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C*-IDEALS GENERATED BY POLYNOMIALS

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The *-algebra A_1 is defined to be the free unital *-algebra with one generator x. A *-ideal I of A_1 is defined to be a C*-ideal if A_1/I may be embedded into a C*-algebra. It is proved that if I is a *-ideal of A_1 generated by polynomials in A_1, then I is a C*-ideal. This is not true for general *-ideals of A_1.

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1. Definitions

Let W_1 be the set of finite length words in the non-commuting elements x and x^*. For $w \in W_1$, let len(w) denote the length of w. Let A_1 be the free unital involutive algebra over $\mathbb{C}$ generated by the element x. So, if y is in A_1 then $y = \sum_{w \in W_1} y_w w$, where $y_w \in \mathbb{C}$ for all w in W_1, and only finitely many are non-zero. If $y \in A_1$ then call y a *-polynomial in x. Let P_1 be the subset of A_1 which consists of the polynomials in x, as opposed to the *-polynomials. Say that a word in W_1 is a syllable if it is of the form x^n or x^{*n} for some $n \in \mathbb{N}$.

Given $I \subseteq A_1$, say that I is a *-ideal of A_1 if I is an ideal of A_1 and is closed under * (the involution on A_1). If $S \subseteq A_1$ then let $\langle S \rangle$ denote the ideal of A_1 generated by S and let $\langle S \rangle_*$ denote the *-ideal of A_1 generated by S. So, $\langle S \rangle_* = \langle S \cup S^* \rangle$. Say that I is a C*-ideal of A_1 if I is a *-ideal of A_1 and the *-algebra A_1/I may be embedded into a C*-algebra.

2. Examples

The *-ideal $I = \langle x^*x \rangle_*$ is not a C*-ideal. This is because, if A_1/I is embedded in some C*-algebra B, then, as $x^*x \in I$, we have $\|x^*x\| = 0$. So, $\|x\| = 0$, but $x \notin I$, so x is non-zero in A_1/I.

It is a result of Goodearl and Menal [2] that A_1 itself may be embedded into a C*-algebra. So, $\langle 0 \rangle_*$ is a C*-ideal of A_1. It is a result of Coburn [1] that the *-algebra $A_1/\langle xx^* - 1 \rangle_*$ may be faithfully *-represented by sending x to the left unilateral shift on $l^2(\mathbb{N})$. So, $\langle xx^* - 1 \rangle_*$ is a C*-ideal of A_1. There are many other related results in [3].

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For the rest of this paper we shall be interested in the question of whether the *-ideals generated by polynomials are C*-ideals.

3. Definitions

Let p be a polynomial in P_1 with $p=(x-c_1)\ldots(x-c_n)$ for some complex numbers $c_1,\ldots,c_n$ (we may take p to have leading coefficient 1). Let $\mathbf{c}$ denote the n-tuple $(c_1,\ldots,c_n)$. Let $I=\langle p\rangle_*$. Then, in the quotient *-algebra A_1/I, we have the identity $x^n=x^n-p$. As x^n-p is a polynomial of degree $n-1$, any element of A_1/I may be written as a linear combination of words which have syllables of length at most $n-1$. Whenever considering an element of A_1/I we shall assume it is in this form.

For all words $w=x^{r_R}x^{*r_{R-1}}\ldots x^{r_3}x^{*r_2}x^{r_1}$ in A_1/I, (with $r_j<n$ for all j), we can make the following definitions. Let $n_j=\sum_{k=1}^{j}r_k$ for all $j\geq 1$, and let $H_w=l^2(\text{len}(w)+1)$ with basis $\{\epsilon_0,\ldots,\epsilon_{\text{len}(w)}\}$. Call $\epsilon_{n_2},\epsilon_{n_4},\ldots$ *sources* (and also ϵ_0 if $n_1>0$) and call $\epsilon_{n_1},\epsilon_{n_3},\ldots$ *sinks*. If we were to think of the ϵ_j lined up in order, then any ϵ_j which was not itself a source or a sink would be between a source and a sink. Say that these are the source and sink to which ϵ_j belongs. Still thinking of the ϵ_j as being lined up, let $\delta_w(j)$ be (informally) the number of places ϵ_j is from the source it belongs to plus 1, with a source having a value 1 and a sink having the value for the further of the two sources it is next to. So, if ϵ_j is itself a source then $\delta_w(j)=1$, and $\delta_w(j+1)=2$, etc. until you go past a sink. For example, if $w=x^2x^{*2}x^3$ then the sources are ϵ_0 and ϵ_5 and the sinks are ϵ_3 and ϵ_7. If we allow ourselves to write δ_w as acting on tuples of values as well as just single values, then $\delta_{x^2x^{*2}x^3}(0,1,2,3,4,5,6,7)=(1,2,3,4,2,1,2,3)$. Note that $\delta_w(j)\leq(n-1)+1=n$ for all j.

