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Non-integrability of the restricted three-body problem

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Abstract. The problem of non-integrability of the circular restricted three-body problem is very classical and important in the theory of dynamical systems. It was partially solved by Poincaré in the nineteenth century: he showed that there exists no real-analytic first integral which depends analytically on the mass ratio of the second body to the total and is functionally independent of the Hamiltonian. When the mass of the second body becomes zero, the restricted three-body problem reduces to the two-body Kepler problem. We prove the non-integrability of the restricted three-body problem both in the planar and spatial cases for any non-zero mass of the second body. Our basic tool of the proofs is a technique developed here for determining whether perturbations of integrable systems which may be non-Hamiltonian are not meromorphically integrable near resonant periodic orbits such that the first integrals and commutative vector fields also depend meromorphically on the perturbation parameter. The technique is based on generalized versions due to Ayoul and Zung of the Morales–Ramis and Morales–Ramis–Simó theories. We emphasize that our results are not just applications of the theories.

Key words: restricted three-body problem, non-integrability, perturbation, Morales–Ramis–Simó theory
2020 Mathematics Subject Classification: 70F07, 37J30, 34E10 (Primary); 34M15, 34M35, 37J40 (Secondary)

1. *Introduction*

In this paper, we study the non-integrability of the circular restricted three-body problem for the planar case,

$$\begin{aligned} \dot{x} &= p_x + y, \quad \dot{p}_x = p_y + \frac{\partial U_2}{\partial x}(x, y), \\ \dot{y} &= p_y - x, \quad \dot{p}_y = -p_x + \frac{\partial U_2}{\partial y}(x, y), \end{aligned} \tag{1.1}$$

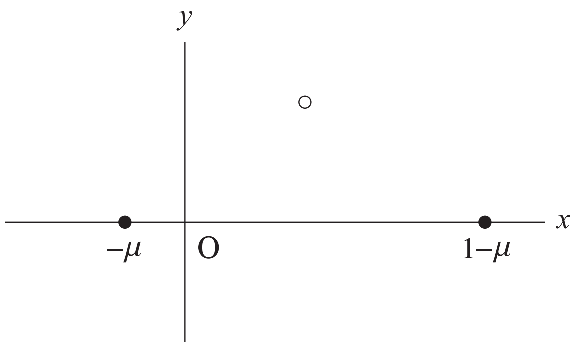

FIGURE 1. Configuration of the circular restricted three-body problem in the rotational frame.

and for the spatial case,

$$\begin{aligned} \dot{x} &= p_x + y, \quad \dot{p}_x = p_y + \frac{\partial U_3}{\partial x}(x, y, z), \\ \dot{y} &= p_y - x, \quad \dot{p}_y = -p_x + \frac{\partial U_3}{\partial y}(x, y, z), \\ \dot{z} &= p_z, \quad \dot{p}_z = \frac{\partial U_3}{\partial z}(x, y, z), \end{aligned} \tag{1.2}$$

where

$$U_2(x, y) = \frac{\mu}{\sqrt{(x-1+\mu)^2 + y^2}} + \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2}},$$
$$U_3(x, y, z) = \frac{\mu}{\sqrt{(x-1+\mu)^2 + y^2 + z^2}} + \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2 + z^2}}.$$

The systems (1.1) and (1.2) are Hamiltonian with the Hamiltonians

$$H_2(x, y, p_x, p_y) = \tfrac{1}{2}(p_x^2 + p_y^2) + (p_x y - p_y x) - U_2(x, y)$$

and

$$H_3(x, y, z, p_x, p_y, p_z) = \tfrac{1}{2}(p_x^2 + p_y^2 + p_z^2) + (p_x y - p_y x) - U_3(x, y, z),$$

respectively, and represent the dimensionless equations of motion of the third massless body subjected to the gravitational forces from the two primary bodies with mass μ and $1-\mu$ which remain at $(1-\mu, 0)$ and $(-\mu, 0)$, respectively, on the xy-plane in the rotational frame, under the assumption that the primaries rotate counterclockwise on the circles about their common center of mass at the origin in the inertial coordinate frame (see Figure 1). Their non-integrability means that equation (1.1) (respectively equation (1.2)) does not have one first integral (respectively two first integrals) which is (respectively are) functionally independent of the Hamiltonian H_2 (respectively H_3). See [**3**, **20**] for the definition of integrability of general Hamiltonian systems, and, e.g., [**19**, §4.1] for more details on the derivation and physical meaning of equations (1.1) and (1.2).

The problem of non-integrability of equations (1.1) and (1.2) is very classical and important in the theory of dynamical systems. In his famous memoir [**29**], which was

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related to a prize competition celebrating the 60th birthday of King Oscar II, Henri Poincaré studied the planar case and discussed the non-existence of a first integral which is analytic in the state variables and parameter μ near $\mu = 0$ and functionally independent of the Hamiltonian. His approach was improved significantly in the first volume of his masterpieces [30] published two years later: he showed the non-existence of such a first integral for the restricted three-body problem in the planar case. See [6] for an account of his work from mathematical and historical perspectives. His result was also explained in [4, 14, 15, 38]. Moreover, remarkable progress has been made on the planar problem (1.1) in a different direction recently: Guardia *et al* [12] showed the occurrence of transverse intersection between the stable and unstable manifolds of the infinity for any $\mu \in (0, 1)$ in a region far from the primaries in which $r = \sqrt{x^2 + y^2}$ and its conjugate momentum are sufficiently large. This implies, e.g., by [26, Theorem 3.10], the real-analytic non-integrabilty of equation (1.1) as well as the existence of oscillatory motions such that $\limsup_{t\to\infty} r(t) = \infty$ while $\liminf_{t\to\infty} r(t) < \infty$. Similar results were obtained much earlier when $\mu > 0$ is sufficiently small in [16] and for any $\mu \in (0, 1)$ except for a certain finite number of the values in [40]. Note that these results immediately say nothing about the non-integrability of the spatial problem (1.2).

Moreover, the non-integrability of the general three-body problem is now well understood, in comparison with the restricted one. Tsygvintsev [32, 33] proved the non-integrability of the general planar three-body problem near the Lagrangian parabolic orbits in which the three bodies form an equilateral triangle and move along certain parabolas, using Ziglin's method [45]. Boucher and Weil [9] also obtained a similar result, using the Morales–Ramis theory [20, 22], which is considered as an extension of the Ziglin method, while it was proven for the case of equal masses a little earlier in [8]. Moreover, Tsygvintsev [34–36] proved the non-existence of a single additional first integral near the Lagrangian parabolic orbits when

$$\frac{m_1m_2 + m_2m_3 + m_3m_1}{(m_1 + m_2 + m_3)^2} \notin \left\{\frac{1}{3}, \frac{2}{9}, \frac{8}{27}\right\},$$

where m_j represents the mass of the jth body for $j = 1, 2, 3$. Subsequently, Morales-Ruiz and Simon [25] succeeded in removing the three exceptional cases and extended the result to the space of three or more dimensions. Ziglin [46] also proved the non-integrability of the general three-body problem near a collinear solution which was used by Yoshida [43] for the problem in the one-dimensional space much earlier, in the space of any dimension when two of the three masses, say m_1, m_2, are nearly equal but neither m_3/m_1 nor $m_3/m_2 \in \{11/12, 1/4, 1/24\}$. Maciejewski and Przybylska [17] discussed the three-body problem with general homogeneous potentials. It should be noted that Ziglin [46] and Morales-Ruiz and Simon [25] also discussed the general N-body problem. We remark that these results say nothing about the non-integrability of the restricted three-body problem obtained by limiting manipulation from the general one. In particular, there exists no non-constant solution corresponding to the Lagrangian parabolic orbits or collinear solutions in the restricted one.

Here we show the non-integrability of the three-body problems (1.1) and (1.2) near the primaries for any $\mu \in (0, 1)$ fixed. To state our result precisely, we use the

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following treatment originally made in [10]. We first introduce the new variables $u_1, u_2 \in \mathbb{C}$ given by

$$u_1^2 - (x - 1 + \mu)^2 - y^2 = 0, \quad u_2^2 - (x + \mu)^2 - y^2 = 0$$

and

$$u_1^2 - (x - 1 + \mu)^2 - y^2 - z^2 = 0, \quad u_2^2 - (x + \mu)^2 - y^2 - z^2 = 0,$$

and regard equations (1.1) and (1.2) as Hamiltonian systems on the four- and six-dimensional complex manifolds (algebraic varieties)

$$\begin{aligned}\mathscr{S}_2 = \{&(x, y, p_x, p_y, u_1, u_2) \in \mathbb{C}^6 \\ &\mid u_1^2 - (x - 1 + \mu)^2 - y^2 = u_2 - (x + \mu)^2 - y^2 = 0\}\end{aligned}$$

and

$$\begin{aligned}\mathscr{S}_3 = \{&(x, y, z, p_x, p_y, p_z, u_1, u_2) \in \mathbb{C}^8 \\ &\mid u_1^2 - (x - 1 + \mu)^2 - y^2 - z^2 = u_2 - (x + \mu)^2 - y^2 - z^2 = 0\},\end{aligned}$$

respectively. Let $\pi_2 : \mathscr{S}_2 \to \mathbb{C}^4$ and $\pi_3 : \mathscr{S}_3 \to \mathbb{C}^6$ be the projections such that

$$\pi_2(x, y, p_x, p_y, u_1, u_2) = (x, y, p_x, p_y)$$

and

$$\pi_3(x, y, z, p_x, p_y, p_z, u_1, u_2) = (x, y, z, p_x, p_y, p_z),$$

and let

$$\begin{aligned}\Sigma(\mathscr{S}_2) &= \{u_1 = (x - 1 + \mu)^2 + y^2 = 0\} \\ &\qquad \cup \{u_2 = (x + \mu)^2 + y^2 = 0\} \subset \mathscr{S}_2, \\ \Sigma(\mathscr{S}_3) &= \{u_1 = (x - 1 + \mu)^2 + y^2 + z^2 = 0\} \\ &\qquad \cup \{u_2 = (x + \mu)^2 + y^2 + z^2 = 0\} \subset \mathscr{S}_3.\end{aligned}$$

Note that π_2 and π_3 are singular on $\Sigma(\mathscr{S}_2)$ and $\Sigma(\mathscr{S}_3)$, respectively. The sets $\Sigma(\mathscr{S}_2)$ and $\Sigma(\mathscr{S}_3)$ are called the *critical sets* of $\mathscr{S}_2$ and $\mathscr{S}_3$, respectively. The systems (1.1) and (1.2) are respectively rewritten as

$$\begin{aligned}&\dot{x} = p_x + y, \quad \dot{p}_x = p_y - \frac{\mu}{u_1^3}(x - 1 + \mu) - \frac{1 - \mu}{u_2^3}(x + \mu), \\ &\dot{y} = p_y - x, \quad \dot{p}_y = -p_x - \frac{\mu}{u_1^3}y - \frac{1 - \mu}{u_2^3}y, \\ &\dot{u}_1 = \frac{1}{u_1}((x - 1 + \mu)(p_x + y) + y(p_y - x)), \\ &\dot{u}_2 = \frac{1}{u_2}((x + \mu)(p_x + y) + y(p_y - x))\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}
&\dot{x} = p_x + y, \quad \dot{p}_x = p_y - \frac{\mu}{u_1^3}(x - 1 + \mu) - \frac{1-\mu}{u_2^3}(x + \mu),\\
&\dot{y} = p_y - x, \quad \dot{p}_y = -p_x - \frac{\mu}{u_1^3}y - \frac{1-\mu}{u_2^3}y,\\
&\dot{z} = p_z, \quad \dot{p}_z = -\frac{\mu}{u_1^3}z - \frac{1-\mu}{u_2^3}z, \qquad (1.4)\\
&\dot{u}_1 = \frac{1}{u_1}((x - 1 + \mu)(p_x + y) + y(p_y - x) + zp_z),\\
&\dot{u}_2 = \frac{1}{u_2}((x + \mu)(p_x + y) + y(p_y - x) + zp_z),
\end{aligned}$$

which are rational on $\mathscr{S}_2$ and $\mathscr{S}_3$. We prove the following theorem.

