ON MEASURES OF SYMMETRY OF CONVEX BODIES

G. D. CHAKERIAN AND S. K. STEIN

1. Introduction and statement of theorems. Let K be a convex body (compact, convex set with interior points) in n-dimensional Euclidean space E_n, and let $V(K)$ denote the volume of K. Let K' be a centrally symmetric body of maximum volume contained in K (in fact, K' is unique; see **2** or **9**), and define

$$c(K) = V(K')/V(K).$$

Let

$$c(n) = \inf\{c(K) : K \subset E_n\}.$$

It is known that $c(n) > 2^{-n}$ for all n (**9**); that is, any n-dimensional convex body K contains a centrally symmetric convex body of volume $> 2^{-n}V(K)$. (Better results are known in E_1, E_2, and E_3: $c(1) = 1$, $c(2) = 2/3$, and $c(3) \geqslant 2/9$. For references, see (**4**, p. 254).) One could also consider K as a non-homogeneous convex solid with an integrable density $f(p)$ at each $p \in K$ and ask for a centrally symmetric convex subset K' of maximum total mass. This leads one to define, for each integrable density f on K,

$$\mu(K;f) = M(K')/M(K),$$

where K' is a centrally symmetric convex body of maximum mass $M(K')$ contained in K, and $M(K)$ is the mass of K. (Note that K' need not be unique. However, if the density is a concave function, then essentially the same argument used in (**9**), applied to the ordinate set of f, shows that K' is unique.) Let

$$\mu(K) = \inf \mu(K;f),$$

where the infimum is taken over all integrable densities on K, and define

$$\mu(n) = \inf\{\mu(K) : K \subset E_n\}.$$

We shall prove in § 3 that

$$\mu(n) \geqslant 2^{-n}, \qquad \text{for all } n, \tag{1.1}$$

and

$$\mu(2) = \tfrac{1}{3}. \tag{1.2}$$

The inequality (1.1) is a result of an obvious generalization of the computation of "mean symmetry" used in (**9**), while (1.2) depends on the following theorem, proved in §2.

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THEOREM 1. *Any plane convex body K can be covered by three translates of $-K$. A triangle T cannot be covered by fewer than three translates of $-T$.*

It is evident that if a convex body K can be covered by r translates of $-K$ (so that K is the union of r centrally convex symmetric sets), then $\mu(K) \geqslant r^{-1}$. This leads us to consider the numbers $g(K)$ and $g(n)$, defined as follows:

$g(K)$ is the least number r such that K can be covered by r translates of $-K$.

$$g(n) = \max\{g(K) : K \subset E_n\}.$$

Theorem 1 simply states that $g(2) = 3$. In **(3)**, numbers $h(K)$ and $h(n)$ are defined:

$h(K)$ is the least number r with the following property: whenever $\mathfrak{F}$ is any family of pairwise intersecting translates of K, then there exist r points such that each member of $\mathfrak{F}$ contains at least one of them.

$$h(n) = \max\{h(K) : K \subset E_n\}.$$

It is proved in **(3)** that $h(n)$ is finite, and that if $K \subset E_2$ with $K = -K$, then $h(K) \leqslant 3$. Moreover, it is conjectured that $h(2) = 3$ and $h(n) \leqslant n + 1$ for all n. The second conjecture is false, as follows from

THEOREM 2. *For all n, $h(n) \geqslant g(n) \geqslant c(n)^{-1}$.*

The first inequality is proved in § 2. The second inequality is immediate, since the definition of $g(n)$ implies that any convex body K contains a centrally symmetric body of volume at least $g(n)^{-1}V(K)$, so that $c(K) \geqslant g(n)^{-1}$; hence, $c(n) \geqslant g(n)^{-1}$. Now, it is shown in **(2)** that if T is a simplex in E_n, then

$$c(n) \leqslant c(T) < \sqrt{\frac{2}{\pi}}\left(\frac{2}{e}\right)^n\left(\frac{n}{n+1}\right)^{n-1}\sqrt{n+1}\,.$$

Thus, by Theorem 2, $h(n)$ is greater than $n + 1$ for all sufficiently large n, and in fact grows faster than any fixed power of n. The conjecture on $h(n)$ is false even for $n = 3$, since, as shown in § 2,

$$g(3) \geqslant 7. \tag{1.3}$$

Since $h(n)$ is finite, so is $g(n)$ by Theorem 2. We shall give an independent proof of the finiteness of $g(n)$ in § 2.

