchy's formula [HY83, Lemma 10] $\|D\Psi(h)\cdot E\|_{\rho-\delta} \leqslant 2\|\Psi(h)\|_{\rho-\delta}\|E\|_{\rho-\delta}$, the estimate (4) and Taylor's formula [HY83, Proposition 7]

$$\|f\circ(h+\varepsilon) - f\circ h - Df\circ h\cdot\varepsilon\|_{\rho-\delta} \leqslant 4\|f\|_1\|\varepsilon\|_{\rho-\delta}^2$$

we deduce from (3) the estimate

$$\|\Psi(h_{i+1})\|_{\rho_{i+1}} \leqslant \frac{K}{(\rho_i-\rho_{i+1})^{2\beta}}\|\Psi(h_i)\|_{\rho_i}^2, \tag{6}$$

where $K > C_2^2$ is a constant depending only on $C$, $\beta$, $\|\Lambda\|$, $\|f\|_1$, cf. [HY83, p. 425].

As $\|f-\Lambda\|_1$ can be made arbitrary small, we may assume that $\|\Psi(h_0)\|_1 < K^{-1}2^{-6\beta}$, where $h_0 = \mathrm{id}$. Let us show by induction that $\|\Psi(h_i)\|_{\rho_i} < K^{-1}2^{-2\beta(i+3)}$. By (6) and the induction assumption, $\|\Psi(h_{i+1})\|_{\rho_{i+1}} < K^{-1}(\rho_i-\rho_{i+1})^{-2\beta}2^{-4\beta(i+3)} = K^{-1}2^{-2\beta(i+4)}$, completing induction. From (5) we get $\|h_{i+1} - h_i\|_{\rho_{i+1}} \leqslant C_2K^{-1}2^{-\beta(i+4)}$, so $\|h_{i+1} - h_i\|_{\rho_{i+1}}$ tends to 0. This completes the proof of Theorem 4.1.



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## 5. Noether normalization

Lemma 5.1. *Let $k$ be an infinite field, $r \geqslant 1$ an $m \geqslant 0$ be integers, $\mathbb{A}^r_k = Y_m \supset Y_{m-1} \supset \cdots \supset Y_0 \supset Y_{-1} := \emptyset$ be a chain of closed embeddings of equidimensional affine varieties over $k$, $\dim Y_j =: d_j$, $r > d_{m-1} > \cdots > d_1 > d_0 \geqslant 0$, and for each $0 \leqslant j \leqslant m$, $q_j$ be a smooth closed point of $Y_j \smallsetminus Y_{j-1}$, smooth also as a point of $Y_s$ for $s > j$. Let $g$ a regular function on $\mathbb{A}^r_k$ with differential non-vanishing at $q_0, \dots, q_m$: $Dg_{q_j} \neq 0$. Then there exists an affine linear morphism $\varphi : \mathbb{A}^r_k \to \mathbb{A}^{r-1}_k$ such that:*

- *$(\varphi, g) : \mathbb{A}^r_k \to \mathbb{A}^{r-1}_k \times_k \mathbb{A}^1_k$ is finite;*
- *$(\varphi, g)$ is étale at $q_0, \dots, q_m$;*
- *$\varphi(q_j) \notin \varphi(Y_{j-1})$ for all $1 \leqslant j < m$.*

*Proof.* Let $x_1, \dots, x_r$ be affine coordinates on $\mathbb{A}^r_k$, so that $g$ becomes a polynomial in $x_1, \dots, x_r$. Set $\varphi = \mathrm{id} + x_r \cdot \mathbf{a}$ for some $\mathbf{a} = (a_1, \dots, a_{r-1}) \in k^{r-1}$. Then the morphism $(\varphi, g)$ is étale at $q_s$ for all $\mathbf{a} \in k^{r-1}$ outside the affine hyperplane $\{\mathbf{a} \in k^{r-1} \mid \sum_{i=1}^{r-1} a_i (\partial g / \partial x_i)(q_s) = (\partial g / \partial x_r)(q_s)\} \subset \mathbb{A}^{r-1}_k$. It is well known after E. Noether (cf. [AM69, ch. 5, Exercise 16], or [Eis95, Theorem 13.2(c)]) that $(\varphi, g)$ is a finite morphism for all $\mathbf{a} \in k^{r-1}$ outside a hypersurface in $\mathbb{A}^{r-1}_k$. The set $\varphi^{-1}(\varphi(q_s))$ is an affine line through $q_s$ (here $q_{si} := x_i(q_s)$):

$$\varphi^{-1}(\varphi(q_s)) = \{(q_{s1} + a_1(q_{sr} - u), q_{s2} + a_2(q_{sr} - u), \dots, q_{s1} + a_{r-1}(q_{sr} - u), u) \mid u \in k\}.$$

For any $j < m$, the codimension of $Y_{j-1}$ in $\mathbb{A}^r_k$ is at least two, so the set of affine lines in $\mathbb{A}^r_k$, passing through $q_j$ and a point of $Y_{j-1}$, is a proper closed subset in $\mathbb{P}^{r-1}_k$. Therefore, for any point $\mathbf{a}$ outside a proper closed subset in $\mathbb{A}^{r-1}_k \subset \mathbb{P}^{r-1}_k$ the set $\varphi^{-1}(\varphi(q_j))$ does not meet $Y_{j-1}$, and thus $\varphi(q_j) \notin \varphi(Y_{j-1})$ for all $1 \leqslant j < m$. $\square$