THEOREM 1.1. *The circular restricted three-body problem (1.1) (respectively (1.2)) does not have a complete set of first integrals in involution that are functionally independent almost everywhere and meromorphic in* $(x, y, p_x, p_y, u_1, u_2)$ *(respectively in* $(x, y, z, p_x, p_y, p_z, u_1, u_2)$*) except on* $\Sigma(\mathscr{S}_2)$ *(respectively on* $\Sigma(\mathscr{S}_3)$*) in punctured neighborhoods of*

$$(x, y) = (-\mu, 0) \quad \textit{and} \quad (1 - \mu, 0)$$
$$(\textit{respectively} \quad (x, y, z) = (-\mu, 0, 0) \quad \textit{and} \quad (1 - \mu, 0, 0))$$

for any $\mu \in (0, 1)$*, as Hamiltonian systems on* $\mathscr{S}_2$ *(respectively on* $\mathscr{S}_3$*).*

Proofs of Theorem 1.1 are given in §3 for the planar case of equation (1.1) and in §4 for the spatial case of equation (1.2). Our basic tool of the proofs is a technique developed in §2 for

$$\dot{I} = \varepsilon h(I, \theta; \varepsilon), \quad \dot{\theta} = \omega(I) + \varepsilon g(I, \theta; \varepsilon), \quad (I, \theta) \in \mathbb{R}^\ell \times \mathbb{T}^m, \tag{1.5}$$

where $\ell, m \in \mathbb{N}$, $\mathbb{T}^m = (\mathbb{R}/2\pi\mathbb{Z})^m$, ε is a small parameter such that $0 < |\varepsilon| \ll 1$, and $\omega : \mathbb{R}^\ell \to \mathbb{R}^m$, $h : \mathbb{R}^\ell \times \mathbb{T}^m \times \mathbb{R} \to \mathbb{R}^\ell$ and $g : \mathbb{R}^\ell \times \mathbb{T}^m \times \mathbb{R} \to \mathbb{R}^m$ are meromorphic in their arguments. The system (1.5) is Hamiltonian if $\varepsilon = 0$ or

$$\mathrm{D}_I h(I, \theta; \varepsilon) \equiv -\mathrm{D}_\theta g(I, \theta; \varepsilon)$$

as well as $\ell = m$, and non-Hamiltonian if not. The developed technique enables us to determine whether the system (1.5) is not meromorphically integrable in the Bogoyavlenskij sense [7] (see Definition 1.2) such that the first integrals and commutative vector fields also depend meromorphically on ε near $\varepsilon = 0$, like the result of Poincaré [29, 30] stated above, when the domains of the independent and dependent variables are extended to regions in $\mathbb{C}$ and $\mathbb{C}^\ell \times (\mathbb{C}/2\pi\mathbb{Z})^m$, respectively. The general definition of integrability adopted here is precisely stated as follows.

Definition 1.2. (Bogoyavlenskij) For $n \in \mathbb{N}$, an n-dimensional dynamical system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \text{or } \mathbb{C}^n,$$

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is called $(q, n-q)$-*integrable* or simply *integrable* if there exist q vector fields $f_1(x)(:= f(x))$, $f_2(x), \ldots, f_q(x)$ and $n-q$ scalar-valued functions $F_1(x), \ldots, F_{n-q}(x)$ such that the following two conditions hold:

(i) $f_1(x), \ldots, f_q(x)$ are linearly independent almost everywhere and commute with each other, that is, $[f_j, f_k](x) := \mathrm{D}f_k(x)f_j(x) - \mathrm{D}f_j(x)f_k(x) \equiv 0$ for $j, k = 1, \ldots, q$, where $[\cdot, \cdot]$ denotes the Lie bracket;

(ii) the derivatives $\mathrm{D}F_1(x), \ldots, \mathrm{D}F_{n-q}(x)$ are linearly independent almost everywhere and $F_1(x), \ldots, F_{n-q}(x)$ are first integrals of $f_1, \ldots, f_q$, that is, $\mathrm{D}F_k(x) \cdot f_j(x) \equiv 0$ for $j = 1, \ldots, q$ and $k = 1, \ldots, n-q$, where '$\cdot$' represents the inner product.

We say that the system is *meromorphically integrable* if the first integrals and commutative vector fields are meromorphic.

Definition 1.2 is considered as a generalization of Liouville-integrability for Hamiltonian systems [3, 20] since an n-degree-of-freedom Liouville-integrable Hamiltonian system with $n \geq 1$ has not only n functionally independent first integrals but also n linearly independent commutative (Hamiltonian) vector fields generated by the first integrals. When $\varepsilon = 0$, the system (1.5) is meromorphically (m, ℓ)-integrable in the Bogoyavlenskij sense: $F_j(I, \theta) = I_j$, $j = 1, \ldots, \ell$, are first integrals and $f_j(I, \theta) = (0, e_j) \in \mathbb{R}^\ell \times \mathbb{R}^m$, $j = 2, \ldots, m$, give m commutative vector fields along with its own vector field, where e_j is the m-dimensional vector of which the jth element is the unit and the other ones are zero. Conversely, a general (m, ℓ)-integrable system is transformed to the form (1.5) with $\varepsilon = 0$ if the level set of the first integrals $F_1(x), \ldots, F_m(x)$ has a connected compact component. See [7, 47] for more details. Thus, the system (1.5) can be regarded as a normal form for perturbations of general (m, ℓ)-integrable systems.

Systems of the form (1.5) have attracted much attention, especially when they are Hamiltonian. See [3, 4, 15] and references therein for more details. In particular, Kozlov [15] extended the famous result of Poincaré [29, 30] for Hamiltonian systems to the general analytic case of equation (1.5) and gave sufficient conditions for the non-existence of additional real-analytic first integrals depending analytically on ε near $\varepsilon = 0$. See also [4, 14] for his result in Hamiltonian systems. Moreover, Motonaga and Yagasaki [27] gave sufficient conditions for real-analytic non-integrability of general nearly integrable systems in the Bogoyavlenskij sense such that the first integrals and commutative vector fields also depend real-analytically on ε near $\varepsilon = 0$. The technique developed in §2 is different from them and based on generalized versions due to Ayoul and Zung [5] of the Morales–Ramis and Morales–Ramis–Simó theories [20, 22, 23]. See [41, Appendix A] for a brief review of the previous results and their comparison with the developed technique. Our technique can also be applied to several nearly integrable systems containing time-periodic perturbation of single-degree-of-freedom Hamiltonian systems such as the periodically forced Duffing oscillator and pendulum [28, 41]. Moreover, it can be used directly to give a new proof of Poincaré's result of [30] on the restricted three-body problem [42]. The systems (1.1) and (1.2) are transformed to the form (1.5) in the punctured neighborhoods in Theorem 1.1 and the technique is applied to them. We emphasize that our results are not just applications of the Morales–Ramis and Morales–Ramis–Simó theories or their generalized versions.

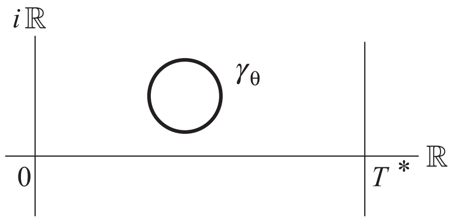

FIGURE 2. Assumption (A2).

2. *Determination of non-integrability for equation (1.5)*

In this section, we give a technique for determining whether the system (1.5) is not meromorphically Bogoyavlenskij-integrable such that the first integrals and commutative vector fields also depend meromorphically on ε near $\varepsilon = 0$.

When $\varepsilon = 0$, equation (1.5) becomes

$$\dot{I} = 0, \quad \dot{\theta} = \omega(I). \tag{2.1}$$

We assume the following on the unperturbed system (2.1).

(A1) For some $I^* \in \mathbb{R}^\ell$, a resonance of multiplicity $m-1$,

$$\dim_{\mathbb{Q}}\langle \omega_1(I^*), \ldots, \omega_m(I^*)\rangle = 1,$$

occurs with $\omega(I^*) \neq 0$, that is, there exists a constant $\omega^* > 0$ such that

$$\frac{\omega(I^*)}{\omega^*} \in \mathbb{Z}^m \setminus \{0\},$$

where $\omega_j(I)$ is the *j*th element of $\omega(I)$ for $j = 1, \ldots, m$.

Note that we can replace ω^* with ω^*/n for any $n \in \mathbb{N}$ in assumption (A1). We refer to the m-dimensional torus $\mathscr{T}^* = \{(I^*, \theta) \mid \theta \in \mathbb{T}^m\}$ as the *resonant torus* and to periodic orbits $(I, \theta) = (I^*, \omega(I^*)t + \theta_0)$, $\theta_0 \in \mathbb{T}^m$, on $\mathscr{T}^*$ as the *resonant periodic orbits*. Let $T^* = 2\pi/\omega^*$. We also make the following assumption.

(A2) For some $k \in \mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}$ and $\theta \in \mathbb{T}^m$, there exists a closed loop γ_θ in a region including $(0, T^*) \subset \mathbb{R}$ in $\mathbb{C}$ such that $\gamma_\theta \cap (i\mathbb{R} \cup (T^* + i\mathbb{R})) = \emptyset$ and

$$\mathscr{I}^k(\theta) := \frac{1}{k!}\mathrm{D}\omega(I^*) \int_{\gamma_\theta} \mathrm{D}_\varepsilon^k h(I^*, \omega(I^*)\tau + \theta; 0)\, d\tau \tag{2.2}$$

is not zero (see Figure 2).

Note that the condition $\gamma_\theta \cap (i\mathbb{R} \cup (T^* + i\mathbb{R})) = \emptyset$ is not essential in assumption (A2), since it always holds by replacing ω^* with ω^*/n for sufficiently large $n \in \mathbb{N}$ if necessary. We prove the following theorem which guarantees that conditions (A1) and (A2) are sufficient for non-integrability of equation (1.5) in a certain meaning.

THEOREM 2.1. *Let Γ be any domain in $\mathbb{C}/T^*\mathbb{Z}$ containing $\mathbb{R}/T^*\mathbb{Z}$ and γ_θ. Suppose that assumptions (A1) and (A2) hold for some $\theta = \theta_0 \in \mathbb{T}^m$. Then the system (1.5) is not*

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meromorphically integrable in the Bogoyavlenskij sense near the resonant periodic orbit $(I, \theta) = (I^*, \omega(I^*)\tau + \theta_0)$ *with* $\tau \in \Gamma$ *such that the first integrals and commutative vector fields also depend meromorphically on* ε *near* $\varepsilon = 0$*, when the domains of the independent and dependent variables are extended to regions in* $\mathbb{C}$ *and* $\mathbb{C}^\ell \times (\mathbb{C}/2\pi\mathbb{Z})^m$*, respectively. Moreover, if assumption (A2) holds for any* $\theta \in \Delta$*, where* Δ *is a dense set in* $\mathbb{T}^m$*, then the conclusion holds for any resonant periodic orbit on the resonant torus* $\mathscr{T}^*$*.*

Our basic idea of the proof of Theorem 2.1 is similar to that of Morales-Ruiz [**21**], who studied time-periodic Hamiltonian perturbations of single-degree-of-freedom Hamiltonian systems and showed a relationship of their non-integrability with a version due to Ziglin [**44**] of the Melnikov method [**18**] when the small parameter ε is regarded as a state variable. Here the Melnikov method enables us to detect transversal self-intersection of complex separatrices of periodic orbits unlike the standard version [**13**, **18**, **39**]. More concretely, under some restrictive conditions, he essentially proved that they are meromorphically non-integrable when the small parameter is taken as one of the state variables if the Melnikov functions are not identically zero, based on a generalized version due to Ayoul and Zung [**5**] of the Morales–Ramis theory [**20**, **22**]. We also use their generalized versions of the Morales–Ramis theory and its extension, the Morales–Ramis–Simó theory [**23**], to prove Theorem 2.1. These generalized theories enable us to show the non-integrability of general differential equations in the Bogayavlenskij sense by using differential Galois groups [**11**, **37**] of their variational or higher-order variational equations along non-constant particular solutions. We extend the idea of Morales-Ruiz [**21**] to higher-dimensional non-Hamiltonian systems near periodic orbits.