A convex body is "k-symmetric" if it coincides with its reflection through some k-dimensional plane. For example, in E_n, "mirror symmetry" means $(n - 1)$-symmetry, and "central symmetry" is 0-symmetry. Let K' be a k-symmetric convex body of maximum volume contained in K. Let

$$c(K; k) = V(K')/V(K),$$

and

$$c(n, k) = \inf\{c(K; k) : K \subset E_n\}.$$

Thus $c(n, 0) = c(n)$. It is proved by Krakowski **(6)** that $c(2, 1) \geqslant 5/8$, the best result known so far in E_2. Nohl **(8)** proves that any centrally symmetric plane convex body K contains a 1-symmetric body of area $2(\sqrt{2} - 1)V(K)$. In § 4 we prove

THEOREM 3.

$$c(n, k) \geqslant \frac{\max\{k!, (n-k)!\}}{2^{n-k}\, n!}, \qquad 0 \leqslant k < n.$$

Thus, for example, any convex body in E_n contains a mirror symmetric body of volume $\geqslant (2n)^{-1}V(K)$. The result of Macbeath **(7)**, that any convex body in E_n contains a rectangular parallelotope of volume $n^{-n}V(K)$, implies that $c(n, k) \geqslant n^{-n}$; however, Theorem 3 gives a considerably better lower bound.

In § 4 we also give an integral-geometric formula involving the "mean k-symmetry" of K; see formula (4.11).

2. Proofs of the covering theorems.

Proof of Theorem 1. Fáry **(1)** proved that any plane convex body K admits an inscribed affine regular hexagon H (image of a regular hexagon under an affine transformation of E_2). K is contained in the interior of the hexagram S formed by extending alternate sides of H. S can be covered by three translates, H_i, $i = 1, 2, 3$, of H. Then K is covered by the three translates of $-K$ circumscribed about the H_i.

The second part of the theorem follows from the observation that a translate of $-T$ covers two vertices of T only when T and $-T$ have a side in common. This completes the proof.

Remark. The proof shows that if P_1, P_2, P_3 are the centres of the sides of a triangle formed by connecting three alternate vertices of H, then K is the union of the three centrally symmetric sets $(2P_i - K) \cap K$ centred at P_i, $i = 1, 2, 3$.

Proof of Theorem 2. We prove the first inequality, the second being trivial, as shown in § 1. Let $\mathfrak{F}$ be the family of all translates of K by elements of K, so $\mathfrak{F} = \{q + K : q \in K\}$. One notices that $\mathfrak{F}$ is a pairwise intersecting family, since

$$q_1 + q_2 \in (q_1 + K) \cap (q_2 + K),$$

whenever $q_1, q_2 \in K$. Hence there exist $r = h(K)$ points $p_1, \ldots, p_r$ such that each $q + K, q \in K$, contains at least one of them. In other words for each $q \in K$, there is at least one of the p_i such that $p_i \in q + K$, or equivalently, $q \in p_i - K$. Thus

$$K \subset \bigcup_{1 \leqslant i \leqslant r} \{p_i - K\},$$

and $g(K) \leqslant r = h(K)$; hence $g(n) \leqslant h(n)$. This completes the proof.

Proof of (1.3). We show that a regular tetrahedron T in E_3 cannot be covered by fewer than seven translates of $-T$; hence $g(3) \geqslant g(T) \geqslant 7$. Assume that T has edges of length one and let E denote the 1-skeleton of T (union of the six edges of T). For each translate $p - T$ of $-T$, $p \in E_3$, let $l(p)$ denote the total length of $E \cap (p - T)$. Consideration of a few cases (we omit the tedious details) shows that $l(p) \leqslant 1$ for all $p \in E_3$. Moreover, $l(p) = 1$ only if the plane of some face of $p - T$ contains a vertex of T.