Theorem 5.2. *Let $k$ be an infinite field, $m \geqslant 0$ be an integer, $Y_m \supset Y_{m-1} \supset \cdots \supset Y_0 \supset Y_{-1} := \emptyset$ be a chain of closed embeddings of equidimensional affine varieties over $k$, $\dim Y_j =: d_j$, $d_m > d_{m-1} > \cdots > d_1 > d_0 \geqslant 0$, and for each $0 \leqslant j \leqslant m$, $q_j$ be a smooth closed point of $Y_j \smallsetminus Y_{j-1}$, smooth also as a point of $Y_s$ for $s > j$.*

*Then there exists a finite morphism $f : Y_m \to \mathbb{A}^d_k$ inducing a morphism onto a flag of affine subspaces (a commutative diagram):*

$$\begin{tikzcd}
Y_m \arrow[d, "f"'] & \supset & Y_{m-1} \arrow[d] & \supset & \cdots & \supset & Y_1 \arrow[d] & \supset & Y_0 \arrow[d] \\
\mathbb{A}^{d_m}_k & \supset & \mathbb{A}^{d_{m-1}}_k & \supset & \cdots & \supset & \mathbb{A}^{d_1}_k & \supset & \mathbb{A}^{d_0}_k
\end{tikzcd}$$

*such that $f(q_j) \notin \mathbb{A}^{d_{j-1}}_k$ for all $1 \leqslant j \leqslant m$, and $f$ is étale at each $q_j$ whenever $q_j$ is a smooth point of $Y_m$. For any closed embedding $\iota : Y_m \hookrightarrow \mathbb{A}^r_k$, $f$ can be chosen to be the composition of $\iota$ with an affine linear projection $\mathbb{A}^r_k \to \mathbb{A}^{d_m}_k$.*

*Proof.* This proof follows closely that of [Eis95, Theorem 13.3]. It is enough to treat the case $Y_m = \mathbb{A}^{d_m}_k$. (If $Y_m$ is a proper closed subset in $\mathbb{A}^r_k$ then one can consider the longer chain $Y_{m+1} := \mathbb{A}^r_k \supset Y_m \supset Y_{m-1} \supset \cdots \supset Y_1 \supset Y_0$ and choose any closed point $q_{m+1} \in Y_{m+1} \smallsetminus Y_m$.) After inserting into the chain $Y_\bullet$ intermediate pointed varieties (and adding a chain of pointed subvarieties of $Y_0$) we may assume that $d_j = j$ for all $0 \leqslant j \leqslant m$. We proceed by induction on $m$. The case $m = 1$ is evident: $f$ can be any polynomial with a non-zero non-critical value at $q_1$



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and vanishing on $Y_0$ with a non-critical value at $q_0$. For arbitrary $m > 1$, let $g : Y_m \to \mathbb{A}^1_k$ be a morphism vanishing on $Y_{m-1}$, non-zero at $q_m$ and non-critical at $q_0, \ldots, q_m$. By Lemma 5.1, there exists a linear morphism $\varphi : Y_m \to \mathbb{A}^{m-1}_k$ such that:

(i) $(\varphi, g)$ is finite;

(ii) $(\varphi, g) : Y_m \to \mathbb{A}^{m-1}_k \times_k \mathbb{A}^1_k$ is étale at $q_0, \ldots, q_m$;

(iii) $\varphi(q_j) \notin \varphi(Y_{j-1})$ for all $1 \leqslant j < m$.

Set $Y'_j := (\varphi, g)(Y_j)$ and $q'_j := (\varphi, g)(q_j)$ for all $1 \leqslant j \leqslant s$. Then $Y'_{m-1} = \mathbb{A}^{m-1}_k \times \{0\}$ is the hyperplane in $\mathbb{A}^{m-1}_k \times_k \mathbb{A}^1_k$. We note that $q'_j \notin Y'_{j-1}$, so the induction assumption can be applied to choose a finite endomorphism $f'$ of $Y_{m-1}$, étale at $q'_0, \ldots, q'_m$ and inducing a commutative diagram of the following type.

$$\begin{tikzcd}
\mathbb{A}^{m-1}_k \times \{0\} \arrow[r, phantom, "="] & Y'_{m-1} \arrow[d, "f'"'] \arrow[r, phantom, "\supset"] & Y'_{m-2} \arrow[d] \arrow[r, phantom, "\supset"] & \cdots \arrow[r, phantom, "\supset"] & Y'_0 \arrow[d] \\
& \mathbb{A}^{m-1}_k \arrow[r, phantom, "\supset"] & \mathbb{A}^{m-2}_k \arrow[r, phantom, "\supset"] & \cdots \arrow[r, phantom, "\supset"] & \mathbb{A}^0_k
\end{tikzcd}$$

Then we set $f = (f' \times \mathrm{id}_{\mathbb{A}^1_k}) \circ (\varphi, g)$. $\square$


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