Define the representation $\mathfrak{T}_{w,\mathbf{c}}:A_1\to B(H_w)$ to be the unital *-homomorphism given by

$$\mathfrak{T}_{w,\mathbf{c}}(x)\epsilon_j=\begin{cases} c_{\delta_w(j)}.\epsilon_j+\epsilon_{j-1}+\epsilon_{j+1} & \text{if } \epsilon_j \text{ is a source}\\ c_{\delta_w(j)}.\epsilon_j+\epsilon_{j+1} & \text{if } 0<j<n_1, n_2<j<n_3,\ldots\\ c_{\delta_w(j)}.\epsilon_j & \text{if } \epsilon_j \text{ is a sink}\\ c_{\delta_w(j)}.\epsilon_j+\epsilon_{j-1} & \text{if } n_1<j<n_2, n_3<j<n_4,\ldots\end{cases}$$

where ϵ_{-1} and $\epsilon_{\text{len}(w)+1}$ are taken to mean zero.

4. Examples

As an example of this definition consider the case when $\mathbf{c}=(1,2,3)$, so $p=(x-1)(x-2)(x-3)$, and $w=x^2x^*x$. The sources are ϵ_0 and ϵ_2 and the sinks are ϵ_1 and ϵ_4. We have $\delta_{x^2x^*x}(0,1,2,3,4)=(1,2,1,2,3)$ and $\mathfrak{T}_{w,\mathbf{c}}(x):\epsilon_0\mapsto 1.\epsilon_0+\epsilon_1$, $\epsilon_1\mapsto 2.\epsilon_1$, $\epsilon_2\mapsto 1.\epsilon_2+\epsilon_1+\epsilon_3$, $\epsilon_3\mapsto 2.\epsilon_3+\epsilon_4$, $\epsilon_4\mapsto 3.\epsilon_4$.

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5. Theorem

Theorem. *For all words w in A_1/I we have* $\mathfrak{T}_{w,c}(p) = 0$.

Proof. If ϵ_j is a sink then write $p = p'(x - c_{\delta_w(j)})$ where $p' \in P_1$. Then,

$$\mathfrak{T}_{w,c}(p)\epsilon_j = \mathfrak{T}_{w,c}(p')\mathfrak{T}_{w,c}(x - c_{\delta_w(j)})\epsilon_j = 0.$$

If ϵ_j is neither a sink nor a source, and the sink to which it belongs is ϵ_k where $k > j$ then write $p = p'(x - c_{\delta_w(k)}) \ldots (x - c_{\delta_w(j+1)})(x - c_{\delta_w(j)})$ where $p' \in P_1$. Note that δ_w has been defined in such a way that p will not run out of linear factors when writing it in this way. Then,

$$\begin{aligned}\mathfrak{T}_{w,c}(p)\epsilon_j &= \mathfrak{T}_{w,c}(p'(x - c_{\delta_w(k)}) \ldots (x - c_{\delta_w(j+1)}))\epsilon_{j+1} \\ &= \ldots = \mathfrak{T}_{w,c}(p'(x - c_{\delta_w(k)}))\epsilon_k = 0.\end{aligned}$$

Similarly, if ϵ_j is neither a sink nor a source, and the sink to which it belongs is ϵ_k with $k < j$ then $\mathfrak{T}_{w,c}(p)\epsilon_j = 0$. Finally, if ϵ_j is a source then write $p = p'(x - c_{\delta_w(j)})$ where $p' \in P_1$. Then,

$$\mathfrak{T}_{w,c}(p)\epsilon_j = \mathfrak{T}_{w,c}(p')\epsilon_{j-1} + \mathfrak{T}_{w,c}(p')\epsilon_{j+1}.$$

By the way we have defined δ_w, the polynomial p' will still have sufficient linear factors to be able to continue separately as in the two previous cases to get $\mathfrak{T}_{w,c}(p)\epsilon_j = 0$ as required.

6. Examples

To illustrate the previous result let $w = x^2 x^{*2} x^3$ and $\mathbf{c} = (c_1, c_2, c_3, c_4)$. Consider $\mathfrak{T}_{w,c}(p)\epsilon_1$. Write $p = (x - c_1).(x - c_4)(x - c_3)(x - c_2)$. As

$$\mathfrak{T}_{w,c}(x - c_2)\epsilon_1 = (c_2\epsilon_1 + \epsilon_2) - c_2\epsilon_1 = \epsilon_2$$

we have

$$\mathfrak{T}_{w,c}(p)\epsilon_1 = \mathfrak{T}_{w,c}((x - c_1).(x - c_4)(x - c_3))\epsilon_2$$

and continuing in a similar fashion we see that

$$\mathfrak{T}_{w,c}(p)\epsilon_1 = \mathfrak{T}_{w,c}((x - c_1)(x - c_4))\epsilon_3 = \mathfrak{T}_{w,c}(x - c_1)0 = 0.$$