For the proof of Theorem 2.1, we first consider systems of the general form

$$\dot{x} = f(x; \varepsilon), \quad x \in \mathbb{C}^n, \tag{2.3}$$

where $f : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ is meromorphic, and describe direct consequences of the generalized versions due to Ayoul and Zung [**5**] of the Morales–Ramis theory [**20**, **22**] when the parameter ε is regarded as a state variable in equation (2.3) near $\varepsilon = 0$. Let $x = \bar{x}(t)$ be a periodic orbit in the unperturbed system

$$\dot{x} = f(x; 0).$$

Taking ε as another state variable, we extend equation (2.3) as

$$\dot{x} = f(x; \varepsilon), \quad \dot{\varepsilon} = 0, \tag{2.4}$$

in which $(x, \varepsilon) = (\bar{x}(t), 0)$ is a periodic orbit. The variational equation (VE) of equation (2.4) along the periodic solution $(\bar{x}(t), 0)$ is given by

$$\dot{y} = \mathrm{D}_x f(\bar{x}(t); 0) y + \mathrm{D}_\varepsilon f(\bar{x}(t); 0)\lambda, \quad \dot{\lambda} = 0. \tag{2.5}$$

We regard equation (2.5) as a linear differential equation on a Riemann surface. Applying the version due to Ayoul and Zung [**5**] of the Morales–Ramis theory [**20**, **22**] to equation (2.4), we obtain the following result.

THEOREM 2.2. *If the system (2.3) is meromorphically integrable in the Bogoyavlenskij sense near* $x = \bar{x}(t)$ *such that the first integrals and commutative vector fields also depend*

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meromorphically on ε near $\varepsilon = 0$, then the identity component of the differential Galois group of equation (2.5) is commutative.

See Appendix A for necessary information on the differential Galois theory.

We can obtain a more general result for equation (2.4) as follows. For simplicity, we assume that $n = 1$. The general case can be treated similarly. Letting

$$\bar{f}^{(j,l)} = \mathrm{D}_x^j \mathrm{D}_\varepsilon^l f(\bar{x}(t); 0),$$

we express the Taylor series of $f(x; \varepsilon)$ about $(x, \varepsilon) = (\bar{x}(t), 0)$ as

$$f(x; \varepsilon) = \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{j!\, l!} \bar{f}^{(j,l)} (x - \bar{x}(t))^j \varepsilon^l.$$

Let

$$x = \bar{x}(t) + \sum_{j=1} \varepsilon^j y^{(j)}.$$

Using these expressions, we write the kth-order VE of equation (2.4) along the periodic orbit $(x, \varepsilon) = (\bar{x}(t), 0)$ as

$$\begin{aligned}
\dot{y}^{(1)} &= \bar{f}^{(1,0)} y^{(1)} + \bar{f}^{(0,1)} \lambda^{(1)}, \quad \dot{\lambda}^{(1)} = 0,\\
\dot{y}^{(2)} &= \bar{f}^{(1,0)} y^{(2)} + \bar{f}^{(0,1)} \lambda^{(2)}\\
&\quad + \bar{f}^{(2,0)} (y^{(1)})^2 + 2\bar{f}^{(1,1)} (y^{(1)}, \lambda^{(1)}) + \bar{f}^{(0,2)} (\lambda^{(1)})^2, \quad \dot{\lambda}^{(2)} = 0,\\
\dot{y}^{(3)} &= \bar{f}^{(1,0)} (y^{(3)}) + \bar{f}^{(0,1)} (\lambda^{(3)}) + 2\bar{f}^{(2,0)} (y^{(1)}, y^{(2)})\\
&\quad + 2\bar{f}^{(1,1)} (y^{(1)}, \lambda^{(2)}) + 2\bar{f}^{(1,1)} (y^{(1)}, \lambda^{(2)}) + 2\bar{f}^{(0,2)} (\lambda^{(1)}, \lambda^{(2)})\\
&\quad + \bar{f}^{(3,0)} (y^{(1)})^3 + 3\bar{f}^{(2,1)} ((y^{(1)})^2, \lambda^{(1)}) + 3\bar{f}^{(1,2)} (y^{(1)}, (\lambda^{(1)})^2)\\
&\quad + \bar{f}^{(0,3)} (\lambda^{(1)})^3, \quad \dot{\lambda}^{(3)} = 0,\\
&\ \ \vdots\\
\dot{y}^{(k)} &= \sum \frac{(j+l)!}{j_1! \cdots j_r!\, l_1! \cdots l_s!}\\
&\quad \times \bar{f}^{(j,l)} ((y^{(i_1)})^{j_1}, \ldots, (y^{(i_r)})^{j_r}, (\lambda^{(i'_1)})^{l_1}, \ldots, (\lambda^{(i'_s)})^{l_s}), \quad \dot{\lambda}^{(k)} = 0,
\end{aligned} \tag{2.6}$$

where such terms as $(y^{(1)})^0, (\lambda^{(1)})^0 = 1$ have been substituted and the summation in the last equation has been taken over all integers

$$j, l, r, s, i_1, \ldots, i_r, i'_1, \ldots, i'_s, j_1, \ldots, j_r, l_1, \ldots, l_s$$

such that

$$1 \le j + l \le k, \quad i_1 < i_2 < \cdots < i_r, \quad i'_1 < i'_2 < \cdots < i'_s,$$
$$\sum_{r'=1}^{r} j_{r'} = j, \quad \sum_{s'=1}^{s} l_{s'} = l, \quad \sum_{r'=1}^{r} j_{r'} i_{r'} + \sum_{s'=1}^{s} l_{s'} i_{s'} = k.$$

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See [**23**] for the details on derivation of higher-order VEs in a general setting. Substituting $y^{(j)} = 0$, $j = 1, \ldots, k-1$, and $\lambda^{(l)} = 0$, $l = 2, \ldots, k$, into equation (2.6), we obtain

$$\dot{\lambda}^{(1)} = 0, \quad \dot{y}^{(k)} = \bar{f}^{(1,0)}(y^{(k)}) + \bar{f}^{(0,k)}(\lambda^{(1)})^k,$$

which is equivalent to

$$\dot{y} = \mathrm{D}_x f(\bar{x}(t); 0)y + \frac{1}{k!}\mathrm{D}_\varepsilon^k f(\bar{x}(t); 0)\lambda, \quad \dot{\lambda} = 0 \tag{2.7}$$

with $y = y^{(k)}$ and $\lambda = k!\,(\lambda^{(1)})^k$. Equation (2.7) is derived for $n > 1$ similarly. We regard equation (2.7) as a linear differential equation on a Riemann surface, again. Such a reduction of higher-order VEs was used for planar systems in [**1**, **2**]. We call equation (2.7) the *kth-order reduced variational equation* (RVE) of equation (2.4) around the periodic orbit $(x, \varepsilon) = (\bar{x}(t), 0)$. Using the version due to Ayoul and Zung [**5**] of the Morales–Ramis–Simó theory [**23**], we obtain the following result.

THEOREM 2.3. *If the system (2.3) is meromorphically integrable in the Bogoyavlenskij sense near $x = \bar{x}(t)$ such that the first integrals and commutative vector fields also depend meromorphically on ε near $\varepsilon = 0$, then the identity component of the differential Galois group of equation (2.7) is commutative.*

Remark 2.4. The statement of Theorem 2.3 is very weak, compared with the original one of [**23**], since the RVE (2.7) is much smaller than the full higher-order VE for equation (2.3). However, it is tractable and enough for our purpose.

We return to the system (1.5) and regard ε as a state variable to rewrite it as

$$\dot{I} = \varepsilon h(I, \theta; \varepsilon), \quad \dot{\theta} = \omega(I) + \varepsilon g(I, \theta; \varepsilon), \quad \dot{\varepsilon} = 0, \tag{2.8}$$

like equation (2.4). We extend the domain of the independent variable t to a region including $\mathbb{R}$ in $\mathbb{C}$, as stated in Theorem 2.1. The $(k+1)$th-order RVE of equation (2.8) along the periodic orbit $(I, \theta, \varepsilon) = (I^*, \omega(I^*)t + \theta_0, 0)$ is given by

$$\begin{aligned} \dot{\xi} &= h^k(I^*, \omega^* t + \theta_0; 0)\lambda, \\ \dot{\eta} &= \mathrm{D}\omega(I^*)\xi + g^k(I^*, \omega^* t + \theta_0; 0)\lambda, \quad (\xi, \eta, \chi) \in \mathbb{C}^\ell \times \mathbb{C}^m \times \mathbb{C}, \\ \dot{\lambda} &= 0, \end{aligned} \tag{2.9}$$

where

$$h^k(I, \theta) = \frac{1}{k!}\mathrm{D}_\varepsilon^k h(I, \theta; 0), \quad g^k(I, \theta) = \frac{1}{k!}\mathrm{D}_\varepsilon^k g(I, \theta; 0).$$

As a Riemann surface, we take any region Γ in $\mathbb{C}/T^*\mathbb{Z}$ such that the closed loop γ_θ in assumption (A2), as well as $\mathbb{R}/T^*\mathbb{Z}$, is contained in Γ, as in Theorem 2.1 (see Figure 3). Let $\mathbb{K}_\theta \neq \mathbb{C}$ be a differential field that consists of T^*-periodic functions and contains the elements of $h^k(I^*, \omega(I^*)t + \theta)$ and $g^k(I^*, \omega(I^*)t + \theta)$ with $t \in \Gamma$. We regard the $(k+1)$th-order RVE (2.9) as a linear differential equation over $\mathbb{K}_\theta$ on the Riemann surface Γ. We obtain a fundamental matrix of equation (2.9) as

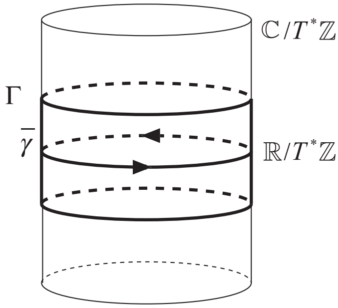

FIGURE 3. Riemann surface Γ. The monodromy matrix M_γ is computed along the loop γ.

$$\Phi^k(t;\theta_0) = \begin{pmatrix} \mathrm{id}_\ell & 0 & \Xi^k(t;\theta_0) \\ \mathrm{D}\omega(I^*)t & \mathrm{id}_m & \Psi^k(t;\theta_0) \\ 0 & 0 & 1 \end{pmatrix},$$

where id_ℓ is the $\ell \times \ell$ identity matrix and

$$\Xi^k(t;\theta) = \int_0^t h^k(I^*, \omega(I^*)\tau + \theta)\, d\tau,$$
$$\Psi^k(t;\theta) = \int_0^t (\mathrm{D}\omega(I^*)\Xi(\tau;\theta) + g^k(I^*, \omega(I^*)\tau + \theta))\, d\tau.$$

Let $\mathscr{G}_\theta$ be the differential Galois group of equation (2.9) and let $\sigma \in \mathscr{G}_\theta$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma(\Xi^k(t;\theta)) = \sigma\left(\frac{\mathrm{d}}{\mathrm{d}t}\Xi^k(t;\theta)\right) = h^k(I^*, \omega(I^*)t + \theta),$$

so that

$$\sigma(\Xi^k(t;\theta)) = \int_0^t h^k(I^*, \omega(I^*)\tau + \theta; 0)\, d\tau + C = \Xi^k(t;\theta) + C, \tag{2.10}$$

where C is a constant ℓ-dimensional vector depending on σ. If $\Xi^k(t;\theta) \in \mathbb{K}_\theta$, then $C = 0$ for any $\sigma \in \mathscr{G}_\theta$. Similarly, we have

$$\sigma(\mathrm{D}\omega(I^*)t) = \mathrm{D}\omega(I^*)t + C',$$

where C' is a constant $m \times \ell$ matrix depending on σ. If $\mathrm{D}\omega(I^*) \neq 0$, then $C' \neq 0$ for some $\sigma \in \mathscr{G}_\theta$ since $\mathrm{D}\omega(I^*)t \notin \mathbb{K}_\theta$. However,

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma(\Psi^k(t;\theta)) = \sigma\left(\frac{\mathrm{d}}{\mathrm{d}t}\Psi^k(t;\theta)\right) = \sigma(\mathrm{D}\omega(I^*)\Xi^k(t;\theta) + g^k(I^*, \omega(I^*)t + \theta))$$
$$= \mathrm{D}\omega(I^*)\Xi^k(t;\theta) + g^k(I^*, \omega(I^*)t + \theta) + \mathrm{D}\omega(I^*)C.$$

Hence,

$$\sigma(\Psi^k(t;\theta)) = \Psi^k(t;\theta) + \mathrm{D}\omega(I^*)Ct + C'',$$

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where C'' is a constant m-dimensional vector depending on σ. If $\Xi^k(t;\theta), \Psi^k(t;\theta) \in \mathbb{K}_\theta$, then $C'' = 0$ for any $\sigma \in \mathscr{G}_\theta$. Thus, we see that

$$\mathscr{G}_\theta \subset \tilde{\mathscr{G}} := \{M(C_1, C_2, C_3) \mid C_1 \in \mathbb{C}^\ell, C_2 \in \mathbb{C}^m, C_3 \in \mathbb{C}^{m\times\ell}\},$$

where

$$M(C_1, C_2, C_3) = \begin{pmatrix} \mathrm{id}_\ell & 0 & C_1 \\ C_3 & \mathrm{id}_m & C_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof of Theorem 2.1. Assume that the hypotheses of the theorem hold. We fix $\theta \in \mathbb{T}^m$ such that the integral (2.2) is not zero. We continue the fundamental matrix $\Phi^k(t;\theta)$ analytically along the loop $\gamma = \gamma_\theta$ to obtain the monodromy matrix as

$$M_\gamma = \begin{pmatrix} \mathrm{id}_\ell & 0 & \hat{C}_1 \\ 0 & \mathrm{id}_m & \hat{C}_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.11}$$

where

$$\hat{C}_1 = \int_{\gamma_\theta} h^k(I^*, \omega(I^*)t + \theta; 0)\, d\tau,$$
$$\hat{C}_2 = \int_{\gamma_\theta} (\mathrm{D}\omega(I^*)\Xi^k(\tau;\theta) + g^k(I^*, \omega(I^*)\tau + \theta))\, d\tau.$$

See Appendix B for basic information on monodromy matrices. In particular, we have $M_\gamma \in \mathscr{G}_\theta$. Note that $\mathrm{D}\omega(I^*)\hat{C}_1 \neq 0$ by assumption (A2).

