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ON THE CODEGREES OF STRONGLY MONOLITHIC CHARACTERS OF FINITE GROUPS

TEMHA ERKOÇ and UTKU YILMAZTÜRK

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Abstract

Let G be a finite group and let χ be an irreducible character of G. The number $|G : \ker\chi|/\chi(1)$ is called the codegree of the character χ. We provide several relations between the structure of G and the codegrees of the characters in a given subset of $\mathrm{Irr}(G)$, where $\mathrm{Irr}(G)$ is the set of all complex irreducible characters of G. For example, we show that if the codegrees of all strongly monolithic characters of G are odd, then G is solvable, analogous to the well-known fact that if all irreducible character degrees of a finite group are odd, then that group is solvable.

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1. Introduction

All groups in this paper are finite. We use the notation of [3]. Let χ be an irreducible character of a group G. Initially, the codegree of χ was defined as $|G|/\chi(1)$ in [1], whereas later, it was defined as $|G : \ker\chi|/\chi(1)$ in [6]. We will follow the latter definition and take advantage of [6, Lemma 2.1]. The codegree of χ will be denoted by $\mathrm{a}(\chi)$.

There are several results connecting the structure of the group G and the codegree values of certain subsets of irreducible characters of the group. For example, [7, Theorem 1] shows that if G is solvable and the codegrees of all irreducible nonlinear, monomial, monolithic characters of G are p-power, where p is a fixed prime, then G has a normal Sylow p-subgroup. Let $N \lhd G$. In [6], the codegree graph $\Gamma(G|N)$ is defined. The vertex set $\mathrm{V}(G|N)$ of $\Gamma(G|N)$ consists of all primes dividing some integer in $\mathrm{cod}(G|N)$, where $\mathrm{cod}(G|N) = \{\mathrm{a}(\chi) : \chi \in \mathrm{Irr}(G), N \nleq \ker(\chi)\}$. There is an edge between distinct primes $p, q \in \mathrm{V}(G|N)$ if pq divides some integer in $\mathrm{cod}(G|N)$. Several connections between this graph and the structure of both G and N are proved.

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For instance, [6, Theorem B] shows that if $\Gamma(G|N)$ is disconnected, then it has exactly two connected components and the vertex set of $\Gamma(G|N)$ coincides with the set of prime divisors of the order of G when $1 < N \lhd G$.

The notion of 'strongly monolithic character' was introduced in [2, Definition 2.2]. Recall that an irreducible character χ is monolithic if $G/\ker\chi$ has a unique minimal normal subgroup.

Definition 1.1. Let G be a group. A monolithic character χ of G is called strongly monolithic if one of the following conditions is satisfied:

(i) $Z(\chi) = \ker\chi$, where $Z(\chi) = \{g \in G : |\chi(g)| = \chi(1)\}$;
(ii) $G/\ker\chi$ is a p-group whose commutator group is its unique minimal normal subgroup.

Groups all of whose nonlinear irreducible characters are monolithic and having exactly two strongly monolithic characters are classified in [5, Theorem C].

In this paper, we provide some relations between the structure of a group G and the codegrees of (monomial) strongly monolithic characters of G.

2. Theorems and proofs

Let $\mathrm{Irr}_{\mathrm{sm}}(G)$ denote the set of strongly monolithic characters of a group G and let $\mathrm{Irr}_{\mathrm{msm}}(G)$ denote the set of monomial characters in $\mathrm{Irr}_{\mathrm{sm}}(G)$. We also set $\mathrm{cod}_{\mathrm{sm}}(G) = \{\mathrm{a}(\chi) : \chi \in \mathrm{Irr}_{\mathrm{sm}}(G)\}$ and $\mathrm{cod}_{\mathrm{msm}}(G) = \{\mathrm{a}(\chi) : \chi \in \mathrm{Irr}_{\mathrm{msm}}(G)\}$.

Let $N \lhd G$ and χ be an irreducible character of G with $N \leq \ker\chi$. Then, χ may be viewed as an irreducible character of G/N. It is known that χ is a (monomial) strongly monolithic character of G if and only if it is a (monomial) strongly monolithic character of G/N. Thus, $\mathrm{Irr}_{\mathrm{sm}}(G/N) \subseteq \mathrm{Irr}_{\mathrm{sm}}(G)$ and $\mathrm{Irr}_{\mathrm{msm}}(G/N) \subseteq \mathrm{Irr}_{\mathrm{msm}}(G)$. In the proofs, we use these facts without further reference.