As another example, consider

$$\begin{aligned}\mathfrak{T}_{w,c}(p)\epsilon_5 &= \mathfrak{T}_{w,c}(p'(x - c_2)(x - c_1))\epsilon_5 \\ &= \mathfrak{T}_{w,c}(p')(\epsilon_3 + \epsilon_7) = \mathfrak{T}_{w,c}(p''(x - c_4))\epsilon_3 + \mathfrak{T}_{w,c}(p'''(x - c_3))\epsilon_7 = 0.\end{aligned}$$

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7. Corollary

Corollary. *If p is a polynomial in P_1 then $\langle p\rangle_*$ is a C^*-ideal.*

Proof. Take $p=(x-c_1)\dots(x-c_n)$ along with all the other previous definitions. Firstly, $\mathfrak{T}_{w,c}$ is well-defined on A_1/I as $\mathfrak{T}_{w,c}(p)=0$ by Theorem 5, (where w is a word in A_1/I). Note that $\|\mathfrak{T}_{w,c}(x)\|\le\max\{|c_j|\}+2$ as $\mathfrak{T}_{w,c}(x)=D+P+Q$ where D is a diagonal operator and P and Q are partial isometries. Therefore, for all $y\in A_1/I$, let $v(y)=\sup\{\|\mathfrak{T}_{w,c}(y)\|: w \text{ a word in } A_1/I\}$ which is a C^*-seminorm on A_1/I. We are seeking to show that v is a C^*-norm on A_1/I. If this is so then we may let B be the C^*-algebra which is the completion of A_1/I with respect to v. Then A_1/I is embedded in B, and we have finished. If v is not a C^*-norm then there exists some non-zero y in A_1/I such that $\mathfrak{T}_{w,c}(y)=0$ for all w in A_1/I.

Let $m=\max\{\text{len}(v): y_v\neq 0\}$ and let w be a word of length m with $y_w\neq 0$. Let $\epsilon=\epsilon_0$ and $\epsilon'=\epsilon_m$. Given α in H_w, let $d(\alpha)=\max\{j:\langle\epsilon_j,\alpha\rangle\neq 0\}$. Informally, this represents the distance along the basis that α contains information. Write $\mathfrak{T}_{w,c}(x)=t$. Considering the action of t on α in H_w we see that both t and t^* can only move information along to the right by at most one basis vector or, more formally, $d(t\alpha)\le d(\alpha)+1$ and $d(t^*\alpha)\le d(\alpha)+1$. If u is a word then $d(u(t)\epsilon)\le\text{len}(u)$ with equality only being attained if each letter of the word u increases d. The $*$-representation $\mathfrak{T}_{w,c}$ is defined in such a way that $d(\mathfrak{T}_{w,c}(w)\epsilon)=m$. Let v be a word other than w. If $y_v=0$ then clearly $\langle y_v\mathfrak{T}_{w,c}(v)\epsilon,\epsilon'\rangle=0$. If $y_v\neq 0$ then either $\text{len}(v)<m$, in which case $d(\mathfrak{T}_{w,c}(v)\epsilon)<m$, or $\text{len}(v)=m$. It is not hard to see that if $\text{len}(v)=m$ and $v\neq w$ then we again have $d(\mathfrak{T}_{w,c}(v)\epsilon)<m$ (informally, in this case the operator turns back, or stops, at some point along the basis). Thus, $\langle\mathfrak{T}_{w,c}(y)\epsilon,\epsilon'\rangle=y_w\langle\mathfrak{T}_{w,c}(w)\epsilon,\epsilon'\rangle\neq 0$, and $\mathfrak{T}_{w,c}(y)\neq 0$ as required.

Note that, for the particular case where $p(x)=x^n$ and $n\in\{1,2,3,\dots\}$, we could replace the operator $\mathfrak{T}_{w,c}$ with $\lambda.\mathfrak{T}_{w,c}$, where λ is any positive number. This would give us the stronger result that, in this case, $A_1/\langle p\rangle_*$ can be embedded into a C^*-algebra so that $\|x\|=M$ for any positive real M.

If $y\in A_1$ and m is the maximum length of a word with non-zero coefficient in y, then taking $p(x)=x^{m+1}$, we get a $*$-representation π of A_1 such that $\pi(y)\neq 0$. This implies the result of Goodearl and Menal referred to in Examples 2.

8. Corollary

Corollary. *If $p_1,\dots,p_r$ are in P_1 then $\langle p_1,\dots,p_r\rangle_*$ is a C^*-ideal.*

Proof. By elementary algebra we know that there exists some polynomial q such that $\langle p_1,\dots,p_r\rangle_*=\langle q\rangle_*$. By Corollary 7 this is a C^*-ideal.

Thus, if I is any $*$-ideal in A_1 which is generated by polynomials then I is a C^*-ideal of A_1 and A_1/I may be embedded into a C^*-algebra.

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