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A CHARACTERIZATION OF BIREGULAR GROUP RINGS

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In this note biregular group rings are characterized and an example is given to show that Renault's conjecture is false.

A ring A with 1 is biregular if for all $a \in A$, AaA is generated by a central idempotent. Equivalently, A is biregular iff all the stalks of its Pierce sheaf are simple.

In [1] Bovdi and Mihovski showed that for a ring A, if the group ring AG is biregular then: (*) A is biregular and G is locally normal with the order of each finite normal sub-group of G invertible in A. A proof is found in Renault [7]. In [6] Renault showed that (*) is necessary and sufficient in case A is a finitely generated module over its centre or if A is right self-injective. He conjectured that (*) is necessary and sufficient in general. In fact (*) is not sufficient as the example below shows. Some familiarity with Pierce sheaf techniques is assumed (see [5] or [2]).

Proposition. Let A be a ring and $X = \operatorname{Spec} B(A)$ where $B(A)$ is the algebra of central idempotents of A. For $x \in X$, A_x denotes the corresponding stalk of the Pierce sheaf of A. For a group G, AG is biregular if and only if: (i) A is biregular, (ii) G is locally normal and the order of every finite normal subgroup of G is invertible in A, (iii) for every stalk A_x of A, the coefficients of each central idempotent of A_xG lift to central elements of A.

Proof. Assume AG is regular. A proof that (i) and (ii) hold is found in [7]. They can be proved directly. Since A is a homomorphic image of AG, (i) holds. For a finite normal subgroup H of G put $n = |H|$ and $r = \Sigma_H h$. Now r is central and $r^2 = nr$. This and biregularity gives that n is invertible. From this it can be shown that if Δ^+ denotes the characteristic subgroup of torsion elements with only finitely many conjugates (see [4, p. 81]) then for each $x \in X$, $A_x(G/\Delta^+)$ is a stalk of the Pierce sheaf of AG. Since it is simple $G = \Delta^+$ and (ii) follows. To show (iii), for $e \in B(A_xG)$, let $r \in AG$ be such that $r + xAG = e$. Then $AGrAG = fAG$ for some $f \in B(AG)$. It follows that $f + xAG = e$ and the coefficients of e lift to the corresponding coefficients of f, which are central in A.

Converse: By [6, Lemma 1.4], it suffices to suppose that G is finite. We need the following remark suggested to me by D. Handelman: Let A be simple

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and $|G|$ invertible in A, then AG is a finite product of simple rings. Indeed let C be the centre of A. Then CG is completely reducible, say $CG \cong \Pi_1^m S_i$, S_i simple (Artinian) rings. Then $AG = A \otimes_C \Pi S_i \cong \Pi A \otimes_C S_i$, a product of simple rings by [3, p. 109].

Now let $r \in AG$. For each $x \in X$, the ideal of A_xG generated by $r + xAG$ is, by the remark, generated by some $e_x \in B(A_xG)$. Now e_x can be lifted to a central idempotent f_x of AG. Indeed the property of being a central idempotent in A_xG can be expressed in terms of the coefficients: by a finite set of equations showing that e_x is an idempotent, by the fact that the coefficients are constant on the conjugacy classes and that they are central. The coefficients of e_x can be lifted to elements of A satisfying all the properties by [5, Prop. 3.4] and by (iii). Call the resulting element $f_x \in B(AG)$. The fact that e_x generated the ideal generated by $r + xAG$ means that there exist r_i^x, $s_i^x \in AG$, $i = 1, \ldots, n_x$, such that $\Sigma r_i^x r s_i^x - f_x \in xAG$ and $rf_x - r \in xAG$. This in turn may be expressed as a finite set of equations involving the coefficients which are satisfied modulo xA. Then there is a central idempotent u_x of A with $u_x \notin x$ such that $\Sigma u_x r_i^x r s_i^x = u_x f_x$ and $u_x r f_x = u_x r$ ([5, Lemma 4.3]). Then using the compactness of X ([5, p. 12]) we can get equations holding globally, giving r_j, $s_j \in AG$, $j = 1, \ldots, N$, $f \in B(AG)$ with $\Sigma r_j r s_j = f$ and $rf = r$ so that $AGrAG = fAG$.

Notice that (iii) is superfluous if A is such that central elements of the stalks of A lift to central elements of A. This occurs, for example, when A is finitely generated as an algebra over its centre.

Example. Let H be the quaternions and C a copy of the complex field lying in H. Put $A = \{(h_1, h_2, h_3, \ldots) \mid a_i \in H$ and the sequence eventually a constant in $C\}$. A is a biregular real algebra. Take $G = \{1, g, g^2\}$. Then AG satisfies (*) but is not biregular. In fact C is one of the stalks of A via the ideal of sequences which are eventually 0. The coefficients of the idempotent $\frac{1}{3}(1 + \omega g + \omega^2 g^2)$ of CG, ω a primitive cube root of 1, do not lift to central elements of A.

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