Let $\mathrm{h}(G)$ denote the Fitting length of a solvable group G.

Lemma 2.1. *Let G be a nonabelian group, $N \lhd G$ and m a fixed positive integer.*

(a) *Then,* $\mathrm{h}(N) \leq m$ *if and only if* $\mathrm{h}(N\ker\chi/\ker\chi) \leq m$ *for all strongly monolithic characters χ of G.*
(b) *Assume further G is solvable. Then,* $\mathrm{h}(N) \leq m$ *if and only if* $\mathrm{h}(N\ker\chi/\ker\chi) \leq m$ *for all monomial, strongly monolithic characters χ of G.*

Proof. Clearly, the 'if' parts of both cases are true. To prove the 'only if' parts, assume that the assertions are false and let G be a minimal counterexample for both cases. First, we will show that G has a unique minimal normal subgroup. To see why this is true, assume that G has two distinct minimal normal subgroups E_1 and E_2. Then, for all strongly monolithic characters χ of G/E_i,

$$(NE_i/E_i)(\ker\chi/E_i)/(\ker\chi/E_i) \cong N\ker\chi/\ker\chi.$$

By the minimality of G, we obtain $\mathrm{h}(NE_i/E_i) \le m$ $(i = 1, 2)$. Note that the 1–1 homomorphism $\theta : G \to G/E_1 \times G/E_1$ given by $g \mapsto (gE_1, gE_2)$ allows us to see N as a subgroup of $NE_1/E_1 \times NE_2/E_2$, which yields

$$\mathrm{h}(N) \le \mathrm{h}(NE_1/E_1 \times NE_2/E_2) \le \max\{\mathrm{h}(NE_1/E_1), \mathrm{h}(NE_2/E_2)\} \le m,$$

which is a contradiction. Thus, G has a unique minimal normal subgroup, say M, and so there exists a faithful irreducible character of G.

Now assume that $1 < \mathrm{Z}(G)$. Then, $\mathrm{h}(N) = \mathrm{h}(N\mathrm{Z}(G)/\mathrm{Z}(G)) \le m$, which contradicts the choice of G. Thus, G is centreless and so all faithful irreducible characters of G are strongly monolithic by [3, Lemma 2.27]. By the hypothesis of the theorem, we obtain the contradiction $\mathrm{h}(N) \le m$ and this completes the proof of item (a).

Now let us assume that G is solvable and prove the 'only if' part of item (b). Now, G has a monomial strongly monolithic irreducible character. If $1 < \Phi(G)$, then $\mathrm{h}(N) = \mathrm{h}(N\Phi(G)/\Phi(G)) \le m$, which contradicts the choice of G. Thus, $\Phi(G) = 1$ and so $\mathrm{F}(G) = M$ has a complement H in G. Let $1 \ne \lambda$ be an irreducible character of M. Note that M is abelian and so $\lambda(1) = 1$. Then, $K := I_G(\lambda) = MI_H(\lambda)$ and $M \cap I_H(\lambda) = 1$ since M is complemented by H in G. By [3, Problem 6.18], there exists an irreducible character α of K such that $\alpha_M = \lambda$. Note that $\alpha(1) = \lambda(1) = 1$. By [3, Theorem 6.11], α^G is irreducible and faithful since $\lambda \ne 1$ is an irreducible constituent of $(\alpha^G)_M$. Thus, $\chi := \alpha^G$ is faithful and monomial. We have already seen that all faithful irreducible characters of G are strongly monolithic. Thus, χ is a monomial, strongly monolithic character of G and so by hypothesis, $\mathrm{h}(N) = \mathrm{h}(N \ker\chi/\ker\chi) \le m$, which is a contradiction. This contradiction completes the proof. □

From [4, Theorem 1.3], $\mathrm{h}(G) \le |\mathrm{cod}(G)| - 1$ for all solvable groups G. Here, we provide an upper bound for $\mathrm{h}(G)$ in terms of the number of the codegrees of just monomial and strongly monolithic characters of a solvable group G.

