2

Parameter-Dependent Monic Polynomials

Definitions, Key Formulas and Other Preliminaries

In Chapter 2 we detail our notation for parameter-dependent monic polynomials and we then report the formulas relating the derivatives—with respect to the parameter—of the *zeros* of such a polynomial to the analogous derivatives of the *coefficients* of this polynomial. Most of the results reported in this book are based on these key formulas. Moreover, we will discuss other results that are also fundamental for the rest of this book: (i) the *periodicity* properties of the *zeros* of a time-dependent polynomial that is itself, *periodic*; (ii) a convenient transformation applicable to a class of nonlinear evolutionary Ordinary Differential Equations (ODEs) which causes the *transformed* ODEs thereby obtained to feature the remarkable property to be *isochronous*, i.e. to possess an *open* set of *initial* data yielding solutions *all* of which are *periodic* with a *fixed* period (independent of the initial data inside that *open* set, which is generally *quite large*); (iii) specific examples of such ODEs.

2.1 Notation

In Section 2.1 we introduce the notation for parameter-dependent polynomials that will be used throughout this book.

A monic polynomial of degree N in its argument z that is dependent on a parameter t is hereafter denoted as follows:

$$p_N\left(z;\ \vec{y}\left(t\right),\ \tilde{x}\left(t\right)\right) = z^N + \sum_{m=1}^{N}\left[y_m\left(t\right)\ z^{N-m}\right] = \prod_{n=1}^{N}\left[z - x_n\left(t\right)\right]\ . \qquad (2.3)$$

Hereafter—unless otherwise indicated—N is an *arbitrary positive integer* (generally, $N \geq 2$), *indices* such as n, m run over the *positive integers* from 1 to N, t is a *real* parameter generally having the significance of "time", the N components $y_m(t)$ of the N-vector $\vec{y}(t)$ are the N *coefficients* of the polynomial $p_N(z;\vec{y}(t),\tilde{x}(t))$, and

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the N elements of the *unordered* set $\tilde{x}(t)$ are the N *zeros* of this polynomial. Note that often, for notational convenience, the t dependence of the various quantities will *not* be explicitly displayed; for instance, we will write simply y_m or x_n rather than $y_m(t)$ or $x_n(t)$. Note that the notation $p_N(z;\vec{y},\tilde{x})$, see (2.3), is *redundant*: this polynomial is completely identified by assigning *either* its N coefficients y_m *or* its N zeros x_n. Indeed the N coefficients y_m of the polynomial $p_N(z;\vec{y},\tilde{x})$ are *explicitly* expressed by well-known formulas in terms of the N *symmetrical* sums $\sigma_m(\tilde{x})$ of the N zeros x_n of this polynomial,

$$y_m = (-1)^m \ \sigma_m(\tilde{x}) \ , \tag{2.4a}$$

$$\sigma_m(\tilde{x}) = \sum_{1\leq n_1<n_2<\cdots<n_m\leq N} \left(x_{n_1}\cdot x_{n_2}\cdots x_{n_m}\right) = \frac{1}{m!}\sum_{n_1,n_2,\ldots,n_m=1}^{N} \left(x_{n_1}\cdot x_{n_2}\cdots x_{n_m}\right) . \tag{2.4b}$$

Likewise the N zeros x_n are uniquely identified—up to their $N!$ permutations—once the N coefficients y_m of the polynomial $p_N(z;\vec{y},\tilde{x})$ are assigned, although *explicit* formulas expressing the *zeros* x_n in terms of the *coefficients* y_m are generally *only* available—in terms of elementary functions such as quadratic, cubic respectively quartic roots—for $N=2$, $N=3$ respectively $N=4$.

Hereafter—unless otherwise indicated—we assume the quantities z, y_m and x_n to be *complex* numbers; and we generally focus—unless otherwise specified—on *generic* polynomials characterized by *generic* values of their N *complex coefficients* and N *complex zeros*—with the N *zeros* being *all different among themselves*. And we adopt hereafter the standard convention, which states that *empty sums vanish* and *empty products equal unity*, $\sum_{\ell=L}^{L'}(\cdot)=0$ and $\prod_{\ell=L}^{L'}(\cdot)=1$ if $L>L'$.

For future reference, we also report the *explicit* expressions of the t-derivatives of the coefficients $y_m(t)$, which clearly follow from the formulas (2.4):

$$\dot{y}_1 = -\sum_{n=1}^{N}\dot{x}_n \ , \tag{2.5a}$$

$$\dot{y}_m = (-1)^m \sum_{j=1}^{m}\left[\dot{x}_j \sum_{1\leq n_1<n_2<\cdots<n_{j-1}<n_{j+1}<\cdots<n_m\leq N;} \left(x_{n_1}x_{n_2}\cdots x_{n_m}\right)\right] ,$$
$$m=2,\ldots,N \ , \tag{2.5b}$$

as well as the obvious identities

$$\sum_{m=1}^{N}\left[y_m\left(x_n\right)^{N-m}\right] = -\left(x_n\right)^N \ . \tag{2.6}$$

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Remark 2.1.1. As emphasized by the notation $\tilde{x}$ denoting the *unordered* set of N (generally *complex*) numbers x_n, the identification of the N zeros x_n—namely, the association of the (generally *complex*) value of the zero x_n to its index n—is *ambiguous* since the N zeros x_n are only defined up to $N!$ permutations of their indices. But—in the case of polynomials *continuously* dependent on a parameter t (say, "continuous time"—as those generally considered throughout this book, except in Chapter 7)—this *ambiguity* is obviously restricted *only* to the N values of the *zeros* at a specific value of the (*real*) parameter t: say, to the "initial" values $x_n(0)$ of the N *zeros* at $t = 0$. Indeed if the N *zeros* of the polynomial $p_N(z;\vec{y}(t),\tilde{x}(t))$ feature a *continuous* dependence on the parameter t throughout their t-evolution from $t = 0$—and moreover do *not* collide, thereby *losing* their identities—then the identity of each of them (as specified by the value of the index n associated to each of them) is determined for *all* values of the parameter t by *continuity* in t from their identity at $t = 0$.

However, when the technique of solution of a system of ODEs that characterizes the evolution of N points $x_n(t)$ moving in the *complex* x-plane is based on their identification as the N *zeros* of a t-dependent polynomial of degree N—the main tool used throughout this book—then this technique of solution provides the *configuration* of the system at time t as an *unordered* set of N coordinates $x_n(t)$ but it does *not* allow to identify, say, which is the value at time t of the specific coordinate $x_1(t)$ that has evolved by *continuity* in t from the initial data (say, $x_1(0)$ and $\dot{x}_1(0)$ for second-order ODEs). One way to do so is by tracing the time evolution of each coordinate over the *Riemann surface* associated with the configuration of the *zeros* of the polynomial (2.3). This is not a trivial endeavor, as demonstrated by various papers where this phenomenology has been studied in considerable detail [76], [89], [55], [77], [56], [75]. Another, more practical way is to integrate *numerically* the equations of motion from the initial data (possibly only with rather poor precision); or to chop up the time interval from 0 to t into several (say, s) subintervals (from 0 to t_1, from t_1 to $t_2, \ldots,$ from t_{s-1} to $t_s = t$), to solve in every subinterval by the technique described below, and to make sure that each subinterval is sufficiently short to allow the identification of each moving point by an argument of *contiguity* (approximating *continuity*) of their positions over their time evolution.

On the other hand, often the technique of solution described below also yields some important *general information* on the behavior of the solutions of the system under consideration, such as their *periodicity properties*: see below. ■

Let us end this section by emphasizing the relevance of this Remark 2.1.1 to most of the following developments.

2.2 Key Formulas

The formulas relating the t-derivatives of the N *zeros* $x_n(t)$ of the t-dependent monic polynomial $p_N(z;\vec{y}(t),\tilde{x}(t))$, see (2.3), to the analogous t-derivatives of the

N coefficients $y_m(t)$ of this polynomial underline essentially *all* the results reported in this book. The formulas for the derivatives of order 1 and 2—which play a major role in the following—are displayed in Subsection 2.2.1 and proven in Subsection 2.2.3; some of those for derivatives of higher order are reported in Subsection 2.2.2, and indications of where their proofs can be found are provided in Section 2.N. Let us emphasize that these formulas are *identities* valid for any parameter-dependent *generic* polynomial; they should be replaced by their limiting forms in the case of *nongeneric* polynomials featuring *multiple zeros*.