Suppose now that there exist p_i, $i = 1, \ldots, 6$, such that

$$T \subset \bigcup_{1 \leqslant i \leqslant 6} (p_i - T).$$

Let $T_i = p_i - T$, $i = 1, \ldots, 6$. Since the total edge length of T is 6, it follows that $l(p_i) = 1$, each T_i has a face on some vertex of T, and it is not possible that $T_i \cap E$ and $T_j \cap E$ "overlap" (have a segment of non-zero length in common) for $i \neq j$. Each vertex of T must be interior to a face of some T_i; otherwise, a neighbourhood in the interior of T near that vertex would not be covered. But a vertex cannot be interior to faces of T_i, T_j, $i \neq j$, since that would yield an "overlap" of $T_i \cap E$ and $T_j \cap E$. Thus four of the T_i, say for $i = 1, 2, 3, 4$, have a vertex of T interior to one of their faces, while T_5 and T_6 do not. But then T_5 and T_6 do not cover any interior points of T, and it is clear that the interior of T is not contained in $\bigcup_{1 \leqslant i \leqslant 4} T_i$. (In fact, $V(T_i \cap T) \leqslant \frac{2}{9} V(T)$, for $1 \leqslant i \leqslant 4$.) This contradiction completes the proof.

Remark. We show how T can be covered by eight translates of $-T$. In the following, assume that the centroid of T is at the origin.

(i) Let p be the mid-point of an edge of $\frac{1}{2}T$ and q the mid-point of the opposite edge. Let T_1 and T_2 be translates of $-T$ with centroids at p and q respectively. One observes that $\frac{1}{2}T \subset T_1 \cup T_2$.

(ii) T is the union of a regular octahedron R and four translates S_i, $1 \leqslant i \leqslant 4$, of $\frac{1}{2}T$ .Each S_i can be covered by two translates of $-T$, by (i). Hence there are points p_i, $1 \leqslant i \leqslant 8$, such that

$$\bigcup_{1 \leqslant i \leqslant 4} S_i \subset \bigcup_{1 \leqslant i \leqslant 8} T_i,$$

where $T_i = p_i - T$. With a judicious choice of the p_i, R is also covered by the T_i, so

$$T \subset \bigcup_{1 \leqslant i \leqslant 8} T_i.$$

In fact, if the vertices of T are $a_1, \ldots, a_4$, it can be verified that the following choice of p_i works:

$$p_i = \begin{cases} \frac{1}{4}(a_i + 3a_{i+1}), & 1 \leqslant i \leqslant 4, \\ \frac{1}{4}(2a_i + a_{i+1} + a_{i+2}), & 5 \leqslant i \leqslant 8, \end{cases}$$

with the convention that $a_i = a_j$ when $i \equiv j \pmod 4$.

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This remark, together with the proof of (1.3), implies that $g(T)$ is either 7 or 8. We conjecture that $g(T) = 8$. Indeed, it seems that the surface of T cannot be covered by fewer than eight translates of $-T$.

Proof that $g(n) < \infty$. Let T be a simplex of maximum volume inscribed in K. Then, as proved in **(7)**, at each vertex of T some support plane of K is parallel to the opposite face of T. These support planes form a simplex containing K, and in fact this simplex is a translate of $-nT$ (one sees this by transforming the entire configuration by an affine transformation of E_n sending T onto a regular simplex). Thus if we can cover $-nT$ by r translates of $-T$, then we can cover K by r translates of $-K$. Hence $g(n) \leqslant r(n)$, where $r(n)$ is the minimum number of translates of a regular simplex T of E_n needed to cover nT.

Remark. A similar argument can be used to prove that $h(n)$ is finite. Indeed, suppose $\mathfrak{F} = \{q + K : q \in Q\}$ is a pairwise intersecting family of translates of K. One easily shows that the condition that $\mathfrak{F}$ is pairwise intersecting is equivalent to

$$Q - Q \subset K - K.$$

Using the simplex T of the last proof, one sees that $-K$ is contained in a translate of nK; hence $K - K$ is contained in a translate of $(n + 1)K$. But some translate of Q is contained in $Q - Q$, thence in $(n + 1)K$. Now $(n+1)K$ is inscribed in a translate of $-n(n + 1)T$, so Q is contained in a translate of $-n(n + 1)T$. Now suppose that $-n(n + 1)T$ is covered by r translates of $-T$. Then Q is covered by r translates of $-K$. That is, there exist r points $p_1, \ldots, p_r$ such that

$$Q \subset \bigcup_{1 \leqslant i \leqslant r} (p_i - K).$$

Then for each $q \in Q$, there exists one of the p_i such that $q \in p_i - K$, or equivalently, $p_i \in q + K$. Hence $h(n) \leqslant s(n)$, where $s(n)$ is the minimum number of translates of a regular simplex T in E_n needed to cover $n(n + 1)T$.