THEOREM 2.2. *We have* $\mathrm{h}(G) \le |\mathrm{cod}_{\mathrm{msm}}(G)| + 1$ *for all finite solvable groups* G.

PROOF. Let G be a minimal counterexample. Assume that G has no faithful, monomial, strongly monolithic character. By the minimality of G,

$$\mathrm{h}(G/\ker\chi) \le |\mathrm{cod}_{\mathrm{msm}}(G/\ker\chi)| + 1 \le |\mathrm{cod}_{\mathrm{msm}}(G)| + 1$$

for all $\chi \in \mathrm{Irr}_{\mathrm{msm}}(G)$. However now, by using Lemma 2.1, $\mathrm{h}(G) \le |\mathrm{cod}_{\mathrm{msm}}(G)| + 1$, which is a contradiction. Thus, G has at least one faithful, monomial, strongly monolithic character and so G has a unique minimal normal subgroup, say M.

Now assume that $N \lhd G$ and $\mathrm{h}(G/N) = \mathrm{h}(G)$. We argue that $N = 1$. Otherwise, by the minimality of G,

$$\mathrm{h}(G) = \mathrm{h}(G/N) \le |\mathrm{cod}_{\mathrm{msm}}(G/N)| + 1 \le |\mathrm{cod}_{\mathrm{msm}}(G)| + 1,$$

which is a contradiction. Thus, $N = 1$. This implies that $\Phi(G) = 1 = \mathrm{Z}(G)$ and so $\mathrm{F}(G) = M$ has a complement H in G.

Let ψ be an irreducible character of G/M with codegree as large as possible. Since $\mathrm{F}(G) = M \le \ker\psi$, G has an irreducible character θ with $a(\psi) < a(\theta)$ by

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[4, Lemma 2.11]. Note that θ is faithful since it does not lie in $\mathrm{Irr}(G/M)$. Now let χ be a faithful irreducible character of G with the largest possible codegree among the faithful irreducible characters of G. We claim that χ is monomial. To see why this is true, let λ be an irreducible constituent of χ_M. Clearly, $\lambda \neq 1$. Now $K := I_G(\lambda) = MI_H(\lambda)$ and $M \cap I_H(\lambda) = 1$ since M is complemented by H in G. Thus, there exists an irreducible character α of K such that $\alpha_M = \lambda$. Note that $\alpha(1) = \lambda(1) = 1$. From [3, Theorem 6.11], α^G is irreducible and faithful since $\lambda \neq 1$ is an irreducible constituent of $(\alpha^G)_M$. Moreover, $\chi = \beta^G$ for some $\beta \in \mathrm{Irr}(K)$. By the choice of χ,

$$|G|/|G:K| = a(\alpha^G) \leq a(\chi) = |G|/\chi(1) = |G|/\beta^G(1) = |G|/\beta(1)|G:K|,$$

which forces $\beta(1) = 1$. This means χ is monomial as desired. Therefore, χ is a monomial strongly monolithic character of G and so $\mathrm{a}(\chi) \in \mathrm{cod}_{\mathrm{msm}}(G)$. However, $a(\chi) \notin \mathrm{cod}_{\mathrm{msm}}(G/M)$ since $a(\psi) < a(\theta) \leq a(\chi)$ which means $|\mathrm{cod}_{\mathrm{msm}}(G/M)| \leq |\mathrm{cod}_{\mathrm{msm}}(G)| - 1$. Thus,

$$\mathrm{h}(G) = \mathrm{h}(G/F(G)) + 1 = \mathrm{h}(G/M) + 1 \leq |\mathrm{cod}_{\mathrm{msm}}(G/M)| + 2 \leq |\mathrm{cod}_{\mathrm{msm}}(G)| + 1.$$

This final contradiction completes the proof. □

Let $G = \mathrm{SL}(2,3)$. Then, $\mathrm{h}(G) = 2 = |\mathrm{cod}_{\mathrm{msm}}(G)| + 1$. This example shows that the upper bound in Theorem 2.2 is the best possible.