Hereafter superimposed dots denote t-derivatives, for instance $\dot{x}_n(t) \equiv d\,x_n(t)/dt$, $\ddot{y}_m(t) \equiv d^2 y_m(t)/dt^2$.

2.2.1 First-Order and Second-Order t Derivatives of the Zeros of a Monic t-Dependent Polynomial

The following formula relates the *first-order* t-derivative $\dot{x}_n(t)$ of the *zero* $x_n(t)$ of a generic t-dependent polynomial $p_N(z;\vec{y}(t),\tilde{x}(t))$, see (2.3), to the N *first-order* t-derivatives $\dot{y}_m(t)$ of the N coefficients $y_m(t)$ of the same polynomial:

$$\dot{x}_n = -\left[\prod_{\ell=1,\ \ell\neq n}^{N}(x_n - x_\ell)\right]^{-1}\sum_{m=1}^{N}\left[\dot{y}_m\ (x_n)^{N-m}\right]. \tag{2.7}$$

The analogous formula for *second-order* derivatives reads as follows:

$$\ddot{x}_n = \sum_{\ell=1,\ \ell\neq n}^{N}\left(\frac{2\,\dot{x}_n\,\dot{x}_\ell}{x_n - x_\ell}\right) - \left[\prod_{\ell=1,\ \ell\neq n}^{N}(x_n - x_\ell)\right]^{-1}\sum_{m=1}^{N}\left[\ddot{y}_m\ (x_n)^{N-m}\right]. \tag{2.8}$$

The inverse formula expressing the t-derivatives of the coefficients $y_m(t)$ in terms of the zeros $x_n(t)$ and their t-derivatives, see (2.5), is an immediate consequence of (2.4). We do not report the analogous formula for second derivatives since we do not use it in the following.

2.2.2 Third-Order and Fourth-Order t-Derivatives of the Zeros of a Monic t-Dependent Polynomial

In Subsection 2.2.2 we display the formulas relating the *third-order* and *fourth-order* t-derivatives of the *zero* $x_n(t)$ of a generic t-dependent polynomial $p_N(z;\vec{y}(t),\tilde{x}(t))$, see (2.3), to the corresponding t-derivatives of the N coefficients $y_m(t)$ of the same polynomial:

$$\dddot{x}_n = 3 \sum_{\ell=1;\ \ell \neq n}^{N} \left(\frac{\ddot{x}_n \ \dot{x}_\ell + \ddot{x}_\ell \ \dot{x}_n}{x_n - x_\ell} \right)$$
$$- 3 \sum_{\ell_1,\ell_2=1;\ \ell_1 \neq n,\ \ell_2 \neq n}^{N} \left[\frac{\dot{x}_n \ \dot{x}_{\ell_1} \ \dot{x}_{\ell_2}}{\left(x_n - x_{\ell_1}\right) \ \left(x_n - x_{\ell_2}\right)} \right]$$
$$- \left[\prod_{\ell=1,\ \ell \neq n}^{N} \left(x_n - x_\ell\right)^{-1} \right] \sum_{m=1}^{N} \left[\dddot{y}_m \ \left(x_n\right)^{N-m} \right], \tag{2.9}$$

$$\ddddot{x}_n = \sum_{\ell=1}^{N} \left(\frac{4 \ \dddot{x}_n \ \dot{x}_\ell + 4 \ \dddot{x}_\ell \ \dot{x}_n + 6 \ \ddot{x}_n \ \ddot{x}_\ell}{x_n - x_\ell} \right)$$
$$- 6 \sum_{\ell_1,\ell_2=1;\ \ell_1 \neq n,\ \ell_2 \neq n}^{N} \left[\frac{\ddot{x}_n \ \dot{x}_{\ell_1} \ \dot{x}_{\ell_2} + 2 \ \ddot{x}_{\ell_1} \ \dot{x}_{\ell_2} \ \dot{x}_n}{\left(x_n - x_{\ell_1}\right) \ \left(x_n - x_{\ell_2}\right)} \right]$$
$$+ 4 \sum_{\ell_1,\ell_2,\ell_3=1;\ \ell_1 \neq n,\ \ell_2 \neq n,\ \ell_2 \neq n}^{N} \left[\frac{\dot{x}_n \ \dot{x}_{\ell_1} \ \dot{x}_{\ell_2} \ \dot{x}_{\ell_3}}{\left(x_n - x_{\ell_1}\right) \ \left(x_n - x_{\ell_2}\right) \ \left(x_n - x_{\ell_3}\right)} \right]$$
$$- \left[\prod_{\ell=1,\ \ell \neq n}^{N} \left(x_n - x_\ell\right)^{-1} \right] \sum_{m=1}^{N} \left[\ddddot{y}_m \ \left(x_n\right)^{N-m} \right]. \tag{2.10}$$

2.2.3 *Proofs*

In Subsection 2.2.3 we report the proofs of (2.7) and (2.8).

The starting point to prove the relation (2.7) are the two formulas

$$\left(\frac{d}{dt}\right) p_N \left(z; \vec{y}\left(t\right), \tilde{x}(t)\right) = \sum_{m=1}^{N} \left[\dot{y}_m \ z^{N-m}\right] , \tag{2.11a}$$

$$\left(\frac{d}{dt}\right) p_N \left(z; \vec{y}\left(t\right), \tilde{x}(t)\right) = - \sum_{n=1}^{N} \left[\dot{x}_n \prod_{\ell=1,\ \ell \neq n}^{N} \left(z - x_\ell\right) \right] , \tag{2.11b}$$

which clearly obtain by t-differentiation of the expressions of the polynomial $p_N(z; \vec{y}(t), \tilde{x}(t))$ in terms of its N *coefficients* $y_m(t)$, respectively of its N *zeros* $x_n(t)$, see (2.3). They imply the relation

$$\sum_{n=1}^{N} \left[\dot{x}_n \prod_{\ell=1,\ \ell \neq n}^{N} \left(z - x_\ell\right) \right] = - \sum_{m=1}^{N} \left[\dot{y}_m \ z^{N-m}\right] , \tag{2.11c}$$

and, for $z = x_n$, this formula yields (2.7), which is thereby proven.

Likewise, an additional t-differentiation of (2.11a) yields

$$\left(\frac{d}{dt}\right)^2 p_N\left(z;\vec{y}\left(t\right),\tilde{x}\left(t\right)\right)=\sum_{m=1}^{N}\left(\ddot{y}_m\, z^{N-m}\right)\ , \tag{2.12a}$$

while an additional t-differentiation of (2.11b) yields

$$\begin{aligned}\left(\frac{d}{dt}\right)^2 p_N\left(z;\vec{y}\left(t\right),\tilde{x}\left(t\right)\right)=&-\sum_{n=1}^{N}\left\{\ddot{x}_n\left[\prod_{\ell=1,\ \ell\neq n}^{N}\left(z-x_\ell\right)\right]\right\}\\&+\sum_{\ell_1,\ell_2=1,\ \ell_1\neq\ell_2}^{N}\left\{\dot{x}_{\ell_1}\,\dot{x}_{\ell_2}\left[\prod_{\ell'=1,\ \ell'\neq\ell_1,\ell_2}^{N}\left(z-x_{\ell'}\right)\right]\right\}\\&=\sum_{m=1}^{N}\left(\ddot{y}_m\, z^{N-m}\right)\ ,\end{aligned} \tag{2.12b}$$

where the second equality is implied by (2.12a). Hence, for $z=x_n$, one gets (2.8), which is thereby proven.

2.3 Periodicity of the Zeros of a Time-Dependent Periodic Polynomial

In Section 2.3 we discuss *tersely* the topic indicated in its title, and we provide a key reference for a more detailed treatment of this topic.