Let $\bar{\gamma} = \{T^*s \mid s \in [0, 1]\}$, which is also a closed loop on the Riemann surface Γ (see Figure 3). We continue $\Phi^k(t;\theta)$ analytically along the loop $\bar{\gamma}$ to obtain the monodromy matrix as

$$M_{\bar{\gamma}} = \begin{pmatrix} \mathrm{id}_\ell & 0 & \Xi^k(T^*;\theta) \\ \mathrm{D}\omega(I^*)T^* & \mathrm{id}_m & \Psi^k(T^*;\theta) \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\bar{C}_1 = \Xi^k(T^*;\theta)$, $\bar{C}_2 = \Psi^k(T^*;\theta)$ and $\bar{C}_3 = \mathrm{D}\omega(I^*)T^*$. We see that $M_{\bar{\gamma}} = M(\bar{C}_1, \bar{C}_2, \bar{C}_3) \in \mathscr{G}_\theta$ and $\bar{C}_3\hat{C}_1 \neq 0$ by $\mathrm{D}\omega(I^*)\hat{C}_1 \neq 0$.

LEMMA 2.5. *Suppose that $M(C_1, C_2, C_3), M(C_1', C_2', C_3') \in \mathscr{G}_\theta$ for some C_j, C_j', $j = 1, 2, 3$, with $C_3C_1' \neq C_3'C_1$. Then the identity component $\mathscr{G}_\theta^0$ of $\mathscr{G}_\theta$ is not commutative.*

Proof. Assume that the hypothesis holds. We easily see that $M(C_1, C_2, C_3)$ and $M(C_1', C_2', C_3')$ is not commutative since

$$M(C_1, C_2, C_3)M(C_1', C_2', C_3) = \begin{pmatrix} \mathrm{id}_\ell & 0 & C_1 + C_1' \\ C_3 + C_3' & \mathrm{id}_m & C_3C_1' + C_2 + C_2' \\ 0 & 0 & 1 \end{pmatrix}$$

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while

$$M(C_1', C_2', C_3)M(C_1', C_2', C_3) = \begin{pmatrix} \mathrm{id}_\ell & 0 & C_1 + C_1' \\ C_3 + C_3' & \mathrm{id}_m & C_3'C_1 + C_2 + C_2' \\ 0 & 0 & 1 \end{pmatrix}.$$

However, we compute

$$\begin{aligned} M(C_1, C_2, C_3)^2 &= \begin{pmatrix} \mathrm{id}_\ell & 0 & C_1 \\ C_3 & \mathrm{id}_m & C_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{id}_\ell & 0 & C_1 \\ C_3 & \mathrm{id}_m & C_2 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id}_\ell & 0 & 2C_1 \\ 2C_3 & \mathrm{id}_m & C_3C_1 + 2C_2 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

and easily show by induction that

$$M(C_1, C_2, C_3)^k = \begin{pmatrix} \mathrm{id}_\ell & 0 & kC_1 \\ kC_3 & \mathrm{id}_m & (k-1)C_3C_1 + kC_2 \\ 0 & 0 & 1 \end{pmatrix}$$

for any $k \in \mathbb{N}$. Since $\mathscr{G}_\theta^0$ is a subgroup of finite index in $\mathscr{G}_\theta$ (see Appendix A), we show that

$$\mathscr{G}_\theta^0 \supset \{M(cC_1, (c-1)C_3C_1 + cC_2, cC_3) \mid c \in \mathbb{C}\}$$

if $C_3C_1 + C_2 \neq 0$ and

$$\mathscr{G}_\theta^0 \supset \{M(cC_1, C_2, cC_3) \mid c \in \mathbb{C}\}$$

if $C_3C_1 + C_2 = 0$. Thus, we show that $\mathscr{G}_\theta^0$ is not commutative in both cases. □

By Lemma 2.5, the identity component $\mathscr{G}_\theta^0$ is not commutative. Applying Theorem 2.3, we see that the system (1.5) is meromorphically non-integrable near the resonant periodic orbit $(I^*, \omega(I^*)t + \theta)$ in the meaning of Theorem 2.1. If this statement holds for θ on a dense set $\Delta \subset \mathbb{T}^m$, then it does so on $\mathbb{T}^m$. Thus, we complete the proof. □

Remark 2.6

(i) When the system (1.5) is Hamiltonian, it is not meromorphically Liouville-integrable such that the first integrals also depend meromorphically on ε near $\varepsilon = 0$, if the hypotheses of Theorem 2.1 hold.

(ii) Assumption (A2) in Theorem 2.1 may be replaced with the following.

(A2′) For some $k \in \mathbb{Z}_{\geq 0}$ and $\theta \in \mathbb{T}^m$,

$$\mathrm{D}\omega(I^*)\Xi^k(t; \theta) \notin \mathbb{K}_\theta(t).$$

This is easily proven as follows. Let $\mathbb{L}$ be the Picard–Vessiot extension of equation (2.9) and let $\hat{\sigma} : \mathbb{L} \to \mathbb{L}$ be a $\mathbb{K}_\theta(t)$-automorphism, that is, $\hat{\sigma} \in \mathrm{Gal}(\mathbb{L}/\mathbb{K}_\theta(t)) \subset \mathscr{G}_\theta$

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(see Appendix A). Since $\Xi^k(t;\theta) \notin \mathbb{K}_\theta(t)$, we have

$$\hat{\sigma}(\Xi^k(t;\theta)) = \Xi^k(t;\theta) + \hat{C}_1$$

as in equation (2.10), so that $\hat{\sigma}$ corresponds to the matrix

$$\begin{pmatrix} \mathrm{id}_\ell & 0 & \hat{C}_1 \\ 0 & \mathrm{id}_m & \hat{C}_2 \\ 0 & 0 & 1 \end{pmatrix} = M(\hat{C}_1, \hat{C}_2, 0).$$

Since $\mathrm{D}\omega(I^*)\hat{C}_1 \neq 0$ for some $\hat{\sigma} \in \mathrm{Gal}(\mathbb{L}/\mathbb{K}_\theta(t))$, we only have to use the above matrix instead of equation (2.11) and apply the same arguments to obtain the desired result.

3. *Planar case*

We prove Theorem 1.1 for the planar case (1.1). We only consider a neighborhood of $(x, y) = (-\mu, 0)$ since we only have to replace x and μ with $-x$ and $1 - \mu$ to obtain the result for a neighborhood of $(1 - \mu, 0)$. We introduce a small parameter ε such that $0 < \varepsilon \ll 1$. Letting

$$\varepsilon^2\xi = x + \mu, \quad \varepsilon^2\eta = y, \quad \varepsilon^{-1} p_\xi = p_x, \quad \varepsilon^{-1} p_\eta = p_y + \mu$$

and scaling the time variable $t \to \varepsilon^3 t$, we rewrite equation (1.1) as

$$\begin{aligned} \dot{\xi} &= p_\xi + \varepsilon^3\eta, \quad \dot{p}_\xi = -\frac{(1-\mu)\xi}{(\xi^2+\eta^2)^{3/2}} + \varepsilon^3 p_\eta - \varepsilon^4\mu - \varepsilon^4\frac{\mu(\varepsilon^2\xi - 1)}{((\varepsilon^2\xi-1)^2+\varepsilon^4\eta^2)^{3/2}}, \\ \dot{\eta} &= p_\eta - \varepsilon^3\xi, \quad \dot{p}_\eta = -\frac{(1-\mu)\eta}{(\xi^2+\eta^2)^{3/2}} - \varepsilon^3 p_\xi - \varepsilon^6\frac{\mu\eta}{((\varepsilon^2\xi-1)^2+\varepsilon^4\eta^2)^{3/2}}, \end{aligned}$$

or up to the order of ε^6,

$$\begin{aligned} \dot{\xi} &= p_\xi + \varepsilon^3\eta, \quad \dot{p}_\xi = -\frac{(1-\mu)\xi}{(\xi^2+\eta^2)^{3/2}} + \varepsilon^3 p_\eta + 2\varepsilon^6\mu\xi, \\ \dot{\eta} &= p_\eta - \varepsilon^3\xi, \quad \dot{p}_\eta = -\frac{(1-\mu)\eta}{(\xi^2+\eta^2)^{3/2}} - \varepsilon^3 p_\xi - \varepsilon^6\mu\eta, \end{aligned} \tag{3.1}$$

where the $O(\varepsilon^8)$ terms have been eliminated. Equation (3.1) is a Hamiltonian system with the Hamiltonian

$$H = \frac{1}{2}(p_\xi^2 + p_\eta^2) - \frac{1-\mu}{\sqrt{\xi^2+\eta^2}} + \varepsilon^3(\eta p_\xi - \xi p_\eta) - \frac{1}{2}\varepsilon^6\mu(2\xi^2 - \eta^2). \tag{3.2}$$

Non-integrability of a system which is similar to equation (3.1) but does not contain a small parameter was proven by using the Morales–Ramis theory [**20**, **22**] in [**24**]. See also Remark 3.1(ii).

We next rewrite equation (3.2) in the polar coordinates. Let

$$\xi = r\cos\phi, \quad \eta = r\sin\phi.$$

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The momenta (p_r, p_ϕ) corresponding to (r, ϕ) satisfy

$$p_\xi = p_r \cos\phi - \frac{p_\phi}{r}\sin\phi, \quad p_\eta = p_r \sin\phi + \frac{p_\phi}{r}\cos\phi.$$

See, e.g., [19, §8.6.1]. The Hamiltonian becomes

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{1-\mu}{r} - \varepsilon^3 p_\phi - \frac{1}{4}\varepsilon^6 \mu r^2 (3\cos 2\phi + 1).$$

Up to $O(1)$, the corresponding Hamiltonian system becomes

$$\dot{r} = p_r, \quad \dot{p}_r = \frac{p_\phi^2}{r^3} - \frac{1-\mu}{r^2}, \quad \dot{\phi} = \frac{p_\phi}{r^2}, \quad \dot{p}_\phi = 0, \tag{3.3}$$

which is easily solved since p_ϕ is a constant. Let $u = 1/r$. From equation (3.3), we have

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{1-\mu}{p_\phi^2},$$

from which we obtain the relation

$$r = \frac{p_\phi^2}{(1-\mu)(1 + e\cos\phi)}, \tag{3.4}$$

where the position $\phi = 0$ is appropriately chosen and e is a constant. We choose $e \in (0, 1)$, so that equation (3.4) represents an elliptic orbit with the eccentricity e. Moreover, its period is given by

$$T = \frac{p_\phi^3}{(1-\mu)^2} \int_0^{2\pi} \frac{\mathrm{d}\phi}{(1 + e\cos\phi)^2} = \frac{2\pi p_\phi^3}{(1-\mu)^2 (1-e^2)^{3/2}}. \tag{3.5}$$