3. Proofs of the results on $\mu(n)$.

Proof of (1.1). Let f be an integrable density on K, and for each $p \in K$, let $S(p) = (2p - K) \cap K$, and $M(p) = \int_{S(p)} f(q)dq$, the mass of $S(p)$. As shown in **(9**, p. 146**)**, one has

$$\int_{S(p)\ni q} dp = 2^{-n} V(K), \qquad \text{for each } q \in K.$$

By Fubini's theorem,

$$\int_K M(p)dp = \int_K \{\int_{S(p)} f(q)dq\} dp = \int_K f(q)\{\int_{S(p)\ni q} dp\} dq = 2^{-n} V(K) M(K).$$

Thus, for some p, $M(p) \geqslant 2^{-n} M(K)$, and it follows that $\mu(K) \geqslant 2^{-n}$. Hence $\mu(n) \geqslant 2^{-n}$, as was to be proved.

Proof of (1.2). Let K be a convex body in E_2, and let f be a density on K. By Theorem 1, K is the union of three centrally symmetric convex sets; hence one of these necessarily has at least one third the total mass of K. Hence $\mu(K) \geqslant \frac{1}{3}$, and $\mu(2) \geqslant \frac{1}{3}$.

If T is an equilateral triangle, it is easy to prove that any translate of $-T$ covers at most one third of the perimeter of T; hence any centrally symmetric subset of T covers at most one-third of the perimeter of T. Thus by concentrating the mass in a uniform strip around the boundary of T, we can obtain, for each $\epsilon > 0$, an f such that $\mu(T;f) \leqslant \frac{1}{3} + \epsilon$. Hence $\mu(T) \leqslant \frac{1}{3}$, and $\mu(2) \leqslant \frac{1}{3}$. This completes the proof.

4. The lower bound on k-symmetry. Before proceeding to the proof of Theorem 3, it will be convenient to introduce the following notation. An "n-frame" is an n-tuple of mutually orthogonal unit vectors $e_1, \ldots, e_n$ at the origin. The subspace spanned by $e_{i_1}, \ldots, e_{i_r}$ will be denoted by $[e_{i_1}, \ldots, e_{i_r}]$. We need a preliminary

LEMMA. *Let K be a convex body of volume V in E_n. Let $1 \leqslant k \leqslant n-1$. Then there is an n-frame $e_1, \ldots, e_n$ such that*

$$A'A'' \leqslant \frac{n!}{(n-k)!} V,$$

where A' is the k-dimensional volume of the projection of K onto $[e_1, \ldots, e_k]$, and A'' is the $(n-k)$-dimensional volume of the projection of K onto $[e_{k+1}, \ldots, e_n]$.

Proof. By a lemma of Macbeath (**7**, p. 59), there is a unit vector e_1 such that

$$b_1 P_1 \leqslant nV,$$

where b_1 is the length of the projection of K onto $[e_1]$ and P_1 is the $(n-1)$-dimensional volume of the projection K_1 of K onto the hyperplane E_1 orthogonal to $[e_1]$. Applying Macbeath's lemma to K_1 in E_1, we next find a unit vector $e_2 \perp e_1$ such that

$$b_2 P_2 \leqslant (n-1)P_1,$$

where b_2 is the length of the projection of K onto $[e_2]$ and P_2 is the $(n-2)$-dimensional volume of the projection K_2 of K onto the $(n-2)$-plane orthogonal to $[e_1, e_2]$. Continuing in this manner, we find for each r, $2 \leqslant r \leqslant k$, a unit vector $e_r \perp e_1, \ldots, e_{r-1}$, such that

$$b_r P_r \leqslant (n-r+1)P_{r-1},$$

where b_r is the length of the projection of K onto $[e_r]$ and P_r is the $(n-r)$-dimensional volume of the projection K_r of K onto the $(n-r)$-plane orthogonal to $[e_1, \ldots, e_r]$. One deduces from the above string of inequalities that

$$b_1 b_2 \ldots b_k P_k \leqslant \frac{n!}{(n-k)!} V. \tag{4.1}$$

Let A' be the k-dimensional volume of the projection K' of K onto $[e_1, \ldots, e_k]$. Since $b_1 b_2 \ldots b_k$ is the k-dimensional volume of a rectangular parallelotope circumscribed about K' in $[e_1, \ldots, e_k]$, we have

$$A' \leqslant b_1 b_2 \ldots b_k. \tag{4.2}$$

The lemma follows from (4.1) and (4.2), with $A'' = P_k$.