Suppose that a time-dependent polynomial $p_N(z;t)$ such as (2.3)—which we assume in this section to *evolve continuously* over time and to be *generic* for *all* time, i.e., such that its N *zeros* $x_n(t)$ are for *all* time *all* different among themselves, $x_n(t)\neq x_\ell(t)$ if $n\neq\ell$—is *completely periodic* with period T, i.e., for *all* values of z:

$$p_N\left(z;\ t+T\right)=p_N\left(z;\ \vec{y}\left(t+T\right),\ \tilde{x}\left(t+T\right)\right)=p_N\left(z;\ t\right)=p_N\left(z;\ \vec{y}(t),\ \tilde{x}(t)\right)\ . \tag{2.13a}$$

This obviously implies that its N *coefficients* are as well *periodic* with the *same* period T:

$$\vec{y}\left(t+T\right)=\vec{y}\left(t\right)\ ,\quad y_m\left(t+T\right)=y_m\left(t\right)\ , \tag{2.13b}$$

and this is also clearly true for the *unordered* set of its N zeros:

$$\tilde{x}(t+T)=\tilde{x}\left(t\right)\ . \tag{2.13c}$$

But this need not be true for *each* of its N zeros $x_n(t)$, which we assume to evolve themselves *continuously* over time; so that the *unordered* character of the set $\tilde{x}(t)$ is only relevant, say, at the initial time $t=0$, because the assignment of its label n to the *zero* x_n can only be *arbitrarily assigned*—say, at the initial time $t=0$—remaining thereafter *determined* by the assumed *continuity* of $x_n(t)$ as a

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function of time t (see Remark 2.1.1). So the last formula—due to the possibility that, as it were, some *zeros* exchange their roles after one period T— does *not* imply $x_n(t+T) = x_n(t)$ but only

$$x_n\,(t+\nu T) = x_n\,(t) \ , \tag{2.14a}$$

with ν a *positive integer* obviously *not larger* than N!, $\nu \leq N!$ ($N!$ being the *maximal* number of *different* permutations of N items). But in fact—because of the *dual* nature of the exchange of roles among *zeros*—the *maximal* value $\nu_{\text{Max}}(N)$ of ν,

$$\nu \leq \nu_{\text{Max}}\,(N) \ , \tag{2.14b}$$

is *much smaller* than $N!$ (for large N). The reader interested in a more detailed discussion of this topic—including a table of *all possible values* of ν for N up to 12 and *asymptotic estimates* of the number $\nu_{\text{Max}}(N)$ for large N—is advised to study the *clear* treatment of this question in [76] (and for a detailed discussion of this phenomenology in analogous many-body contexts see [55, 89, 56, 75]).

Remark 2.3.1. Clearly the regions of initial data $\tilde{x}(0)$ yielding time evolutions with *different* periods are *separated* from each other by boundaries out of which emerge solutions such that the evolution equations—at some time t_s such that $0 < t_s < T$—feature a *singularity* due to a collision of two (or exceptionally more) different zeros, $x_n(t_s) = x_\ell(t_s)$, $n \neq \ell$. ■

Remark 2.3.2. Essentially everything that has been written thus far about the *periodicity* of the N *zeros* $x_n(t)$ of a time-dependent polynomial $p_N(z;t)$ which is itself *periodic*, is equally valid if the *periodicity* property—see (2.13)—is replaced by the property of *asymptotic periodicity*, i.e.,

$$\lim_{t\to\infty} \left[p_N\,(z;\ t+T) - p_N\,(z;\ t)\right] = 0\ . \ ■ \tag{2.15}$$

2.4 How Certain Evolution Equations Can Be Made Periodic

In Section 2.4 we tersely review a *trick* that allows to modify certain evolution equations so that the modified equations thereby obtained feature many, or perhaps only or almost only, *periodic* solutions. For simplicity we illustrate this *trick* in the simple context of a *single scalar autonomous* ODE of *second* order, but the alert reader shall immediately understand how this finding can be extended to more general contexts: for instance, ODEs of *different* orders, *systems* of such equations, or possibly *nonautonomous* equations. These results provide the basis for many of the findings reported in Chapter 4.

Let us therefore focus on the following (*autonomous*) ODE:

$$\gamma'' = f\left(\gamma',\ \gamma;\ r\right) \ , \tag{2.16a}$$

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where $\gamma \equiv \gamma(\tau)$ is the dependent variable, τ is the independent variable, appended primes denote differentiations with respect to τ, and there is a *real rational* value of the parameter r,

$$r = \frac{q}{k} \tag{2.16b}$$

with q and k two *coprime integers* and, for definiteness, $k \geq 1$, for which the function $f(w, v; r)$ satisfies the following *scaling* property:

$$f\left(a^{1+r}\, w,\ a^r\, v;\ r\right) = a^{2+r} f\,(w,\ v;\ r)\ , \tag{2.16c}$$

with a an *arbitrary* parameter. Hereafter we assume to always work with *complex* numbers, unless otherwise indicated.

The typical example of function $f(w, v;\ r)$ satisfying the condition (2.16c) is

$$f\,(w,\ v;\ r) = c\, w^\alpha\ v^{1-\alpha+(2-\alpha)/r}\ , \tag{2.17}$$

where we generally assume the parameter c to be an arbitrary *complex* number and the parameter α, as the parameter r, to be a *real rational* number—although this last condition is not actually necessary in order that (2.17) satisfy (2.16c)). Indeed, the diligent reader will have no difficulty to verify that this function $f(w, v;\ r)$ satisfies the condition (2.16c)—and this would remain true if in the right-hand side of (2.17) there appeared a sum of an *arbitrary* number of other analogous terms with different assignments of the parameters c and α (but the same parameter r).

Let us now perform a change of dependent and independent variables on the ODE (2.16) by introducing a new function of the *real* variable t ("time") via the position

$$y\,(t) = \exp\,(\mathbf{i}\; r\,\omega\, t)\ \ \gamma\,(\tau)\ ,\quad \tau \equiv \tau\,(t) = \frac{\exp\,(\mathbf{i}\;\omega\, t) - 1}{\mathbf{i}\;\omega}\ , \tag{2.18a}$$

where (here and throughout) $\mathbf{i}$ is the *imaginary unit* (so that $\mathbf{i}^2 = -1$) and ω is an *arbitrary real nonvanishing* parameter to which we associate the basic period

$$T = \frac{2\pi}{|\omega|}\ . \tag{2.18b}$$

It is then easily seen that the ODE (2.16) implies that the function $y(t)$ satisfies—thanks to the *scaling* property (2.16c)—the *autonomous* ODE

$$\ddot{y} = (2\, r + 1)\ \mathbf{i}\,\omega\,\dot{y} + r\ \ (r + 1)\ \ \omega^2\, y + f\,(\dot{y} - \mathbf{i}\; r\,\omega\, y,\ y;\ r)\ . \tag{2.19a}$$

It is then clear—see (2.18)—that this ODE (2.19a) is likely to possess lots of periodic solutions. Indeed *all* its solutions $y(t)$ which correspond via (2.18) to

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solutions $\gamma(\tau)$ of the ODE (2.16) which are themselves *entire* functions of the *complex* variable τ are clearly *periodic* with period kT (see (2.16b) and (2.18)):

$$y(t + k\,T) = y(t)\ . \tag{2.19b}$$

And solutions $\gamma(\tau)$ which are *not entire* functions of τ but have a *simple analytic* dependence on the variable τ shall also entail *simple periodicity* properties for the corresponding solutions of the ODE (2.19a). In the following subsections of this Section 2.4 we identify several *solvable* ODEs of type (2.16) and discuss the *periodicity* properties of their variant (2.19a); these findings shall play an important role below.

Remark 2.4.1. It is also of interest for future developments to replace the assumption made above that ω is an arbitrary *real* nonvanishing parameter with the hypothesis that ω is instead an arbitrary *complex* nonvanishing parameter with, say, *positive imaginary part*, $\mathrm{Im}[\omega] > 0$. This possibility will be tersely explored at the end of each of the following subsections. ■

Remark 2.4.2. In the following subsections of this Section 2.4, we discuss tersely several examples of ODEs belonging to the class (2.16) that are *explicitly solvable* in terms of *elementary* functions. In these treatments we forsake the possibility to introduce additional *free* parameters in these equations via the trick to rescale and shift the *dependent* variable—although this trivial trick might play a less trivial role below, by increasing the number of *free* parameters featured by the models under consideration. ■

Let us finally mention that at this point some readers might well prefer to skip the following subsections of this Section 2.4 and only return to them when their findings are referred to below (mainly in the later sections of Chapter 4).