Now we introduce the Delaunay elements obtained from the generating function

$$W(r, \phi, I_1, I_2) = I_2\phi + \chi(r, I_1, I_2), \tag{3.6}$$

where

$$\begin{aligned}\chi(r, I_1, I_2) &= \int_{r_-}^{r} \left(\frac{2(1-\mu)}{\rho} - \frac{(1-\mu)}{I_1^2} - \frac{I_2^2}{\rho^2}\right)^{1/2} \mathrm{d}\rho \\ &= -2I_1^2 \arcsin\sqrt{\frac{r_+ - r}{r_+ - r_-}} + \sqrt{(r_+ - r)(r - r_-)} \\ &\quad + \frac{2I_1 I_2}{\sqrt{1-\mu}} \arctan\sqrt{\frac{r_-(r_+ - r)}{r_+(r - r_-)}}\end{aligned} \tag{3.7}$$

with

$$r_\pm = I_1\left(I_1 \pm \sqrt{I_1^2 - \frac{I_2^2}{1-\mu}}\right)$$

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(see, e.g., [19, §8.9.1]). We have

$$p_r = \frac{\partial W}{\partial r} = \frac{\partial \chi}{\partial r}(r, I_1, I_2), \quad p_\phi = \frac{\partial W}{\partial \phi} = I_2,$$
$$\theta_1 = \frac{\partial W}{\partial I_1} = \chi_1(r, I_1, I_2), \quad \theta_2 = \frac{\partial W}{\partial I_2} = \phi + \chi_2(r, I_1, I_2),$$

where

$$\chi_1(r, I_1, I_2) = \frac{\partial \chi}{\partial I_1}(r, I_1, I_2), \quad \chi_2(r, I_1, I_2) = \frac{\partial \chi}{\partial I_2}(r, I_1, I_2).$$

Since the transformation from (r, ϕ, p_r, p_ϕ) to $(\theta_1, \theta_2, I_1, I_2)$ is symplectic, the transformed system is also Hamiltonian and its Hamiltonian is given by

$$\begin{aligned} H = &-\frac{(1-\mu)}{2I_1^2} - \varepsilon^3 I_2 \\ &- \frac{1}{4}\varepsilon^6 \mu R(\theta_1, I_1, I_2)^2 (3\cos 2(\theta_2 - \chi_2(R(\theta_1, I_1, I_2), I_1, I_2)) + 1), \end{aligned}$$

where $r = R(\theta_1, I_1, I_2)$ is the r-component of the symplectic transformation satisfying

$$\theta_1 = \chi_1(R(\theta_1, I_1, I_2), I_1, I_2). \tag{3.8}$$

Thus, we obtain the Hamiltonian system

$$\begin{aligned} \dot{I}_1 = &\frac{1}{2}\varepsilon^6 \mu \frac{\partial R}{\partial \theta_1}(\theta_1, I_1, I_2) R(\theta_1, I_1, I_2)\bigg(3\cos 2(\theta_2 - \chi_2(R(\theta_1, I_1, I_2), I_1, I_2)) \\ &+ 1 + 3R(\theta_1, I_1, I_2)\frac{\partial \chi_2}{\partial r}(R(\theta_1, I_1, I_2), I_1, I_2) \\ &\times \sin 2(\theta_2 - \chi_2(R(\theta_1, I_1, I_2), I_1, I_2))\bigg), \\ \dot{I}_2 = &-\frac{3}{2}\varepsilon^6 \mu R(\theta_1, I_1, I_2)^2 \sin 2(\theta_2 - \chi_2(R(\theta_1, I_1, I_2), I_1, I_2)), \\ \dot{\theta}_1 = &\frac{1-\mu}{I_1^3} + O(\varepsilon^6), \quad \dot{\theta}_2 = -\varepsilon^3 + O(\varepsilon^6). \end{aligned} \tag{3.9}$$

Similarly to the treatment for equation (1.1) stated just above Theorem 1.1, the new variables $(v_1, v_2, v_3) \in \mathbb{C} \times (\mathbb{C}/2\pi\mathbb{Z})^2$ given by

$$\begin{aligned} V_1(v_1, r, I_1, I_2) &:= v_1^2 + r^2 - 2I_1^2 r + \frac{I_1^2 I_2^2}{1-\mu} = 0, \\ V_2(v_2, r, I_1, I_2) &:= I_1^2\bigg(I_1^2 - \frac{I_2^2}{1-\mu}\bigg)(2\sin^2 v_2 - 1)^2 - (r - I_1^2)^2 = 0, \\ V_3(v_3, r, I_1, I_2) &:= I_1^2\bigg(r - \frac{I_2^2}{1-\mu}\bigg)^2 (\tan^2 v_3 + 1)^2 \\ &\quad - r^2\bigg(I_1^2 - \frac{I_2^2}{1-\mu}\bigg)^2 (\tan^2 v_3 - 1)^2 = 0 \end{aligned}$$

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are introduced, so that the generating function (3.6) is regarded as an analytic one on the four-dimensional complex manifold

$$\bar{\mathscr{S}}_2 = \{(r, \phi, I_1, I_2, v_1, v_2, v_3) \in \mathbb{C} \times (\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C}^3 \times (\mathbb{C}/2\pi\mathbb{Z})^2 \\ \mid V_j(v_j, r, I_1, I_2) = 0,\ j = 1, 2, 3\}$$

since equation (3.7) is represented by

$$\chi(I_1, I_2, v_1, v_2, v_3) = v_1 - 2I_1^2 v_2 + \frac{2I_1 I_2 v_3}{\sqrt{1-\mu}}.$$

Hence, we can regard equation (3.9) as a meromorphic two-degree-of-freedom Hamiltonian system on the four-dimensional complex manifold

$$\hat{\mathscr{S}}_2 = \{(I_1, I_2, \theta_1, \theta_2, r, v_1, v_2, v_3) \in \mathbb{C}^2 \times (\mathbb{C}/2\pi\mathbb{Z})^2 \times \mathbb{C}^2 \times (\mathbb{C}/2\pi\mathbb{Z})^2 \\ \mid \theta_1 - \chi_1(r, I_1, I_2) = V_j(v_j, r, I_1, I_2) = 0,\ j = 1, 2, 3\},$$

like equation (1.3) on $\mathscr{S}_2$ for equation (1.1). Actually, we have

$$\frac{\partial V_j}{\partial v_j}\frac{\partial v_j}{\partial r} + \frac{\partial V_j}{\partial r} = 0, \quad \frac{\partial V_j}{\partial v_j}\frac{\partial v_j}{\partial I_l} + \frac{\partial V_j}{\partial I_l} = 0, \quad j = 1, 2, 3,\ l = 1, 2,$$

to express

$$\frac{\partial \chi}{\partial r} = \sum_{j=1}^{3} \frac{\partial \chi}{\partial v_j}\frac{\partial V_j}{\partial r}, \quad \chi_l = \frac{\partial \chi}{\partial I_l} + \sum_{j=1}^{3} \frac{\partial \chi}{\partial v_j}\frac{\partial V_j}{\partial I_l}, \quad l = 1, 2$$

as meromorophic functions of $(r, I_1, I_2, v_1, v_2, v_3)$ on $\bar{\mathscr{S}}_2$. In particular, the Hamiltonian system has an additional first integral that is meromorphic in $(I_1, I_2, \theta_1, \theta_2, v_1, v_2, v_3, \varepsilon)$ on $\hat{\mathscr{S}}_2 \setminus \Sigma(\hat{\mathscr{S}}_2)$ near $\varepsilon = 0$ if the system (1.1) has an additional first integral that is meromorphic in $(x, y, p_x, p_y, u_1, u_2)$ on $\mathscr{S}_2 \setminus \Sigma(\mathscr{S}_2)$ near $(x, y) = (-\mu, 0)$, as in [**10**, Theorem 2], since the corresponding Hamiltonian system has the same expression as equation (3.9) on $\hat{\mathscr{S}}_2 \setminus \Sigma(\hat{\mathscr{S}}_2)$, where $\Sigma(\hat{\mathscr{S}}_2)$ is the critical set of $\hat{\mathscr{S}}_2$ on which the projection $\hat{\pi}_2 : \hat{\mathscr{S}}_2 \to \mathbb{C}^2 \times (\mathbb{C}/2\pi\mathbb{Z})^2$ given by

$$\hat{\pi}_2(I_1, I_2, \theta_1, \theta_2, r, v_1, v_2, v_3) = (I_1, I_2, \theta_1, \theta_2)$$

is singular.

We next estimate the $O(\varepsilon^6)$-term in the first equation of equation (3.9) for the unperturbed solutions. When $\varepsilon = 0$, we see that I_1, I_2, θ_2 are constants and can write $\theta_1 = \omega_1 t + \theta_{10}$ for any solution to equation (3.9), where

$$\omega_1 = \frac{1-\mu}{I_1^3} \tag{3.10}$$

and $\theta_{10} \in \mathbb{S}^1$ is a constant. Since $r = R(\omega_1 t + \theta_{10}, I_1, I_2)$ and

$$\phi = -\chi_2(R(\omega_1 t + \theta_{10}, I_1, I_2), I_1, I_2)$$

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respectively become the r- and ϕ-components of a solution to equation (3.3), we have

$$\begin{aligned} R(\omega_1 t+\theta_{10}, I_1, I_2) &= \frac{I_2^2}{(1-\mu)(1+e\cos(\phi(t)+\bar{\phi}(\theta_{10})))}, \\ -\chi_2(R(\omega_1 t+\theta_{10}, I_1, I_2), I_1, I_2) &= \phi(t)+\bar{\phi}(\theta_{10}) \end{aligned} \tag{3.11}$$

by equation (3.4), where $\phi(t)$ is the ϕ-component of a solution to equation (3.3) and $\bar{\phi}(\theta_{10})$ is a constant depending on θ_{10}. Differentiating both equations in equation (3.11) with respect to t yields

$$\begin{aligned} \omega_1\frac{\partial R}{\partial\theta_1}(\omega_1 t+\theta_{10}, I_1, I_2) &= \frac{eI_2^2\sin(\phi(t)+\bar{\phi}(\theta_{10}))\dot{\phi}(t)}{(1-\mu)(1+e\cos(\phi(t)+\bar{\phi}(\theta_{10})))^2}, \\ -\omega_1\frac{\partial\chi_2}{\partial r}(R(\omega_1 t+\theta_{10}, I_1, I_2), I_1, I_2)\frac{\partial R}{\partial\theta_1}(\omega_1 t+\theta_{10}, I_1, I_2) &= \dot{\phi}(t). \end{aligned} \tag{3.12}$$

Using equations (3.11) and (3.12), we can obtain the necessary expression of the $O(\varepsilon^6)$-term.

We are ready to check the hypotheses of Theorem 2.1 for the system (3.9). Assumption (A1) holds for any $I_1>0$. Fix the values of I_1, I_2 at some $I_1^*, I_2^*>0$, and let $\omega^*=\omega_1/3$. Since by the second equation of equation (3.11) $\phi(t)$ is $2\pi/\omega_1$-periodic, we have

$$\frac{2\pi I_2^{*3}}{(1-\mu)^2(1-e^2)^{3/2}}=\frac{2\pi I_1^3}{1-\mu}$$

by equations (3.5) and (3.10), so that

$$I_2^*=I_1^*(1-\mu)^{1/3}\sqrt{1-e^2}.$$

From equation (3.10), we also have

$$\mathrm{D}\omega(I^*)=\begin{pmatrix}-3(1-\mu)/I_1^{*4} & 0\\ 0 & 0\end{pmatrix},$$

where $I^*=(I_1^*, I_2^*)$. We write the first component of equation (2.2) with $k=5$ for $I=I^*$ as

$$\begin{aligned} \mathscr{I}_1^5(\theta) = -\frac{3\mu(1-\mu)}{2I_1^{*4}}\int_{\gamma_\theta}\bigg(&\frac{\partial R}{\partial\theta_1}(\omega_1 t+\theta_1, I_1^*, I_2^*)R(\omega_1 t+\theta_1, I_1^*, I_2^*) \\ &\times\bigg(3\cos 2(\theta_2-\chi_2(R(\omega_1 t+\theta_1, I_1^*, I_2^*), I_1^*, I_2^*))+1 \\ &+3R(\omega_1 t+\theta_1, I_1^*, I_2^*)\frac{\partial\chi_2}{\partial r}(R(\omega_1 t+\theta_1, I_1^*, I_2^*), I_1^*, I_2^*) \\ &\times\sin 2(\theta_2-\chi_2(R(\omega_1 t+\theta_1, I_1^*, I_2^*), I_1^*, I_2^*))\bigg)\,dt, \end{aligned}$$

where the closed loop γ_θ is specified below. Using equations (3.11) and (3.12), we compute

$$\begin{aligned} \mathscr{I}_1^5(\theta) = \frac{3\mu I_2^{*4}}{2(1-\mu)^2 I_1^*}\int_{\gamma_\theta}\dot{\phi}(t)\bigg(&\frac{3\sin 2(\phi(t)+\bar{\phi}(\theta_1)+\theta_2)}{(1+e\cos(\phi(t)+\bar{\phi}(\theta_1)))^2} \\ &-\frac{e\sin(\phi(t)+\bar{\phi}(\theta_1))(3\cos 2(\phi(t)+\bar{\phi}(\theta_1)+\theta_2)+1)}{(1+e\cos(\phi(t)+\bar{\phi}(\theta_1)))^3}\bigg)\,dt. \end{aligned} \tag{3.13}$$

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By equations (3.3) and (3.4), we have

$$\frac{\dot{\phi}(t)}{(1+e\cos\phi(t))^2}=\frac{(1-\mu)^2}{I_2^{*3}}=\frac{\omega_1}{(1-e^2)^{3/2}}. \tag{3.14}$$

Using integration by substitution and the relation (3.14), we rewrite the above integral as

$$\mathscr{I}_1^5(\theta)=\frac{9\mu I_2^*}{2I_1^*}\int_{\gamma_\theta}\left(\sin 2(\phi(t)+\theta_2)-\frac{e\sin\phi(t)(\cos 2(\phi(t)+\theta_2)+1/3)}{1+e\cos\phi(t)}\right)dt, \tag{3.15}$$

where the path of integration might change but the same notation γ_θ has still been used for it.

