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NORMAL $\mathfrak{S}_1$-FITTING CLASSES

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The authors together with M. J. Karbe [*Ill. J. Math.* **33** (1989) 333–359] have considered Fitting classes $\mathfrak{X}$ of $\mathfrak{S}_1$-groups and, under some rather strong restrictions, obtained an existence and conjugacy theorem for $\mathfrak{X}$-injectors. Results of Menegazzo and Newell show that these restrictions are, in fact, necessary.

The Fitting class $\mathfrak{X}$ is *normal* if, for each $G \in \mathfrak{S}_1$, $G_{\mathfrak{X}}$ is the unique $\mathfrak{X}$-injector of G. $\mathfrak{X}$ is *abelian normal* if, for each $G \in \mathfrak{S}_1$, $G_{\mathfrak{X}} \geqq G'$. For finite soluble groups these two concepts coincide but the class of Černikov-by-nilpotent $\mathfrak{S}_1$-groups is an example of a nonabelian normal Fitting class of $\mathfrak{S}_1$-groups. In all known examples in which $\mathfrak{X}$-injectors exist $\mathfrak{X}$ is closely associated with some normal Fitting class (the Černikov-by-nilpotent groups arise from studying the locally nilpotent injectors).

Here we investigate normal Fitting classes further, paying particular attention to the distinctions between abelian and nonabelian normal Fitting classes. Products and intersections with (abelian) normal Fitting classes lead to further examples of Fitting classes satisfying the conditions of the existence and conjugacy theorem.

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1. Introduction

Throughout we shall work in a subclass $\mathfrak{R}$ of $\mathfrak{S}_1$ which is closed under subgroups, finite direct products and finite soluble extensions. Some of our results can be proved in more generality but the above conditions are satisfied by the more important subclasses of $\mathfrak{S}_1$; in particular $\mathfrak{R}$ can be taken to be the class $\mathfrak{C}$ of Černikov groups, the class $\mathfrak{P}$ of polycyclic groups, the class $\mathfrak{M}$ of soluble minimax groups, the class $\mathfrak{S}_1$ itself, or the class of abelian-by-finite groups contained in any of these subclasses.

We introduced the notion of a *Fitting class* of $\mathfrak{R}$-groups, or $\mathfrak{R}$-Fitting class, in [2] and, under some rather strong restrictions, obtained an existence and conjugacy theorem for $\mathfrak{X}$-injectors [2, Theorem 4.4]. A $\mathfrak{R}$-Fitting class $\mathfrak{X}$ is said to be a *normal* $\mathfrak{R}$-Fitting class if the radical $G_{\mathfrak{X}}$ is a maximal $\mathfrak{X}$-subgroup of G, for each $G \in \mathfrak{R}$, or equivalently, $G_{\mathfrak{X}}$ is the unique $\mathfrak{X}$-injector of G, for each $G \in \mathfrak{R}$. The $\mathfrak{R}$-Fitting class $\mathfrak{X}$ is said to be an *abelian normal* $\mathfrak{R}$-Fitting class if $G_{\mathfrak{X}} \geqq G'$, for each $G \in \mathfrak{R}$.

It is clear from the definitions that an abelian normal $\mathfrak{R}$-Fitting class is a normal $\mathfrak{R}$-Fitting class and a well known result of Blessenohl and Gaschütz [5] asserts that every normal Fitting class of finite soluble groups is abelian. However, in [4], we saw that the class $\mathfrak{C}\mathfrak{N}$ of Černikov-by-nulpotent $\mathfrak{S}_1$-groups is a nonabelian normal $\mathfrak{S}_1$-Fitting class. It was also seen that this class plays an important role in the existence and conjugacy of the hypercentral injectors. If $\mathfrak{H}$ denotes the class of hypercentral $\mathfrak{S}_1$-groups and $G_{\mathfrak{H}}$ is the hypercentral radical of the $\mathfrak{S}_1$-group G, then $G_{\mathfrak{C}\mathfrak{N}}/G_{\mathfrak{H}}$ is finite and the hypercentral injectors of G are precisely the hypercentral injectors of $G_{\mathfrak{C}\mathfrak{N}}$.

This last result suggests that the normal Fitting classes play an important role in the theory of Fitting classes of $\mathfrak{S}_1$-groups and our aim here is to investigate these further, paying particular attention to the distinctions between abelian and nonabelian normal Fitting classes and also to normal Fitting classes related to Fitting classes for which injectors exist.