2.4.1 The ODE $\gamma'' = c\left(\gamma'\right)^2 \gamma^{-1}$ and Its Isochronous Variant

In Subsection 2.4.1 the solution of the ODE

$$\gamma'' = c\ \left(\gamma'\right)^2\ \gamma^{-1} \tag{2.20}$$

is reported. Here $\gamma \equiv \gamma(\tau)$, primes denote differentiations with respect to the independent variable τ, and c is an *a priori arbitrary complex* number (but see below for special values of this parameter).

Remark 2.4.1.1. This equation of motion, (2.20), is *Hamiltonian*, being yielded by the Hamiltonian

$$h(p;\ \gamma) = a\ \left[p\,\gamma^{c}\right]^{\beta}\ , \tag{2.21}$$

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where, in the Hamiltonian context, the variable $p \equiv p(\tau)$ plays the role of *canonical momentum* conjugated to the *canonical variable* $\gamma \equiv \gamma(\tau)$, and the (*nonvanishing*) parameters a and β may be *arbitrarily assigned* ($a \neq 0$, $\beta \neq 0$, $\beta \neq 1$). ■

The modified variant (see (2.19a)) of this equation (2.20) reads as follows:

$$\ddot{y} = [2\, r\; (1-c)+1]\; \mathbf{i}\, \omega\, \dot{y} + r\; [r\; (1-c)+1]\; \omega^2\, y + c\, \frac{(\dot{y})^2}{y}\,. \tag{2.22}$$

It features the *real* independent variable t ("time"), $y \equiv y(t)$ (here superimposed dots denote time-differentiations). Let us discuss it, with particular attention to the *periodicity* properties of its solutions. Note that the ODE (2.20) corresponds to (2.17) with $\alpha = 2$ and r essentially *arbitrary*; hence hereafter (in Subsection 2.4.1) r is an *arbitrary real rational* number, see (2.16b). Of course in the following discussion of the ODEs (2.20) we exclude from consideration the initial datum $\gamma(0) = 0$, and likewise we exclude the initial datum $y(0) = 0$ in the following discussion of the ODE (2.22).

It is easily seen that the solution of the initial-value problem of the ODE (2.20) reads as follows: For $c \neq 1$:

$$\gamma(\tau) = \gamma(0)\; \left(1 - \frac{\tau}{\bar{\tau}}\right)^{1/(1-c)}\;, \quad \bar{\tau} = \frac{\gamma(0)}{(c-1)\;\gamma'(0)}\;; \tag{2.23a}$$

for $c = 1$

$$\gamma(\tau) = \gamma(0)\;\exp\left[\frac{\gamma'(0)\;\tau}{\gamma(0)}\right]. \tag{2.23b}$$

Hence the solution of the initial-value problem of the ODE (2.22) reads as follows: for $c \neq 1$

$$y(t) = \tilde{y}(\eta)\;, \quad \tilde{y}(\eta) = y(0)\;\eta^r\;\left(\frac{\bar{\eta}-\eta}{\bar{\eta}-1}\right)^{1/(1-c)}\;,$$
$$\eta \equiv \eta(t) = \exp(\mathbf{i}\,\omega\,t)\;, \quad \bar{\eta} = 1 + \frac{\mathbf{i}\,\omega\,y(0)}{(c-1)\;\left[\dot{y}(0) - \mathbf{i}\,r\,\omega\,y(0)\right]}\;; \tag{2.24a}$$

for $c = 1$

$$y(t) = \tilde{y}(\eta)\;, \quad \eta \equiv \eta(t) = \exp(\mathbf{i}\,\omega\,t)\;,$$
$$\tilde{y}(\eta) = y(0)\;\eta^r\;\exp\left\{\left[\frac{\dot{y}(0)}{\mathbf{i}\,\omega\,y(0)} - r\right]\;(\eta-1)\right\}. \tag{2.24b}$$

These formulas imply that, for $c = 1$, the ODE (2.22)—which, in this case, reads simply

$$\ddot{y} = \mathbf{i}\,\omega\,\dot{y} + r\,\omega^2\,y + \frac{(\dot{y})^2}{y} \tag{2.25}$$

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—is *isochronous: all* its solutions are *nonsingular* and *periodic* with period kT, see (2.19b), (2.18b) and (2.16b).

The situation in the $c \neq 1$ case is more nuanced. There clearly still is an *open* set of initial data—characterized by the *inequality* $|\bar{\eta}| > 1$, i.e.,

$$|\bar{\eta}| = \left| 1 + \frac{\mathbf{i}\, \omega\, y\,(0)}{(c-1)\ \left[\dot{y}\,(0) - \mathbf{i}\, r\, \omega\, y\,(0)\right]} \right| > 1 \tag{2.26}$$

—such that, again, *all* the solutions of the ODE (2.22) are *nonsingular* and *periodic* with period kT, see (2.19b), (2.18b) and (2.16b) (and note, incidentally, that this *inequality* is automatically satisfied for arbitrary $y(0) \neq 0$ if $\dot{y}(0) = 0$ and $c = 1 + \mathbf{i}R$ with R an *arbitrary nonvanishing real* number). On the other hand, the solutions (2.24a) characterized by the *complementary inequality* $|\bar{\eta}| < 1$, i.e.,

$$|\bar{\eta}| = \left| 1 + \frac{\mathbf{i}\, \omega\, y\,(0)}{(c-1)\ \left[\dot{y}\,(0) - \mathbf{i}\, r\, \omega\, y\,(0)\right]} \right| < 1\,, \tag{2.27a}$$

while also *nonsingular* for all time, are *not periodic* because, as a function of real t, the *complex* number $\tilde{y}(\eta) \equiv \tilde{y}\,(\eta\,(t))$, see (2.24a), travels on an infinitely-sheeted Riemann surface; unless the parameter c is itself a *real rational* number,

$$c = \frac{q_c}{k_c} \quad \text{implying} \quad \frac{1}{1-c} = \frac{k_c}{k_c - q_c} \tag{2.27b}$$

with q_c and k_c two *different coprime integers* (*a priori arbitrary*, but for definiteness $k_c \geq 1$), in which case the function $\tilde{y}(\eta)$ of the *complex* variable η features a *rational* branch point at $\eta = \bar{\eta}$ (see (2.24a)). Hence clearly these solutions (2.24a) are *again periodic*, but now with, at most, period KT,

$$y\,(t + K\,T) = y\,(t)\ , \tag{2.27c}$$

where (see (2.16b) and (2.27b))

$$K = \text{MinimumCommonMultiple}\left[k,\ \ |k_c - q_c|\right]\ . \tag{2.28}$$

Remark 2.4.1.2. Here—and occasionally below—we assert that a solution is periodic with, "at most", a certain period. This means that we leave to the interested reader the, generally easy, task to ascertain more precisely what the exact period in question might be. For instance, the function $a \cos\left[(6/5)\, \omega t\right] + b \sin\left[(3/7)\, \omega t\right]$ is periodic in t with "at most" period $5 \cdot 7 \cdot T = 35\ T$ with $T = 2\pi/\,|\omega|$, but—more precisely—it is periodic with the smaller period $(35/3)\ \ T$—which of course implies that it is also periodic with the larger period $35\ T$. ■

There remains to consider, in the $c \neq 1$ case, what happens in the *nongeneric* case when the initial data satisfy the *equality*

$$|\bar{\eta}| = \left| 1 + \frac{\mathbf{i}\,\omega\, y(0)}{(c-1)\ \left[\dot{y}(0) - \mathbf{i}\, r\,\omega\, y(0)\right]} \right| = 1\,. \tag{2.29}$$

Then, the solution (2.24a) of the ODE (2.22) features a *singularity* at the (*real*) times $t = t_s$, such that $\exp(\mathbf{i}\,\omega\, t_s) = \bar{\eta}$ (see (2.24a)), unless the parameter c is rational with $q_c = k_c - 1$ so that $1/(1-c) = k_c$ is a *positive integer* (see (2.27b)): in which case—as in the case $c = 1$—*all* the solutions $\gamma(\tau)$ of the ODE (2.20) are *entire* functions of the *complex* variable τ and *all* the solutions $y(t)$ of the ODE (2.22) are *nonsingular* and *periodic* with, at most (see Remark 2.4.1.2), period kT, see (2.19b), (2.18b) and (2.16b).