THEOREM 2.3. *Let G be a finite nonabelian group. If there exists a fixed prime number p such that $\chi(1)_p = |G : \ker\chi|_p > 1$ for all strongly monolithic characters χ of G, then* $\mathrm{h}(G) \leq |\mathrm{cod}_{\mathrm{sm}}(G)| + 1$. *In particular, G is solvable.*

PROOF. Let G be a minimal counterexample and note that the hypothesis is inherited by factor groups. Assume that G has no faithful strongly monolithic character. By the minimality of G,

$$\mathrm{h}(G/\ker\chi) \leq |\mathrm{cod}_{\mathrm{sm}}(G/\ker\chi)| + 1 \leq |\mathrm{cod}_{\mathrm{sm}}(G)| + 1$$

for every $\chi \in \mathrm{Irr}_{\mathrm{sm}}(G)$. However now, by using Lemma 2.1, we obtain $\mathrm{h}(G) \leq |\mathrm{cod}_{\mathrm{sm}}(G)| + 1$, which is a contradiction. Thus, G has at least one faithful strongly monolithic character and this implies that all faithful irreducible characters of G are strongly monolithic. We also deduce that G has a unique minimal normal subgroup, say M.

Let χ be an irreducible character of G which does not contain M in its kernel. Then, χ is strongly monolithic since it is faithful and so p does not divide $\mathrm{a}(\chi)$ by hypothesis. Therefore, p does not divide the order of M and the action of P on M is Frobenius by [6, Theorem A], where P is a Sylow p-subgroup of G. Hence, G is solvable since M is nilpotent and G/M is solvable by the minimality of G.

Now we argue that $\Phi(G) = 1$. Otherwise, we would have

$$\mathrm{h}(G) = \mathrm{h}(G/\Phi(G)) \leq |\mathrm{cod}_{\mathrm{sm}}(G/\Phi(G))| + 1 \leq |\mathrm{cod}_{\mathrm{sm}}(G)| + 1,$$

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which is a contradiction. Thus, $\Phi(G) = 1$ which yields $\mathrm{F}(G) = M$. By using [4, Lemma 2.11], we deduce that $|\mathrm{cod}_{\mathrm{sm}}(G/M)| \leq |\mathrm{cod}_{\mathrm{sm}}(G)| - 1$ and so

$$\mathrm{h}(G) = \mathrm{h}(G/F(G)) + 1 = \mathrm{h}(G/M) + 1 \leq |\mathrm{cod}_{\mathrm{sm}}(G/M)| + 2 \leq |\mathrm{cod}_{\mathrm{sm}}(G)| + 1,$$

which is the final contradiction completing the proof. □

It is known that if all irreducible character degrees of a finite group G are odd, then G is solvable. We provide an analogue of this fact in terms of codegrees by having an assumption on just the strongly monolithic characters of G.

THEOREM 2.4. *Let G be a group and assume that $4 \nmid \mathrm{a}(\chi)$ for all strongly monolithic characters χ of G. Then, G is solvable. In particular, if $\mathrm{a}(\chi)$ is odd for all strongly monolithic characters χ of G, then G is solvable.*

PROOF. Assume that the theorem is false and let G be a minimal counterexample. Let $1 < N \lhd G$. Then, since G/N is solvable by the minimality of G, we conclude that N is not solvable. In particular, N cannot be abelian. Thus, $\mathrm{Z}(G) = 1$.

It is not difficult to see that G has a unique minimal normal subgroup, say M. Note that M is not abelian. Let $1 \neq \lambda$ be an irreducible character of M and choose an irreducible character χ of G with $[\chi_M, \lambda] \neq 0$. Note that χ is faithful and so strongly monolithic. Since $4 \nmid \mathrm{a}(\chi)$, we see that $4 \nmid \mathrm{a}(\lambda)$ by [6, Lemma 2.1(2)], which means M also satisfies the hypothesis of the theorem. It turns out that $M = G$ is a simple group. From the equality

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = 1 + \sum_{1 \neq \chi \in \mathrm{Irr}(G)} \chi(1)^2,$$

we deduce that G has a nonprincipal irreducible character, say χ, with odd degree. Since G is a simple group, we see that χ is a strongly monolithic character and so 4 does not divide $\mathrm{a}(\chi) = |G|/\chi(1)$ by hypothesis. This forces the order of the Sylow 2-subgroup of G to be 2 since $\chi(1)$ is odd. This implies that G has a normal 2-complement. However, this contradicts the simplicity of G. □

COROLLARY 2.5. *Let G be a group and assume that $\mathrm{a}(\chi)$ is a prime power for all irreducible characters χ of G. Then, G is solvable.*

PROOF. Note that all vertices of the graph $\Gamma(G)$ in [6] are isolated and so G has at most two prime divisors by [6, Theorem E(2)]. Hence, G is solvable. □

Now we generalise Corollary 2.5 by obtaining the solvability of G with the assumption that the codegrees of only the strongly monolithic characters of G are prime powers.