Here we integrate equation (3.14) to obtain

$$\omega_1 t=2\arctan\left(\frac{(1-e)\tan\phi/2}{\sqrt{1-e^2}}\right)-\frac{e\sqrt{1-e^2}\sin\phi}{1+e\cos\phi}\quad\text{for }\phi\in(-\pi,\pi), \tag{3.16}$$

which is rewritten as

$$\omega_1 t=2\,\mathrm{arccot}\left(\frac{(1-e)\cot(\phi+\pi)/2}{\sqrt{1-e^2}}\right)-\frac{e\sqrt{1-e^2}\sin\phi}{1+e\cos\phi}+\pi\quad\text{for }\phi\in(0,2\pi), \tag{3.17}$$

when $\phi(0)=0$ or $\lim_{t\to 0}\phi(t)=0$. From equation (3.16), we see that as $\mathrm{Im}\,\phi\to+\infty$, $\omega_1 t\to iK_1$, where

$$K_1=2\,\mathrm{arctanh}\left(\frac{1-e}{\sqrt{1-e^2}}\right)-\sqrt{1-e^2}>0.$$

See Figure 4. So the integrand in equation (3.15) is singular at $t=iK_1$. Let $K_2=\mathrm{arccosh}(1/e)$. Then $1+e\cos\phi=0$ at $\phi=\pi+iK_2$, and by equation (3.17),

$$\frac{1}{\omega_1 t}=\frac{1}{\sqrt{1-e^2}}\Delta\phi+o(\Delta\phi)$$

near $\phi=\pi+iK_2$, where $\Delta\phi=\phi-(\pi+iK_2)$. Moreover, near $\phi=\pi+iK_2$,

$$\sin\phi=-\frac{i\sqrt{1-e^2}}{e}+O(\Delta\phi),\quad \cos\phi=-\frac{1}{e}+\frac{i\sqrt{1-e^2}}{e}\Delta\phi+O(\Delta\phi^2),$$
$$\sin 2\phi=\frac{2i\sqrt{1-e^2}}{e^2}+O(\Delta\phi),\quad \cos 2\phi=\frac{2-e^2}{e^2}+O(\Delta\phi).$$

We take a closed path starting and ending at $t=\frac{1}{3}T^*$ and passing through $t=\frac{2}{3}T^*$, $\frac{2}{3}T^*+i(K_1\mp\delta)$, $\frac{2}{3}T^*+iM$, $\frac{1}{3}T^*+iM$, and $\frac{1}{3}T^*+i(K_1\pm\delta)$ as γ_θ in $\mathbb{C}/T^*\mathbb{Z}$, where δ and M are respectively sufficiently small and large positive constants (see Figure 5). Here γ_θ passes along the left circular arc centered at $\frac{2}{3}T^*+iK_1$ (respectively at $\frac{1}{3}T^*+iK_1$) with radius δ between $\frac{2}{3}T^*+i(K_1-\delta)$ and $\frac{2}{3}T^*+i(K_1+\delta)$ (respectively

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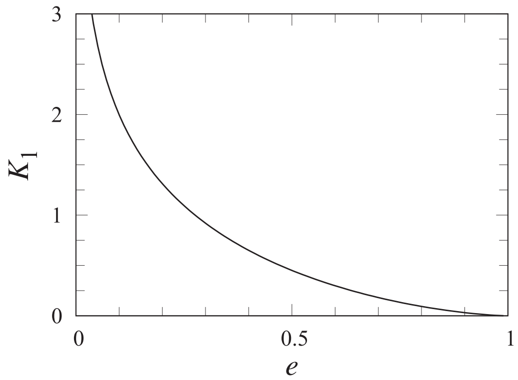

FIGURE 4. Dependence of K_1 on e.

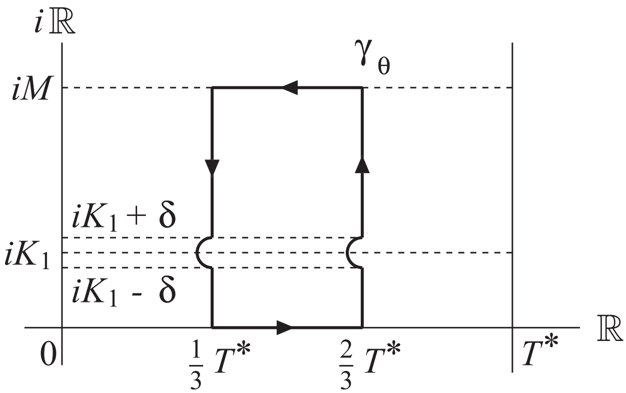

FIGURE 5. Closed path γ_θ.

between $\frac{1}{3}T^* + i(K_1 + \delta)$ and $\frac{1}{3}T^* + i(K_1 - \delta)$). We compute

$$\begin{aligned}\int_{2T^*/3+iM}^{T^*/3+iM} &\frac{\sin\phi(t)(\cos 2(\phi(t)+\theta_2)+1/3)}{1+e\cos\phi(t)}\,dt\\ &= -\int_{2T^*/3}^{T^*/3}\bigg(\frac{2-e^2}{e^3\sqrt{1-e^2}}\cos 2\theta_2 - \frac{2i}{e^3}\sin 2\theta_2 + \frac{1}{3e\sqrt{1-e^2}}\bigg)i\omega_1 M\,dt + O(1)\\ &= \frac{2\pi}{e^3}\bigg(\frac{2-e^2}{\sqrt{1-e^2}}i\cos 2\theta_2 + 2\sin 2\theta_2 + \frac{ie^2}{3\sqrt{1-e^2}}\bigg)M + O(1),\end{aligned}$$

while

$$\int_{2T^*/3+iM}^{T^*/3+iM}\sin 2(\phi(t)+\theta_2)\,dt = O(1).$$

Moreover, the integral on $[\frac{1}{3}T^*, \frac{2}{3}T^*]$ in equation (3.15) is $O(1)$, and the integrals from $\frac{2}{3}T^*$ to $\frac{2}{3}T^* + iM$ and from $\frac{1}{3}T^* + iM$ to $\frac{1}{3}T^*$ cancel since the integrand is $\frac{1}{3}T^*$-periodic. Thus, we see that the integral (3.15) is not zero for $M > 0$ sufficiently large, so that assumption (A2) holds.

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Finally, we apply Theorem 2.1 to show that the meromorphic Hamiltonian system corresponding to equation (3.9) is not meromorphically integrable such that the first integral depends meromorphically on ε near $\varepsilon = 0$ even if any higher-order terms are included. Thus, we obtain the conclusion of Theorem 1.1 for the planar case.

Remark 3.1

(i) The reader may think that a small circle centered at $t = \frac{1}{3}T^* + iK_1$ or $\frac{2}{3}T^* + iK_1$ can be taken as γ_θ in the proof, since the integrand in equation (3.15) is singular there. However, the integral (3.15) for the path is estimated to be zero (cf. [**42**, §3]).

(ii) The different change of coordinates

$$\varepsilon\xi = x + \mu, \quad \varepsilon\eta = y, \quad p_\xi = p_x, \quad p_\eta = p_y + \mu$$

in equation (1.1) yields

$$\begin{aligned} \dot{\xi} &= p_\xi + \varepsilon\eta, \quad \dot{p}_\xi = \varepsilon p_\eta - \varepsilon^{-1}\frac{(1-\mu)\xi}{(\xi^2+\eta^2)^{3/2}} + 2\varepsilon^2\mu\xi, \\ \dot{\eta} &= p_\eta - \varepsilon\xi, \quad \dot{p}_\eta = -\varepsilon p_\xi - \varepsilon^{-1}\frac{(1-\mu)\eta}{(\xi^2+\eta^2)^{3/2}} - \varepsilon^2\mu\eta \end{aligned} \tag{3.18}$$

up to $O(\varepsilon^2)$ after the time scaling $t \to t/\varepsilon$. As in [**24**], we use the Levi–Civita regularization

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1\end{pmatrix}\begin{pmatrix} q_1 \\ q_2\end{pmatrix}, \quad \begin{pmatrix} p_\xi \\ p_\eta\end{pmatrix} = \frac{2}{q_1^2+q_2^2}\begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1\end{pmatrix}\begin{pmatrix} p_1 \\ p_2\end{pmatrix},$$

(see, e.g., [**31**] or [**19**, §8.8.1]) to obtain

$$\begin{aligned} H + C_0 = \frac{4}{q_1^2+q_2^2}\bigg(&\frac{1}{4}C_0(q_1^2+q_2^2) + \frac{1}{2}(p_1^2+p_2^2) + \frac{1}{2}\varepsilon(q_1^2+q_2^2)(q_2p_1 - q_1p_2) \\ &- \frac{1}{2}\varepsilon^2\mu(q_1^2+q_2^2)(q_1^4q_1^2q_2^2 - 4 + q_2^4)) - \frac{1}{4}\varepsilon^{-1}(1-\mu)\bigg), \end{aligned}$$

which yields

$$\begin{aligned} H + C_0 = \frac{4}{q_1^2+q_2^2}\bigg(&\frac{1}{4}C_0(q_1^2+q_2^2) + \frac{1}{2}(p_1^2+p_2^2) + \frac{1}{2}(q_1^2+q_2^2)(q_2p_1 - q_1p_2) \\ &- \frac{1}{2}\mu(q_1^2+q_2^2)(q_1^4q_1^2q_2^2 - 4 + q_2^4)) - \frac{1}{4}(1-\mu)\bigg) \end{aligned}$$

after the scaling $(q, p) \to (q, p)/\varepsilon^{3/2}$. Using the approach of [**24**], we can show that the Hamiltonian system with the Hamiltonian

$$\begin{aligned} \tilde{H} = &\tfrac{1}{4}C_0(q_1^2+q_2^2) + \tfrac{1}{2}(p_1^2+p_2^2) + \tfrac{1}{2}(q_1^2+q_2^2)(q_2p_1 - q_1p_2) \\ &- \tfrac{1}{2}\mu(q_1^2+q_2^2)(q_1^4q_1^2q_2^2 - 4 + q_2^4) \end{aligned}$$

is meromorphically non-integrable. This implies that the Hamiltonian system (3.18) is also meromorphically non-integrable for $\varepsilon > 0$ fixed.