A case with $c \neq 1$ worth of special notice is that with

$$c = 1 + \frac{1}{2\,r} = \frac{1 + 2\,r}{2\,r} \tag{2.30a}$$

when the ODE (2.22) reads

$$\ddot{y} = \frac{r}{2}\,\omega^2\, y + \left(\frac{1+2\,r}{2\,r}\right)\,\frac{(\dot{y})^2}{y}\,. \tag{2.30b}$$

Note the *absence* from this ODE of the *imaginary* unit $\mathbf{i}$. In this case the solution of the initial-value problem in the *real* case—with both $y(0)$ and $\dot{y}(0)$ *real*—reads as follows:

$$y(t) = y(0)\ \left\{\frac{\sin\left[\omega\ (\theta - t)\ /\ 2\right]}{\sin(\omega\,\theta\ /\ 2)}\right\}^{-2r}\,,$$

$$\tan(\omega\,\theta) = \frac{2\,r\,\omega\,\dot{y}(0)\ \ y(0)}{\left[\dot{y}(0)\right]^2 + \left[r\,\omega\, y(0)\right]^2}\,; \tag{2.31}$$

it is therefore *nonsingular* iff $-2r$ is a *positive integer*; and then *periodic* with period $T = 2\pi/|\omega|$ if $-2r$ is an *even integer*, with period $T = 4\pi/|\omega|$ if $-2r$ is an *odd integer*.

We end Subsection 2.4.1 with a terse discussion of the change in the behavior of the solution (2.24) of the ODE (2.22) if the parameter ω, instead of being *real*, is *complex*, featuring, say, a *positive imaginary* part, $\mathrm{Im}[\omega] > 0$. Then, if $r > 0$, in the *remote future* $y(t)$ vanishes,

$$\lim_{t\to+\infty}\left[y(t)\right] = 0\,, \tag{2.32a}$$

while if $r = 0$ it tends to a *generally finite* value,

$$\lim_{t\to+\infty}\left[y(t)\right] = \tilde{y}(0)\ , \tag{2.32b}$$

where the value of $\tilde{y}(0)$ can be immediately read from (2.24). In the *remote past*—when $\eta(t)$ diverges exponentially—the behavior of the solution depends on the values of both parameters, r and c, and also on the initial data in the $c = 1$ case. Indeed, clearly in the $c = 1$ case the solution $y(t)$, see (2.24b), *diverges* respectively *vanishes* in the *remote past* if the *real part* of the (generally *complex*) number $\frac{\dot{y}(0)}{\mathbf{i}\,\omega\, y(0),} - r$ is *positive* respectively *negative*; while in the $c \neq 1$ case, the solution $y(t)$, see (2.24a), *diverges* respectively *vanishes* in the *remote past* if the (generally *real*) number $r + 1/(1 - c)$ is *positive* respectively *negative*.

Clearly these behaviors of the solution $y(t)$ in the *remote future* and the *remote past* are exchanged if the *imaginary part* of ω is *negative* rather than *positive*.

2.4.2 The ODE $\gamma'' = (\bar{y})^{-1}\left(\gamma'\right)^2$ and Its Isochronous Variant

In Subsection 2.4.2 the solution of the ODE

$$\gamma'' = (\bar{y})^{-1} \ \left(\gamma'\right)^2 \ , \tag{2.33}$$

is reported, with $\gamma \equiv \gamma\,(\tau)$, primes denoting differentiations with respect to the independent variable τ, and with $\bar{y}$ an *a priori arbitrary complex* parameter ($\bar{y} \neq 0$); and the modified variant (see (2.19a), in this case with $r = 0$) of this equation,

$$\ddot{y} = \mathbf{i}\,\omega\,\dot{y} + (\bar{y})^{-1} \ \ (\dot{y})^2 \ , \tag{2.34}$$

featuring the *real* independent variable t ("time"), the *complex* dependent variable $y \equiv y(t)$ (and where superimposed dots denote time-differentiations) is then discussed, with particular attention to the *periodicity* properties of its solutions. Note that the ODE (2.33) may be obtained from (2.17) by firstly setting in it $\alpha = 2 + r$ and then letting $r \to 0$.

Remark 2.4.2.1. The equation of motion (2.33) is yielded by the *Hamiltonian*

$$h\,(p;\ \gamma) = a\,p^{\beta}\ \exp\left(\frac{\beta\,\gamma}{\bar{y}}\right)\ , \tag{2.35}$$

where, in the Hamiltonian context, the variable $p \equiv p\,(\tau)$ plays the role of *canonical momentum* conjugated to the *canonical variable* $\gamma \equiv \gamma\,(\tau)$, and the (*nonvanishing*) parameters a and β may be *arbitrarily assigned* ($a \neq 0$, $\beta \neq 0$, $\beta \neq 1$). ■

It is easily seen that the solution of the initial-value problem of the ODE (2.33) reads as follows:

$$\gamma\,(\tau) = \gamma\,(0) - \bar{y}\ \ln\left(1 - \frac{\tau}{\bar{\tau}}\right)\ ,\quad \bar{\tau} = \frac{\bar{y}}{\gamma'\,(0)}\ , \tag{2.36}$$

and the solution of the initial-value problem of the ODE (2.34) reads as follows:

$$y\,(t) = y\,(0) - \bar{y}\,\ln\left[\frac{\bar{\eta} - \exp\,(\mathbf{i}\,\omega\,t)}{\bar{\eta} - 1}\right]\,, \quad \bar{\eta} = 1 + \frac{\mathbf{i}\,\omega\,\bar{y}}{\dot{y}\,(0)}\,. \tag{2.37}$$

Clearly *all* these solutions (2.37) of the ODE (2.34)—characterized by *any initial* data $y(0)$, $\dot{y}(0)$—are (as functions of the *real* variable t) *nonsingular*; moreover, if the initial datum $\dot{y}(0)$ implies the *inequality* $|\bar{\eta}| > 1$, see (2.37), this solution is *periodic* with the basic period T (see (2.18b)), $y\,(t+T) = y(t)$; while if the initial datum $\dot{y}\,(0)$ implies the *complementary inequality* $|\bar{\eta}| < 1$, this solution (2.37) is *not periodic*, featuring instead the property $y\,(t+T) = y(t) - 2\mathbf{i}\pi\,\bar{y}$ (hence in this case this solution diverge *almost linearly* in the *remote* past and future); and note, finally, that the *nongeneric* initial datum $\dot{y}(0) = 0$ yields the trivial solution $y(t) = y(0)$.

We end Subsection 2.4.2 with a terse discussion of the change in the behavior of the solution (2.24) if the parameter ω, instead of being *real*, is *complex*, featuring, say, a *positive imaginary* part, $\mathrm{Im}\,[\omega] > 0$. Then, in the *remote future* (i.e., as $t \to +\infty$)

$$y\,(t) = y\,(0) - \bar{y}\,\ln\left[1 + \frac{\dot{y}\,(0)}{\mathbf{i}\,\omega\,\bar{y}}\right] + O\left[\exp\,(-\,|\mathrm{Im}\,[\omega]\;\,t|)\right]\,, \tag{2.38a}$$

while in the *remote past* (i.e., as $t \to -\infty$)

$$y\,(t) = -\mathbf{i}\,\omega\,\bar{y}\,t + y\,(0) + \bar{y}\,\ln\left(-\frac{\mathbf{i}\,\omega\,\bar{y}}{\dot{y}\,(0)}\right) + O\left[\exp\,(-\,|\mathrm{Im}\,[\omega]\;\,t|)\right]\,. \tag{2.38b}$$

These behaviors in the *remote future* and *past* are exchanged if instead $\mathrm{Im}\,[\omega] < 0$.