Proof of Theorem 3. Let the convex body $K \subset E_n$ have volume V. Let $e_1, \ldots, e_n$ be the n-frame of co-ordinate vectors

$$e_i = (\delta_{i1}, \ldots, \delta_{in}), \qquad i = 1, \ldots, n; \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Through each point $p \in [e_1, \ldots, e_{n-k}]$, there is a k-plane $E_k(p)$ orthogonal to $[e_1, \ldots, e_{n-k}]$. Let $K(p)$ be the reflection of K through $E_k(p)$, and let $V(p)$ be the volume of $K \cap K(p)$. $K \cap K(p)$ is a k-symmetric body contained in K. We proceed to find a positive lower bound for its maximum volume.

Through each point $q \in [e_{n-k+1}, \ldots, e_n]$, there is an orthogonal $(n-k)$-plane $E_{n-k}(q)$. Let $S(p, q)$ be the $(n-k)$-dimensional volume of

$$E_{n-k}(q) \cap K \cap K(p).$$

Then

$$V(p) = \int_{K''} S(p, q) dq,$$

where the integration is over the projection K'' of K onto $[e_{n-k+1}, \ldots, e_n]$. If K' is the projection of K onto $[e_1, \ldots, e_{n-k}]$, then

$$\int_{K'} V(p) dp = \int_{K'} dp \int_{K''} S(p, q) dq = \int_{K''} dq \int_{K'} S(p, q) dp. \tag{4.3}$$

Now $E_{n-k}(q) \cap K(p)$ is the reflection of $E_{n-k}(q) \cap K$ through the *point* $(p, q) = E_k(p) \cap E_{n-k}(q)$; hence, by the formula of (**9**, p. 145),

$$\int_{K'} S(p, q) dp = [S(q)]^2 / 2^{n-k}, \tag{4.4}$$

where $S(q)$ is the $(n-k)$-dimensional volume of $E_{n-k}(q) \cap K$.

Let A' be the $(n-k)$-dimensional volume of K' and A'' the k-dimensional volume of K''. Then, by Schwarz's inequality,

$$\int_{K''} [S(q)]^2 dq \geqslant (1/A'') [\int_{K''} S(q) dq]^2 = V^2 / A''. \tag{4.5}$$

Moreover,

$$A' \max_{p \in K'} V(p) > \int_{K'} V(p) dp. \tag{4.6}$$

Then (4.3), (4.4), (4.5), and (4.6) yield

$$\max_{p \in K'} V(p) > \frac{V^2}{2^{n-k} A' A''}. \tag{4.7}$$

In (4.7), $A'A''$ is the product of the projections of K onto any preassigned pair of orthogonal k-plane and $(n-k)$-plane. The lemma implies that for some such pair

$$A'A'' \leqslant n!\, V \min\left\{\frac{1}{(n-k)!}, \frac{1}{k!}\right\}. \tag{4.8}$$

The theorem follows from (4.7) and (4.8), since

$$c(K;k) \geqslant V^{-1} \max_{p \in K'} V(p).$$

Remark. From (4.3) and (4.4), we have

$$\int_{K'} V(p)dp = 2^{k-n} \int_{K''} [S(q)]^2 dq. \tag{4.9}$$

If $d\tilde{E}_k$ is the integral-geometric "rotational density" for k-planes, then $d\tilde{E}_k = d\tilde{E}_{n-k}$ for orthogonal planes. The kinematic densities for k-planes and orthogonal $(n-k)$-planes are given by

$$dE_k = dp d\tilde{E}_k, \qquad dE_{n-k} = dq d\tilde{E}_{n-k}, \tag{4.10}$$

where dp is the volume element in an $(n-k)$-plane orthogonal to E_k, and dq is the volume element in a k-plane orthogonal to E_{n-k} (Hadwiger **5**, p. 227). For each k-plane E_k, let $K(E_k)$ be the reflection of K across E_k and $V(E_k)$ the volume of $K \cap K(E_k)$. Then it follows from (4.9) and (4.10) that

$$\int V(E_k)dE_k = 2^{k-n} \int [V(E_{n-k} \cap K)]^2 dE_{n-k}, \tag{4.11}$$

$k = 1, 2, \ldots, n-1$, where the integrations are over all k-planes and $(n-k)$-planes respectively. (In (4.11), $V(E_{n-k} \cap K)$ is the $(n-k)$-dimensional volume of $E_{n-k} \cap K$.)

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University of California, Davis