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4. *Spatial case*

We prove Theorem 1.1 for the spatial case (1.2). As in the planar case, we only consider a neighborhood of $(x, y, z) = (-\mu, 0, 0)$ and introduce a small parameter ε such that $0 < \varepsilon \ll 1$. Letting

$$\varepsilon^2\xi = x + \mu, \quad \varepsilon^2\eta = y, \quad \varepsilon^2\zeta = z,$$
$$\varepsilon^{-1}p_\xi = p_x, \quad \varepsilon^{-1}p_\eta = p_y + \mu, \quad \varepsilon^{-1}p_\zeta = p_z$$

and scaling the time variable $t \to \varepsilon^3 t$, we rewrite equation (1.2) as

$$\begin{aligned}
&\dot\xi = p_\xi + \varepsilon^3\eta, \quad \dot\eta = p_\eta - \varepsilon^3\xi, \quad \dot\zeta = p_\zeta,\\
&\dot p_\xi = -\frac{(1-\mu)\xi}{(\xi^2+\eta^2+\zeta^2)^{3/2}} + \varepsilon^3 p_\eta - \varepsilon^4\mu - \varepsilon^4\frac{\mu(\varepsilon^2\xi - 1)}{((\varepsilon^2\xi-1)^2+\varepsilon^4(\eta^2+\zeta^2))^{3/2}},\\
&\dot p_\eta = -\frac{(1-\mu)\eta}{(\xi^2+\eta^2+\zeta^2)^{3/2}} - \varepsilon^3 p_\xi - \varepsilon^6\frac{\mu\eta}{((\varepsilon^2\xi-1)^2+\varepsilon^4(\eta^2+\zeta^2))^{3/2}},\\
&\dot p_\zeta = -\frac{(1-\mu)\zeta}{(\xi^2+\eta^2+\zeta^2)^{3/2}} - \varepsilon^6\frac{\mu\zeta}{((\varepsilon^2\xi-1)^2+\varepsilon^4(\eta^2+\zeta^2)))^{3/2}},
\end{aligned}$$

or up to the order of ε^6,

$$\begin{aligned}
&\dot\xi = p_\xi + \varepsilon^3\eta, \quad \dot p_\xi = -\frac{(1-\mu)\xi}{(\xi^2+\eta^2+\zeta^2)^{3/2}} + \varepsilon^3 p_\eta + 2\varepsilon^6\mu\xi,\\
&\dot\eta = p_\eta - \varepsilon^3\xi, \quad \dot p_\eta = -\frac{(1-\mu)\eta}{(\xi^2+\eta^2+\zeta^2)^{3/2}} - \varepsilon^3 p_\xi - \varepsilon^6\mu\eta,\\
&\dot\zeta = p_\zeta, \quad \dot p_\zeta = -\frac{(1-\mu)\zeta}{(\xi^2+\eta^2+\zeta^2)^{3/2}} - \varepsilon^6\mu\zeta,
\end{aligned} \tag{4.1}$$

like equation (3.1), where the $O(\varepsilon^8)$ terms have been eliminated. Equation (4.1) is a Hamiltonian system with the Hamiltonian

$$\begin{aligned}
H = {}&\tfrac{1}{2}(p_\xi^2 + p_\eta^2 + p_\zeta^2) - \frac{1-\mu}{\sqrt{\xi^2+\eta^2+\zeta^2}}\\
&+ \varepsilon^3(\eta p_\xi - \xi p_\eta) - \tfrac{1}{2}\varepsilon^6(2\xi^2 - \eta^2 - \zeta^2).
\end{aligned} \tag{4.2}$$

We next rewrite equation (4.2) in the spherical coordinates (see Figure 6). Let

$$\xi = r\sin\psi\cos\phi, \quad \eta = r\sin\psi\sin\phi, \quad \zeta = r\cos\psi.$$

The momenta (p_r, p_ϕ, p_ψ) corresponding to (r, ϕ, ψ) satisfy

$$\begin{aligned}
p_\xi &= p_r\cos\phi\sin\psi - \frac{p_\phi\sin\phi}{r\sin\psi} + \frac{p_\psi}{r}\cos\phi\cos\psi,\\
p_\eta &= p_r\sin\phi\sin\psi + \frac{p_\phi\cos\phi}{r\sin\psi} + \frac{p_\psi}{r}\sin\phi\cos\psi,\\
p_\zeta &= p_r\cos\psi - \frac{p_\psi}{r}\sin\psi
\end{aligned}$$

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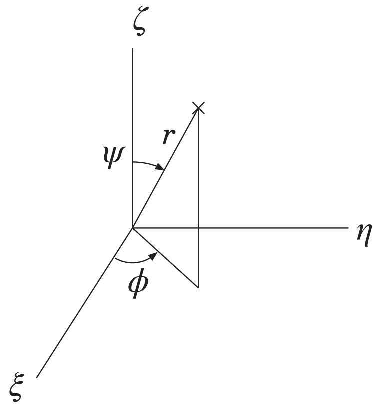

FIGURE 6. Spherical coordinates.

(see, e.g., [19, §8.7]). The Hamiltonian becomes

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\psi^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\psi}\right) - \frac{1-\mu}{r}$$
$$- \varepsilon^3 p_\phi - \varepsilon^6 \mu r^2\left(\frac{1}{4}\sin^2\psi(3\cos 2\phi + 1) - \frac{1}{2}\cos^2\psi\right).$$

Up to $O(1)$, the corresponding Hamiltonian system becomes

$$\begin{aligned} &\dot r = p_r, \quad \dot p_r = \frac{p_\psi^2}{r^3} + \frac{p_\phi^2}{r^3\sin^2\psi} - \frac{1-\mu}{r^2}, \\ &\dot\phi = \frac{p_\phi}{r^2\sin^2\psi}, \quad \dot p_\phi = 0, \quad \dot\psi = \frac{p_\psi}{r^2}, \quad \dot p_\psi = \frac{p_\phi^2\cos\psi}{r^2\sin^3\psi}. \end{aligned} \tag{4.3}$$

We have the relation (3.4) for periodic orbits on the (ξ, η)-plane since equation (4.3) reduces to equation (3.3) when $\psi = \frac{1}{2}\pi$ and $p_\psi = 0$.

As in the planar case, we introduce the Delaunay elements obtained from the generating function

$$\hat W(r, \phi, \psi, I_1, I_2.I_3) = I_3\phi + \chi(r, I_1, I_2) + \hat\chi(\psi, I_2, I_3), \tag{4.4}$$

where

$$\begin{aligned} \hat\chi(\psi, I_2, I_3) &= \int_{\psi_0}^{\psi}\left(I_2^2 - \frac{I_3^2}{\sin^2 s}\right)^{1/2} ds \\ &= I_2 \arctan\frac{\sqrt{I_2^2\sin^2\psi - I_3^2}}{I_2\cos\psi} - I_3\arctan\frac{\sqrt{I_2^2\sin^2\psi - I_3^2}}{I_3\cos\psi} \end{aligned} \tag{4.5}$$

with $\psi_0 = \arcsin(I_3/I_2)$. See, e.g., [19, §8.9.3], although a slightly modified generating function is used here. We have

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$$
\begin{aligned}
p_r &= \frac{\partial \hat{W}}{\partial r} = \frac{\partial \chi}{\partial r}(r, I_1, I_2), \quad p_\phi = \frac{\partial \hat{W}}{\partial \phi} = I_3, \\
p_\psi &= \frac{\partial \hat{W}}{\partial \psi} = \frac{\partial \hat{\chi}}{\partial \psi}(\psi, I_2, I_3), \quad \theta_1 = \frac{\partial \hat{W}}{\partial I_1} = \chi_1(r, I_1, I_2), \\
\theta_2 &= \frac{\partial \hat{W}}{\partial I_2} = \chi_2(r, I_1, I_2) + \hat{\chi}_2(\psi, I_2, I_3), \quad \theta_3 = \frac{\partial \hat{W}}{\partial I_3} = \phi + \hat{\chi}_3(\psi, I_2, I_3),
\end{aligned} \tag{4.6}
$$

where

$$
\hat{\chi}_2(\psi, I_2, I_3) = \frac{\partial \hat{\chi}}{\partial I_2}(\psi, I_2, I_3), \quad \hat{\chi}_3(\psi, I_2, I_3) = \frac{\partial \hat{\chi}}{\partial I_3}(\psi, I_2, I_3).
$$

Since the transformation from $(r, \phi, \psi, p_r, p_\phi, p_\psi)$ to $(\theta_1, \theta_2, \theta_3, I_1, I_2, I_3)$ is symplectic, the transformed system is also Hamiltonian and its Hamiltonian is given by

$$
\begin{aligned}
H = &-\frac{(1-\mu)}{2I_1^2} - \varepsilon^3 I_3 - \varepsilon^6 \mu R(\theta_1, I_1, I_2)^2 \bigg(\frac{1}{4} \sin^2 \Psi(\theta_1, \theta_2, I_1, I_2, I_3) \\
&\times (3 \cos 2(\theta_3 - \hat{\chi}_3(\Psi(\theta_1, \theta_2, I_1, I_2, I_3), I_1, I_2)) + 1) \\
&- \frac{1}{2} \cos^2 \Psi(\theta_1, \theta_2, I_1, I_2, I_3) \bigg),
\end{aligned}
$$

where $r = R(\theta_1, I_1, I_2)$ and $\psi = \Psi(\theta_1, \theta_2, I_1, I_2, I_3)$ are the r- and ψ-components of the symplectic transformation satisfying equation (3.8) and

$$
\hat{\chi}_2(\Psi(\theta_1, \theta_2, I_1, I_2, I_3), I_2, I_3) + \chi_2(R(\theta_1, I_1, I_2), I_1, I_2) = \theta_2,
$$

respectively. Thus, we obtain the Hamiltonian system as

$$
\begin{aligned}
\dot{I}_1 &= \frac{1}{2}\varepsilon^6 \mu \hat{h}_1(I, \theta), \quad \dot{I}_2 = O(\varepsilon^6), \quad \dot{I}_3 = O(\varepsilon^6), \\
\dot{\theta}_1 &= \frac{1-\mu}{I_1^3} + O(\varepsilon^6), \quad \dot{\theta}_2 = O(\varepsilon^6), \quad \dot{\theta}_3 = -\varepsilon^3 + O(\varepsilon^6),
\end{aligned} \tag{4.7}
$$

where

$$
\begin{aligned}
\hat{h}_1(I, \theta) = &\frac{\partial R}{\partial \theta_1}(\theta_1, I_1, I_2) R(\theta_1, I_1, I_2)(\sin^2 \Psi(\theta_1, \theta_2, I_1, I_2, I_3) \\
&\quad \times (3 \cos 2(\theta_3 - \hat{\chi}_3(\Psi(\theta_1, \theta_2, I_1, I_2, I_3), I_2, I_3)) + 1) \\
&\quad - 2 \cos^2 \Psi(\theta_1, \theta_2, I_1, I_2, I_3)) \\
&+ R(\theta_1, I_1, I_2)^2 \frac{\partial \Psi}{\partial \theta_1}(\theta_1, \theta_2, I_1, I_2, I_3) \\
&\quad \times 3 \sin \Psi(\theta_1, \theta_2, I_1, I_2, I_3) \cos \Psi(\theta_1, \theta_2, I_1, I_2, I_3) \\
&\quad \times (\cos 2(\theta_3 - \hat{\chi}_3(\Psi(\theta_1, \theta_2, I_1, I_2, I_3), I_2, I_3)) + 1) \\
&+ 3R(\theta_1, I_1, I_2)^2 \sin^2 \Psi(\theta_1, \theta_2, I_1, I_2, I_3) \\
&\quad \times \frac{\partial \Psi}{\partial \theta_1}(\theta_1, \theta_2, I_1, I_2, I_3) \frac{\partial \hat{\chi}_3}{\partial \psi}(\Psi(\theta_1, \theta_2, I_1, I_2, I_3), I_2, I_3) \\
&\quad \times \sin 2(\theta_3 - \hat{\chi}_3(\Psi(\theta_1, \theta_2, I_1, I_2, I_3), I_2, I_3)).
\end{aligned}
$$