2.4.3 The ODE $\gamma'' = c\,(\gamma')^{\alpha}$ and Its Isochronous Variant

In Subsection 2.4.3, the solution of the ODE

$$\gamma'' = c\;\left(\gamma'\right)^{\alpha} \tag{2.39}$$

is reported, with $\gamma \equiv \gamma\,(\tau)$, primes denoting differentiations with respect to the independent variable τ, c an arbitrary (possibly *complex*) parameter, and α an *arbitrary real rational* parameter ($\alpha \neq 2$, since the case with $\alpha = 2$ has already been treated in the preceding Subsection 2.4.2). Note that this ODE corresponds to the special case of (2.16) with

$$r = \frac{\alpha - 2}{1 - \alpha}\,, \quad \alpha = \frac{r+2}{r+1} \tag{2.40}$$

in (2.17); hence, hereafter we also exclude from consideration the values $\alpha = 1$ and $r = -1$ (besides the value $\alpha = 2$, treated in the preceding Subsection 2.4.2); and we use hereafter—within Subsection 2.4.3—the parameter r (with the restriction

https://doi.org/10.1017/9781108553124.003 Published online by Cambridge University Press

$r \neq 0$, $r \neq -1$) instead of the parameter α, on the understanding that they are always related by these formulas (2.40). The corresponding version of the modified equation satisfied by the time-dependent function $y(t)$ related to $\gamma(\tau)$ by (2.18) reads as follows:

$$\ddot{y} = \mathbf{i}\ (2\,r+1)\ \omega\,\dot{y} + r\ (r+1)\ \omega^2\,y + c\,\left[\dot{y} - \mathbf{i}\,r\,\omega\,y\right]^{(r+2)/(r+1)}\ . \tag{2.41}$$

It is easily seen that the solution of the initial-value problem of the ODE (2.39) reads as follows:

$$\gamma(\tau) = \gamma(0) + \frac{\bar{\tau}\,\gamma'(0)}{r}\left[\left(1 - \frac{\tau}{\bar{\tau}}\right)^{-r} - 1\right]\,, \quad \bar{\tau} = \left(\frac{1+r}{c}\right)\ \left[\gamma'(0)\right]^{-1/(1+r)}\,, \tag{2.42}$$

and the solution of the initial-value problem of the ODE (2.41) reads as follows:

$$y(t) = \tilde{y}(\eta)\ , \quad \eta \equiv \eta(t) = \exp(\mathbf{i}\,\omega\,t)\ , \tag{2.43a}$$

$$\tilde{y}(\eta) = \eta^r\ \left\{y(0) + \left(\frac{1+r}{c\,r}\right)\ \left[\dot{y}(0) - \mathbf{i}\,r\,\omega\,y(0)\right]^{r/(1+r)} \cdot \right. \\ \left. \cdot \left[\left(\frac{\bar{\eta}-\eta}{\bar{\eta}-1}\right)^{-r} - 1\right]\right\}\,, \tag{2.43b}$$

$$\bar{\eta} = 1 + \mathbf{i}\,\omega\ \left(\frac{1+r}{c}\right)\ \left[\dot{y}(0) - \mathbf{i}\,r\,\omega\,y(0)\right]^{-1/(1+r)}\ . \tag{2.43c}$$

Hence these solutions (2.43)—as a function of the time t—are *nonsingular* for *all* initial data $\dot{y}(0)$, $y(0)$ if r is a *negative integer*, and also for *arbitrary real* r ($r \neq 0$, $r \neq -1$) except for the *nongeneric* set of initial data implying $|\bar{\eta}| = 1$, in which case (unless r is a *negative integer* other than -1) they hit a singularity at $t = t_s$, with $t_s = (\mathbf{i}\,\omega)^{-1}\ln(\bar{\eta})$; they are otherwise *periodic*, with at most (see Remark 2.4.1.2) period kT, provided r is a *real rational* number (see (2.16b), (2.18b) and (2.19b)).

We end this Subsection 2.4.3 with a terse discussion of the change in the behavior of the solution (2.43) if the parameter ω, instead of being *real*, features, say, a *positive imaginary* part, $\mathrm{Im}[\omega] > 0$. Then, depending on the sign of r, in the remote future or past $y(t)$ *vanishes*,

$$\lim_{t \to +r\,\infty}\left[y(t)\right] = 0\,, \tag{2.44a}$$

or tends to a *finite* value,

$$\lim_{t \to -r\,\infty}\left[y(t)\right] = \tilde{y}(0)\ , \tag{2.44b}$$

where the value of $\tilde{y}(0)$ can be easily read from (2.43b) with (2.43c).

2.4.4 The Solvable ODE $\gamma'' = \rho\,(\gamma')^2\,\gamma^{-1}+c\gamma^{\rho}$ and Related Isochronous Versions

In Subsection 2.4.4 the solution of the ODE

$$\gamma'' = \rho\ \left(\gamma'\right)^2\ \gamma^{-1}+c\,\gamma^{\rho}\ , \tag{2.45}$$

is reported. Here $\gamma \equiv \gamma\,(\tau)$, primes denote differentiations with respect to the independent variable τ, c is an *arbitrary* (*nonvanishing*, possibly *complex*) parameter, and ρ an *arbitrary real rational* parameter. Note that this ODE features in its right-hand side the sum of two terms (with the *same* parameter ρ playing a *different* role in these two terms). Also note that both terms satisfy the scaling property (2.16c), see (2.17): the first with $\alpha = 2$ and r *unrestricted*, the second with $\alpha = 0$ and

$$r = \frac{2}{\rho - 1}\ ,\quad \rho = 1 + \frac{2}{r}\ , \tag{2.46}$$

a relationship that is maintained throughout this Subsection 2.4.4 (implying the assumption $\rho \neq 1$).

Remark 2.4.4.1. The ODE (2.45) is yielded by the *Hamiltonian*

$$h\,(p;\ \gamma) = a\,p^2\,\gamma^{2\,\rho} - \frac{c\,\gamma^{1-\rho}}{2\,a\ (1-\rho)}\ , \tag{2.47}$$

where, in the Hamiltonian context, the variable $p \equiv p\,(\tau)$ plays the role of *canonical momentum* conjugated to the *canonical variable* $\gamma \equiv \gamma\,(\tau)$, and the (*nonvanishing*) parameter a may be *arbitrarily assigned* ($a \neq 0$, $\rho \neq 0$, $\rho \neq 1$). ■

The diligent reader will easily verify that the solution of the initial value problem for this ODE, (2.45), reads as follows: if $\rho \neq 1$,

$$\gamma\,(\tau) = \gamma\,(0)\ \left[1 + (1-\rho)\ \left\{\frac{\gamma'\,(0)\ \tau}{\gamma\,(0)} + \frac{c}{2}\ \left[\gamma\,(0)\right]^{\rho-1}\ \tau^2\right\}\right]^{1/(1-\rho)}\ ; \tag{2.48a}$$

if $\rho = 1$,

$$\gamma\,(\tau) = \gamma\,(0)\ \exp\left[\frac{\gamma'\,(0)\ \tau}{\gamma\,(0)} + \frac{c}{2}\,\tau^2\right]. \tag{2.48b}$$

But, as already noted, $\rho = 1$ implies $r = \infty$, see (2.46); so—as already indicated above—we restrict our consideration to the case with $\rho \neq 1$.

The corresponding version of the modified equation satisfied by the time-dependent function $y(t)$ related to $\gamma\,(\tau)$ by (2.18) reads as follows:

$$\ddot{y} = -3\,\mathbf{i}\,\omega\,\dot{y} - r\,\omega^2\,y + \left(\frac{r+2}{r}\right)\ \frac{(\dot{y})^2}{y} + c\,y^{(r+2)/r}\ , \tag{2.49}$$

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and the solution of the corresponding initial-value problem reads as follows:

$$y\,(t)=\tilde{y}\,(\eta)\ ,\quad \eta\equiv\eta\,(t)=\exp\,(\mathbf{i}\,\omega\,t)\ , \tag{2.50a}$$

$$\tilde{y}\,(\eta)=y\,(0)\,\left[b\,\left(1-\frac{\eta_+}{\eta}\right)\,\left(1-\frac{\eta_-}{\eta}\right)\right]^{-r/2}\ , \tag{2.50b}$$

$$\eta_\pm=\frac{-2+a+2\,b\pm\Delta}{2\,b}\ ,\quad \Delta^2=(a-2)^2-4\,b\ , \tag{2.50c}$$

$$a=\frac{2\,\dot{y}\,(0)}{\mathbf{i}\,r\,\omega\,y\,(0)}\ ,\quad b=\frac{c\,\left[y\,(0)\right]^{2/r}}{r\,\omega^2}\ . \tag{2.50d}$$

As for the time evolution of this solution, there are clearly various possibilities depending on the parameters and on the initial data.