THEOREM 2.6. *Let G be a group and assume that $\mathrm{a}(\chi)$ is a prime power for all strongly monolithic characters χ of G. Then, G is solvable.*

PROOF. Assume that the theorem is false and let G be a minimal counterexample. It is not difficult to see that G has a unique minimal normal subgroup, say M.

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Let $1 < N \lhd G$. Then, since G/N is solvable by the minimality of G, we conclude that N is not solvable. Thus, $Z(G) = 1$ and M is nonsolvable. It follows that G has a faithful irreducible character and all such characters are strongly monolithic.

Let $1 \neq \lambda$ be an irreducible character of M and choose an irreducible character χ of G with $[\chi_M, \lambda] \neq 0$. Note that χ is faithful and so strongly monolithic since M is the unique minimal normal subgroup of G. Thus, $a(\chi)$ is a prime power by hypothesis. Now, $a(\lambda)$ is a prime power too, since $a(\lambda) \mid a(\chi)$ by [6, Lemma 2.1(2)]. Thus, M also satisfies the hypothesis of the theorem which means $G = M$ is a simple group. However, this contradicts [6, Lemma 2.3]. □

Let p be a prime divisor of the order of a group G and let $\mathscr{A}$ be either the set of nonlinear, monomial, monolithic characters in $\mathrm{Irr}(G)$ or the set of nonlinear, monomial, monolithic characters in $\mathrm{IBr}(G)$, where $\mathrm{IBr}(G)$ denotes the set of irreducible p-Brauer characters of G. If G is solvable and $a(\chi)$ is a power of p for all χ in $\mathscr{A}$, then G has a normal Sylow p-subgroup by [7, Theorem 1]. We give an analogue of this theorem. Note that we do not assume that G is solvable. In fact, under the hypothesis of the following theorem, we deduce the solvability of G from Theorem 2.6.

THEOREM 2.7. *Let G be a group and let p be a fixed prime number. If* $a(\chi)$ *is a power of p for all strongly monolithic characters* χ *of G, then G has a normal Sylow p-subgroup.*

PROOF. Assume that the theorem is false and let G be a minimal counterexample. First, we argue that G has a unique minimal normal subgroup. To see why this is true, let M and N be two different minimal normal subgroups of G. By the minimality of G, the factor groups G/M, G/N and so $G/M \times G/N$ have normal Sylow p-subgroups. Thus, G, which is isomorphic to a subgroup of $G/M \times G/N$, also has a normal Sylow p-subgroup, which is a contradiction with the choice of G. Thus, G has a unique minimal normal subgroup, say M, and so has a faithful irreducible character.

Now we claim that $Z(G) = 1$. Otherwise, M is contained in $Z(G)$ and so normalises P, where P is a Sylow p-subgroup of G. Since G is a minimal counterexample, we obtain $PM \lhd G$ and so, by using a Frattini argument, we see that $G = N_G(P)M$, which means P is normal in G which is not the case. Thus, $Z(G) = 1$ as desired. This means all faithful irreducible characters of G are strongly monolithic. It turns out that G has a faithful strongly monolithic character, say χ. Then, $|G|/\chi(1) = a(\chi)$ is a power of p by hypothesis. Thus, $O_p(G) \neq 1$ by [1, Theorem 4] and it follows that $M \leq O_p(G) \leq P$, which means P/M is a Sylow p-subgroup of G/M. By the minimality of G, we see that $P/M \lhd G/M$, which is equivalent to $P \lhd G$. However, this is a contradiction. □

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TEMHA ERKOÇ, Department of Mathematics,
Faculty of Science, Istanbul University, 34134 Istanbul, Turkey
e-mail: erkoct@istanbul.edu.tr

UTKU YILMAZTÜRK, Department of Mathematics,
Faculty of Science, Istanbul University, 34134 Istanbul, Turkey
e-mail: utku@istanbul.edu.tr

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