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As in the planar case, the new variables $w_1, w_2 \in (\mathbb{C}/2\pi)$ given by

$$W_1(w_1, \psi, I_2, I_3) := I_2^2 \cos^2 \psi \tan^2 w_1 - I_2^2 \sin^2 \psi + I_3^2 = 0,$$
$$W_2(w_2, \psi, I_2, I_3) := I_3^2 \cos^2 \psi \tan^2 w_2 - I_2^2 \sin^2 \psi + I_3^2 = 0$$

are introduced, so that the generating function (4.4) is regarded as an analytic one on the six-dimensional complex manifold

$$\bar{\mathscr{S}}_3 = \{(r, \phi, \psi, I_1, I_2, I_3, v_1, v_2, v_3, w_1, w_2) \in \mathbb{C} \times (\mathbb{C}/2\pi\mathbb{Z})^2 \times \mathbb{C}^4 \times (\mathbb{C}/2\pi\mathbb{Z})^4$$
$$\mid V_j(v_j, r, I_1, I_2) = W_l(w_l, \psi, I_2, I_3) = 0,\ \ j = 1, 2, 3,\ \ l = 1, 2\}$$

since equation (4.5) is represented by

$$\hat{\chi}(I_2, I_3; v_1, v_2) = I_2 w_1 - I_3 w_2.$$

Moreover, we can regard equation (4.7) as a meromorphic three-degree-of-freedom Hamiltonian systems on the six-dimensional complex manifold

$$\hat{\mathscr{S}}_3 = \{(I_1, I_2, I_3, \theta_1, \theta_2, \theta_3, r, v_1, v_2, v_3, w_1, w_2) \in \mathbb{C}^3 \times (\mathbb{C}/2\pi\mathbb{Z})^3 \times \mathbb{C}^2 \times (\mathbb{C}/2\pi\mathbb{Z})^4$$
$$\mid \theta_1 - \chi_1(r, I_1, I_2) = V_j(v_j, r, I_1, I_2) = W_l(w_l, \psi, I_2, I_3) = 0,$$
$$j = 1, 2, 3,\ \ l = 1, 2\},$$

like equation (1.4) on $\mathscr{S}_3$ for equation (1.2). Actually, we have

$$\frac{\partial W_j}{\partial w_j}\frac{\partial w_j}{\partial \psi} + \frac{\partial W_j}{\partial \psi} = 0, \quad \frac{\partial W_j}{\partial w_j}\frac{\partial w_j}{\partial I_l} + \frac{\partial W_j}{\partial I_l} = 0, \quad j = 1, 2,\ l = 2, 3$$

to express

$$\frac{\partial \hat{\chi}}{\partial \psi} = \sum_{j=1}^{2} \frac{\partial \hat{\chi}}{\partial w_j}\frac{\partial W_j}{\partial \psi}, \quad \hat{\chi}_l = \frac{\partial \hat{\chi}}{\partial I_l} + \sum_{j=1}^{2} \frac{\partial \hat{\chi}}{\partial w_j}\frac{\partial W_j}{\partial I_l}, \quad l = 2, 3$$

as meromorophic functions of $(\psi, I_2, I_3, w_1, w_2)$ on $\bar{\mathscr{S}}_3$. In particular, the Hamiltonian system has two additional meromorphic integrals that are meromorphic in $(I_1, I_2, I_3, \theta_1, \theta_2, \theta_3, r, v_1, v_2, v_3, w_1, w_2, \varepsilon)$ on $\hat{\mathscr{S}}_3 \setminus \Sigma(\hat{\mathscr{S}}_3)$ near $\varepsilon = 0$, if the system (1.2) has two additional meromorphic integrals that are meromorphic in $(x, y, z, p_x, p_y, p_z, u_1, u_2)$ on $\mathscr{S}_3 \setminus \Sigma(\mathscr{S}_3)$ near $(x, y, z) = (-\mu, 0, 0)$, as in the planar case. Here $\Sigma(\hat{\mathscr{S}}_3)$ is the critical set of $\hat{\mathscr{S}}_3$ on which the projection $\hat{\pi}_3 : \hat{\mathscr{S}}_3 \to \mathbb{C}^3 \times (\mathbb{C}/2\pi\mathbb{Z})^3$ given by

$$\hat{\pi}_3(I_1, I_2, I_3, \theta_1, \theta_2, \theta_3, r, v_1, v_2, v_3, w_1, w_2) = (I_1, I_2, I_3, \theta_1, \theta_2, \theta_3)$$

is singular.

We next estimate the function $\hat{h}_1(I, \theta)$ for solutions to equation (4.7) with $\varepsilon = 0$ on the plane of $\psi = \frac{1}{2}\pi$. When $\varepsilon = 0$, we see that $I_1, I_2, I_3, \theta_2, \theta_3$ are constants and can write $\theta_1 = \omega_1 t + \theta_{10}$ for any solution to equation (4.7) with equation (3.10), where $\theta_{10} \in \mathbb{S}^1$ is a constant. Note that if $\psi = \frac{1}{2}\pi$ and $p_\psi = 0$, then $I_2 = I_3$ by equation (4.6). Since $r = R(\omega_1 t + \theta_{10}, I_1, I_2)$ and

$$\phi = -\hat{\chi}_3(\Psi(\omega_1 t + \theta_{10}, \theta_2, I_1, I_3, I_3), I_3, I_3), \quad \Psi(\omega_1 t + \theta_{10}, \theta_2, I_1, I_3, I_3) = \tfrac{1}{2}\pi$$

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respectively become the r- and ϕ-components of a solution to equation (4.3) with $\psi = \frac{1}{2}\pi$ and $p_\psi = 0$, we have the first equation of equation (3.11) with

$$-\hat{\chi}_3(\Psi(\omega_1 t + \theta_{10}, \theta_2, I_1, I_3, I_3), I_3, I_3) = \phi(t) + \bar{\phi}(\theta_{10}), \tag{4.8}$$

where $\phi(t)$ is the ϕ-component of a solution to equation (3.3) and $\bar{\phi}(\theta_1)$ is a constant depending only on θ_1 as in the planar case. Differentiating equation (4.8) with respect to t yields

$$\begin{aligned} -\omega_1 \frac{\partial \hat{\chi}_3}{\partial \psi}&(\Psi(\omega_1 t + \theta_{10}, \theta_2, I_1, I_3, I_3), I_3, I_3) \\ &\times \frac{\partial \Psi}{\partial \theta_1}(\omega_1 t + \theta_{10}, \theta_2, I_1, I_3, I_3) = \dot{\phi}(t). \end{aligned} \tag{4.9}$$

Using equations (3.11), (3.12), (4.8), and (4.9), we can obtain the necessary expression of $\hat{h}_1(I, \theta)$.

We are ready to check the hypotheses of Theorem 2.1 for the system (4.7). Assumption (A1) holds for any $I_1 > 0$. Fix the value of I_1 at some $I_1^* > 0$, and let $\omega^* = \omega_1/3$. By the first equation of equation (4.8), $\phi(t)$ is $2\pi/\omega_1$-periodic, so that by equations (3.5) and (3.10),

$$I_2 = I_3 = I_1^*(1-\mu)^{1/3}\sqrt{1-e^2}\ (= I_2^*).$$

From equation (3.10), we have

$$\mathrm{D}\omega(I^*) = \begin{pmatrix} -3(1-\mu)/I_1^{*4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $I^* = (I_1^*, I_2^*, I_2^*)$. Using the first equations of equations (3.11) and (3.12), equations (4.8) and (4.9), we compute the first component of equation (2.2) with $k = 5$ for $I = I^*$ as

$$\begin{aligned} \mathscr{I}_1^5(\theta) &= -\frac{3(1-\mu)}{I_1^{*4}} \int_{\gamma_\theta} h_1(I^*, \omega^* t + \theta_1, \theta_2, \theta_3)\, d\omega^* t \\ &= \frac{3\mu I_2^{*4}}{2(1-\mu)^2 I_1^*} \int_{\gamma_\theta} \dot{\phi}(t) \bigg(\frac{3 \sin 2(\phi(t) + \bar{\phi}(\theta_1) + \theta_3)}{(1 + e\cos(\phi(t) + \bar{\phi}(\theta_1)))^2} \\ &\quad + \frac{e \sin(\phi(t) + \bar{\phi}(\theta_1))(3\cos 2(\phi(t) + \bar{\phi}(\theta_1) + \theta_3) + 1)}{(1 + e\cos(\phi(t) + \bar{\phi}(\theta_1)))^3} \bigg)\, dt, \end{aligned}$$

which has the same expression as equation (3.13) with $\theta_2 = \theta_3$. Repeating the arguments given in §3, we can show that assumption (A2) holds as in the planar case. Finally, we apply Theorem 2.1 to show that the meromorphic Hamiltonian system corresponding to equation (4.7) is not meromorphically integrable such that the first integrals depend meromorphically on ε near $\varepsilon = 0$. Thus, we complete the proof of Theorem 1.1 for the spatial case.

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A. *Appendix. Differential Galois theory*

In this appendix, we give necessary information on differential Galois theory for linear differential equations, which is often referred to as the Picard–Vessiot theory. See the textbooks [**11**, **37**] for more details on the theory.

Consider a linear system of differential equations

$$y' = Ay, \quad A \in \mathrm{gl}(n, \mathbb{K}), \tag{A.1}$$

where $\mathbb{K}$ is a differential field and $\mathrm{gl}(n, \mathbb{K})$ denotes the ring of $n \times n$ matrices with entries in $\mathbb{K}$. Here a *differential field* is a field endowed with a derivation ∂, which is an additive endomorphism satisfying the Leibniz rule. The set $\mathrm{C}_{\mathbb{K}}$ of elements of $\mathbb{K}$ for which ∂ vanishes is a subfield of $\mathbb{K}$ and called the *field of constants of* $\mathbb{K}$. In our application of the theory in this paper, the differential field $\mathbb{K}$ is the field of meromorphic functions on a Riemann surface, so that the field of constants is $\mathbb{C}$.

A *differential field extension* $\mathbb{L} \supset \mathbb{K}$ is a field extension such that $\mathbb{L}$ is also a differential field and the derivations on $\mathbb{L}$ and $\mathbb{K}$ coincide on $\mathbb{K}$. A differential field extension $\mathbb{L} \supset \mathbb{K}$ satisfying the following two conditions is called a *Picard–Vessiot extension* for equation (A.1):

(PV1) the field $\mathbb{L}$ is generated by $\mathbb{K}$ and elements of a fundamental matrix of equation (A.1);

(PV2) the fields of constants for $\mathbb{L}$ and $\mathbb{K}$ coincide.

The system (A.1) admits a Picard–Vessiot extension which is unique up to isomorphism.

We now fix a Picard–Vessiot extension $\mathbb{L} \supset \mathbb{K}$ and fundamental matrix Φ with entries in $\mathbb{L}$ for equation (A.1). Let σ be a $\mathbb{K}$-*automorphism* of $\mathbb{L}$, which is a field automorphism of $\mathbb{L}$ that commutes with the derivation of $\mathbb{L}$ and leaves $\mathbb{K}$ pointwise fixed. Obviously, $\sigma(\Phi)$ is also a fundamental matrix of equation (A.1) and consequently there is a matrix M_σ with constant entries such that $\sigma(\Phi) = \Phi M_\sigma$. This relation gives a faithful representation of the group of $\mathbb{K}$-automorphisms of $\mathbb{L}$ on the general linear group as

$$R\colon \mathrm{Aut}_{\mathbb{K}}(\mathbb{L}) \to \mathrm{GL}(n, \mathrm{C}_{\mathbb{L}}), \quad \sigma \mapsto M_\sigma,$$

where $\mathrm{GL}(n, \mathrm{C}_{\mathbb{L}})$ is the group of $n \times n$ invertible matrices with entries in $\mathrm{C}_{\mathbb{L}}$. The image of R is a linear algebraic subgroup of $\mathrm{GL}(n, \mathrm{C}_{\mathbb{L}})$, which is called the *differential Galois group* of equation (A.1) and often denoted by $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$. This representation is not unique and depends on the choice of the fundamental matrix Φ, but a different fundamental matrix only gives rise to a conjugated representation. Thus, the differential Galois group is unique up to conjugation as an algebraic subgroup of the general linear group.

Let $G \subset \mathrm{GL}(n, \mathrm{C}_{\mathbb{L}})$ be an algebraic group. Then it contains a unique maximal connected algebraic subgroup G^0, which is called the *connected component of the identity* or *identity component*. The identity component $G^0 \subset G$ is the smallest subgroup of finite index, that is, the quotient group G/G^0 is finite.

B. *Appendix. Monodromy matrices*

In this appendix, we give general information on monodromy matrices for the reader's convenience.

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Let $\mathbb{K}$ be the field of meromorphic functions on a Riemann surface Γ, and consider the linear system (A.1). Let $t_0 \in \Gamma$ be a non-singular point for equation (A.1). We prolong the fundamental matrix $\Phi(t)$ analytically along any loop γ based at t_0 and containing no singular point, and obtain another fundamental matrix $\gamma * \Phi(t)$. So there exists a constant non-singular matrix $M_{[\gamma]}$ such that

$$\gamma * \Phi(t) = \Phi(t) M_{[\gamma]}.$$

The matrix $M_{[\gamma]}$ depends on the homotopy class $[\gamma]$ of the loop γ and is called the *monodromy matrix* of $[\gamma]$.

Let $\mathbb{L}$ be a Picard–Vessiot extension of equation (A.1) and let $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ be the differential Galois group, as in Appendix A. Since analytic continuation commutes with differentiation, we have $M_{[\gamma]} \in \mathrm{Gal}(\mathbb{L}/\mathbb{K})$.

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