Let us consider firstly the *nongeneric* set of initial data such that either $|\eta_+|=1$ or $|\eta_-|=1$ or $|\eta_+|=|\eta_-|=1$, see (2.50c). Then in its time evolution the solution $y(t)$ shall hit a *singularity*—at the time t_S such that $\eta\,(t_S)=\eta_+$ or $\eta\,(t_S)=\eta_-$, as the case may be; unless the parameter r is a *negative even integer* (or *any negative integer* if moreover $\eta_+=\eta_-$, i.e., $\Delta=0$; see (2.50c)), in which case clearly—for *any* initial data—the solution $y(t)$ is *nonsingular* for *all* time and *periodic* with at most (see Remark 2.4.1.2) period kT (see (2.18b), (2.16b) and (2.19b)).

Let us then turn our attention to the *generic* case of initial data such that $|\eta_+|\neq 1$ and $|\eta_-|\neq 1$, and moreover r is *not* an *even integer* and $\eta_+\neq\eta_-$. There are then *three* possibilities, depending on the values of the initial data $y(0)$ and $\dot{y}(0)$.

Case (i). If the initial data $y(0)$ and $\dot{y}(0)$ imply

$$|\eta_+|>1\ ,\quad |\eta_-|>1\ , \tag{2.51a}$$

the two branch points of $\tilde{y}(\eta)$—in the *complex* η-plane, at $\eta=\eta_+$ and $\eta=\eta_-$, see (2.50b)—fall *outside* the circle $\tilde{C}$, centered at the origin and of *unit* radius, on which travels the point $\eta(t)=\exp\,(\mathbf{i}\,\omega\,t)$ as a function of the time t. Hence in this case the solution $y(t)$ is *periodic* with period kT (see (2.18b) and (2.16b)), due to the *rational* branch point (of order k) at the center, $\eta=0$, of the circle $\tilde{C}$.

Case (ii). If the initial data $y\,(0)$ and $\dot{y}\,(0)$ imply

$$|\eta_+|>1\ ,\quad |\eta_-|<1\ ,\quad \text{or}\quad |\eta_+|<1\ ,\quad |\eta_-|>1\ , \tag{2.51b}$$

then the right-hand side of (2.50b) features, inside the circle $\tilde{C}$—besides the branch point at $\eta=0$—another *rational* branch point (of order $1/\,(-2k)$), hence the solution $y(t)$ is *periodic* with period $2kT$ (see (2.18b) and (2.16b)).

Case (iii). If the initial data $y\,(0)$ and $\dot{y}(0)$ imply

$$|\eta_+|<1\ ,\quad |\eta_-|<1\ , \tag{2.51c}$$

then the right-hand side of (2.50b) features, inside the circle $\tilde{C}$—besides the branch point at $\eta = 0$—two branch points at $\eta = \eta_+$ and $\eta = \eta_-$, and since due to each of these two branch points after each round around the circle $\tilde{C}$—which takes a time T, see (2.18b) and the definition $\eta(t) = \exp(\mathbf{i}\,\omega\,t)$, see (2.50a)—the solution $y(t)$ gets multiplied by a factor $\exp(\mathbf{i}r\omega t/2)$ while due to the branch point at $\eta = 0$ the solution $y(t)$ gets multiplied by a factor $\exp(-\mathbf{i}r\omega t)$ in this case the solution $y(t)$ turns out to be periodic with the basic period T, see (2.18b).

Note that this implies that, for *arbitrary nongeneric* initial data—such that $|\eta_+| \neq 1$ and $|\eta_-| \neq 1$—the solution $y(t)$ of the ODE (2.49), with an *arbitrary rational* assignment of the parameter $r = q/k$ (see (2.16b)), is *periodic* with at most (see Remark 2.4.1.2) period kT; this ODE is therefore *isochronous* (provided the parameter ω is an *arbitrary nonvanishing real* number).

If, instead, the parameter ω is *not real* but does feature, say, a *positive imaginary* part, $\mathrm{Im}\,[\omega] > 0$, then clearly the periodicity properties of the solution $y(t)$ mentioned above do not hold any more. In this case, as $t \to +\infty$, $\eta\,(t) = \exp(\mathbf{i}\,\omega\,t)$ tends exponentially to zero, while as $t \to -\infty$, $\eta\,(t) = \exp(\mathbf{i}\,\omega\,t)$ diverges exponentially; and the corresponding *asymptotic behaviors* as $t \to \pm\infty$ of the solution $y(t)$ can be read immediately from its expression (2.50).

2.4.5 Other Solvable Cases of the ODE $\gamma'' = c\left(\gamma'\right)^{\alpha}\gamma^{\beta}$ and Related Isochronous Variants

In Subsection 2.4.5 we consider the ODE

$$\gamma'' = c\,\left(\gamma'\right)^{\alpha}\,\gamma^{\beta} \tag{2.52a}$$

(where, as above, $\gamma \equiv \gamma\,(\tau)$ and appended primes indicate τ-derivatives), mainly in the special case with

$$\alpha = \frac{3+\beta}{2+\beta}\,, \quad \beta = \frac{2\,\alpha - 3}{1-\alpha}\,. \tag{2.52b}$$

In this case this ODE is *solvable* in terms of elementary functions.

Remark 2.4.5.1. In Subsection 2.4.5 we exclude from consideration the case with $\alpha = 2$, $\beta = -1$, which has already been treated in Subsection 2.4.1; while the case with $\alpha = 0$, $\beta = -3$ is a special case of the model discussed below in Subsection 4.2.4. These are the only cases in which both α and β are *integers* (consistently with (2.52b)). For obvious reasons (see (2.52b)), in the first part of this Subsection 2.4.5 we assume $\alpha \neq 1$ and $\beta \neq -2$; but at the end of this Subsection 2.4.5, we also identify some *solvable* ODEs of type (2.52a) with $\alpha = 1$ but without (2.52b). ■

The diligent reader will have no difficulty to verify that the solution of the initial value problem for this ODE, (2.52), reads as follows:

$$\gamma\,(\tau) = a^r\,\left\{a^{-1/r}\,\left(\tau + b\right)^{-1/r} - C\right\}^{-r}\,, \tag{2.53a}$$

$$a = \left[\gamma'\,(0)\right]^{1/(1+r)} - C\,\left[\gamma\,(0)\right]^{1/r}\,, \tag{2.53b}$$

$$b = a^{-1}\,\left\{a\,\left[\gamma\,(0)\right]^{-1/r} + C\right\}^{-r}\,, \tag{2.53c}$$

$$C = \frac{c\,r}{1+r}\,,\quad r = \frac{1-\alpha}{\alpha-2} = \frac{1}{1+\beta}\,. \tag{2.54}$$

The (autonomous!) *isochronous* variant of this ODE obtains via the usual position (see above, at the beginning of this Section 2.4),

$$y\,(t) = \exp\left(\mathbf{i}\,r\,\omega\,t\right)\,\gamma\,(\tau)\,,\quad \tau \equiv \tau\,(t) = \frac{\exp\left(\mathbf{i}\,\omega\,t\right) - 1}{\mathbf{i}\,\omega} \tag{2.55}$$

with r defined as above, see (2.55), and it reads as follows:

$$\begin{aligned}\ddot{y} = \mathbf{i}\,&\left(2\,r+1\right)\,\omega\,\dot{y} + r\,\left(r+1\right)\,\omega^2\,y\\ &+ c\,\left(\dot{y} - \mathbf{i}\,r\,\omega\,y\right)^{(1+2r)/(1+r)}\,y^{(1-r)/r}\,.\end{aligned} \tag{2.56}$$

The solution of this ODE reads as follows:

$$y\,(t) = \tilde{y}\,(\eta)\,,\quad \eta \equiv \eta\,(t) = \exp\left(\mathbf{i}\,\omega\,t\right)\,, \tag{2.57a}$$

$$\tilde{y}\,(\eta) = \left(\frac{a\,\eta}{C}\right)^r\,\left[\left(\frac{\eta_1 - \eta_2}{\eta_1 - \eta}\right)^{1/r} - 1\right]^{-r}\,, \tag{2.57b}$$

$$\eta_1 = 1 - \mathbf{i}\,\omega\,b\,,\quad \eta_2 = \eta_1 + \mathbf{i}\,\omega\,a^{-1}\,C^{-r}\,. \tag{2.57c}$$

Let us tersely discuss the time evolution of this solution. Note that the function $\tilde{y}\,(\eta)$ generally features 4 *rational* branch points, at $\eta = 0$, $\eta = \eta_1$, $\eta = \eta_2$, $\eta = \infty$, while as a function of the time t the *complex* variable $\eta\,(t)$ travels on the *circle* $\tilde{C}$—centered at the origin of the *complex* η-plane, and having *unit* radius—taking the time T, see (2.18b), to perform a full round on this circle. Hence, if the two branch points at $\eta = \eta_1$ and $\eta = \eta_2$ fall *outside* this circle $\tilde{C}$—i.e., if the initial data $y(0)$ and $\dot{y}(0)$ imply

$$|\eta_1| > 1\,,\quad |\eta_2| > 1\,, \tag{2.58a}$$

see (2.57c)—then the only relevant branch point is that at the origin hence the solution $y(t)$ is periodic with period kT, see (2.18b), (2.16b) and (2.19b). And since the set of initial data corresponding to the *inequalities* (2.58a) is clearly an *open* set, this is enough to qualify the ODE (2.56) as *isochronous*.

Likewise, if the initial data imply that the two inequalities (2.58a) are reversed, i.e.,

$$|\eta_1| < 1 \, , \quad |\eta_2| < 1 \, , \tag{2.58b}$$

then clearly the only relevant branch point is that at infinity (associated with the exponent $r+1 = (q+k)/k$, see (2.16b), hence of order k), hence again the solution $y(t)$ is periodic with period kT, see (2.18b), (2.16b) and (2.19b).

The interested reader will have no difficulty to analyze the cases when these *equalities* are replaced by *equalities* or only one of them is *reversed*.

We end this discussion of the solutions (2.57) of the ODE (2.56) with a terse mention of the drastic change in their behaviors that is obtained if the restriction that the parameter ω be *real* is abandoned and is instead replaced, say, by the assumption that it feature a *positive imaginary* part, $\text{Im}[\omega] > 0$. Then clearly, if $r > 0$, in the remote future $y(t)$ tends (exponentially) to zero, $y(\infty) = 0$, while in the remote past it diverges *exponentially*; with these behaviors exchanged if $r < 0$.

Finally, let us consider the variant of the (autonomous) ODE (2.52a) (without (2.52b)) with $\alpha = 1$ and β *a priori arbitrary* (but see below),

$$\gamma'' = c\,\gamma'\,\gamma^{\beta} \, . \tag{2.59a}$$

Its (autonomous!) *isochronous* variant obtains via the change of dependent variables (2.55) with

$$r = 1/\beta \tag{2.59b}$$

hence it reads as follows:

$$\ddot{y} = \mathbf{i}\;(2\,r+1)\;\omega\,\dot{y} + r\;(r+1)\;\omega^2\,y + c\;(\dot{y} - \mathbf{i}\,r\,\omega\,y)\;y^{1/r} \, . \tag{2.60}$$

Remark 2.4.5.2. The ODE (2.59a) is yielded, for $\beta \neq -1$ (hence $r \neq -1$) by the Hamiltonian

$$h(p,\,\gamma) = f(\gamma)\;\exp(a\,p) - \frac{c\,\gamma^{\beta+1}}{a\;(\beta+1)} \, , \tag{2.61a}$$

and, for $\beta = -1$ (hence $r = -1$) by the Hamiltonian

$$h(p,\,\gamma) = f(\gamma)\;\exp(a\,p) + \ln\left(\gamma^{-c/a}\right) \, . \tag{2.61b}$$

In these formulas, p is the canonical momentum conjugated to the canonical variable γ, while a is an *arbitrary nonvanishing* parameter and $f(\gamma)$ an *arbitrary differentiable* function (of course *not identically vanishing*). ■

The ODE (2.59a) can be immediately integrated once, yielding

$$\gamma'(\tau) = \left(\frac{c}{1+\beta}\right) \left[\gamma(\tau)\right]^{1+\beta} + B \,,$$
$$B = \gamma'(0) - \left(\frac{c}{1+\beta}\right) \left[\gamma(0)\right]^{1+\beta} \,, \tag{2.62}$$

and a second integration can be performed in more or less *explicit* form in terms of elementary functions for various values of β. For simplicity, we hereafter restrict attention to the $\beta = 1$ (hence $r = 1$) case, leaving to the interested reader the treatment of other cases. Then the two ODEs (2.59a) respectively (2.60) read simply

$$\gamma'' = c\,\gamma'\,\gamma \,, \tag{2.63}$$

respectively (since $\beta = 1$ implies $r = 1$)

$$\ddot{y} = 3\,\mathbf{i}\,\omega\,\dot{y} + 2\,\omega^2\,y + c\,\left(\dot{y} - \mathbf{i}\,\omega\,y\right)\,y \,. \tag{2.64}$$

The complete integration of the first, (2.63), of these two ODEs is then a trivial task, yielding

$$\gamma(\tau) = \frac{\gamma(0) + a\,\tanh\left(\frac{a\,c\,\tau}{2}\right)}{1 + \frac{\gamma(0)}{a}\,\tanh\left(\frac{a\,c\,\tau}{2}\right)} \,,$$
$$a^2 = \left[\gamma(0)\right]^2 - \frac{2\,\gamma'(0)}{c} \,. \tag{2.65}$$

Note that this expression of $\gamma(\tau)$ depends on a^2 rather than a.

The solution of the initial-value problem for the second, (2.64), of these two ODEs reads

$$y(t) = \exp\left(\mathbf{i}\,\omega\,t\right)\,\gamma(\tau) \,, \quad \tau = \frac{\exp\left(\mathbf{i}\,\omega\,t\right) - 1}{\mathbf{i}\,\omega} \,, \tag{2.66a}$$

where $\gamma(\tau)$ is defined as above, see (2.65), but with

$$\gamma(0) = y(0) \,, \quad a^2 = \left[y(0)\right]^2 - \frac{2\,\left[\dot{y}(0) - \mathbf{i}\,\omega\,y(0)\right]}{c} \,. \tag{2.66b}$$

Hence *all* the solutions $y(t)$ of this ODE (2.64) are *periodic*, as functions of the *real* variable t, with the basic period $T = 2\pi/|\omega|$, and only a subset of these solutions feature a *polar* singularity as t evolves from $-\infty$ to $+\infty$.

2.N Notes on Chapter 2

The formulas (2.7) and (2.8) are reported and proven in [34]; the formulas (2.9) and (2.10) are reported and proven in [3]; analogous formulas for higher derivatives (of arbitrary order k; and, in explicit form, for k up to 6) are reported and proven in [12].

For the notion of *asymptotic periodicity* mentioned in Remark 2.3.2—and the related notion of *asymptotic isochrony*—see [54] and Chapter 6 of [27].

For the extension of the main treatment from (monic) *polynomials* of degree N hence featuring N *coefficients* and N *zeros* to *entire functions* featuring an *infinite* number of *coefficients* and *zeros*, respectively to appropriately normalized *rational* functions featuring N *poles* and N *residues*, see F. Calogero, "Zeros of entire functions and related systems of infinitely many nonlinearly coupled evolution equations", Theor. Math. Phys. (in press), respectively F. Calogero, "Zeros of Rational Functions and Solvable Nonlinear Evolution Equations", J. Math. Phys. (submitted to, 2018.04.06).

The *trick* used in Section 2.4 to generate ODEs featuring periodic solutions was introduced in [19] and has been extensively used since, see in particular [27] and references therein.

Finally, let us emphasize that the ODEs discussed in Section 2.4—*solvable* in terms of elementary functions—by no means exhaust the universe of such examples: the alert and interested reader will easily identify many others. The few examples reported and tersely discussed in Section 2.4 are mainly introduced as material to be utilized below, see in particular Chapter 4.

An interesting variant of the models discussed in Section 2.4 provides a Hamiltonian treatment of the motion of a charged particle in a plane in the presence of a constant magnetic field orthogonal to that plane and of friction: see F. Calogero and F. Leyvraz, "Time-independent Hamiltonians describing systems with friction: the 'cyclotron with friction'", J. Phys. A (submitted to, 2018.03.01) and "A Hamiltonian yielding damped motion in an homogeneous magnetic field: quantum treatment", (to be published).