In Section 3 we consider intersections of (abelian) normal Fitting classes with each other and with other Fitting classes. This enables us to construct further examples of Fitting classes $\mathfrak{X}$ for which injectors always exist. We observe that every $\mathfrak{K}$-Fitting class $\mathfrak{X}$ constructed in this way has the following property:

(A) *there is a normal Fitting class $\mathfrak{Y}$ such that, for each $G \in \mathfrak{K}$, $G_{\mathfrak{Y}}/G_{\mathfrak{X}}$ is finite and each $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$ is contained in $G_{\mathfrak{Y}}$.*

The results of [4] show that the $\mathfrak{S}_1$-Fitting class $\mathfrak{H}$ also has the property (A), $\mathfrak{C}\mathfrak{N}$ being the required normal $\mathfrak{S}_1$-Fitting class. We have not been able to show that every Fitting class for which injectors always exist must necessarily have property (A) and we leave this as:

Question 1. If $\mathfrak{X}$ is a $\mathfrak{K}$-Fitting class such that every $\mathfrak{K}$-group has $\mathfrak{X}$-injectors, does $\mathfrak{X}$ have property (A)?

This question is related to the result of Menegazzo and Newell [7, Theorem A] which says that if $\mathfrak{X}$ is a $\mathfrak{K}$-Fitting class such that every $\mathfrak{K}$-group has $\mathfrak{X}$-injectors then each $\mathfrak{K}$-group G contains a normal subgroup M such that $M/G_{\mathfrak{X}}$ is finite and every $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$ is contained in M. It follows from Theorem B of [7] that the $\mathfrak{X}$-injectors of G are necessarily conjugate.

One of the useful tools in considering normal Fitting classes of finite soluble groups is the Lockett $*$-construction. This is also the case here, particularly for abelian normal Fitting classes, and to some extent it helps to illustrate some of the basic differences between abelian and nonabelian normal Fitting classes. We begin by discussing this construction and refer to [3] for further details of the construction.

2. Lockett's $*$-construction

Recall that if $\mathfrak{X}$ is a $\mathfrak{K}$-Fitting class than $\mathfrak{X}^* = \{G \in \mathfrak{K}\colon (G \times G)_{\mathfrak{X}}$ is subdirect in $G \times G\}$ is also a $\mathfrak{K}$-Fitting class and $G_{\mathfrak{X}^*}/G_{\mathfrak{X}}$ is central in G.

Theorem 2.1. *Let $\mathfrak{X}$ be a $\mathfrak{K}$-Fitting class. Then*

(i) *$\mathfrak{X}$ is normal if and only if $\mathfrak{X}^*$ is normal,*

(ii) *$\mathfrak{X}$ is abelian normal if and only if $\mathfrak{X}^* = \mathfrak{K}$.*

Proof. (i) Let $\mathfrak{X}$ be a normal $\mathfrak{R}$-Fitting class, let $G \in \mathfrak{R}$ and consider $G \times G$. Since $\mathfrak{X}$ is normal, $(G \times G)_{\mathfrak{X}}$ is a maximal $\mathfrak{X}$-subgroup of $G \times G$. But, by [3, Theorem 2.3(b)], $(G \times G)_{\mathfrak{X}} = (G_{\mathfrak{X}} \times G_{\mathfrak{X}})\langle (g^{-1}, g) : g \in G_{\mathfrak{X}^*} \rangle$. Let π_1: $G \times G \to G$ be the projection map onto the first component; then $G_{\mathfrak{X}^*} = \pi_1((G \times G)_{\mathfrak{X}})$. Let Y be an $\mathfrak{X}^*$-subgroup of G containing $G_{\mathfrak{X}^*}$. Then $(G \times G)_{\mathfrak{X}} \leqq G_{\mathfrak{X}^*} \times G_{\mathfrak{X}^*} \leqq Y \times Y \in \mathfrak{X}^*$. Since $(G \times G)_{\mathfrak{X}}$ is a maximal $\mathfrak{X}$-subgroup of $G \times G$, we have

$$(G \times G)_{\mathfrak{X}} = (Y \times Y)_{\mathfrak{X}} = (Y_{\mathfrak{X}} \times Y_{\mathfrak{X}})\langle (y^{-1}, y) : y \in Y \rangle.$$

Thus $(y, y^{-1}) \in (G \times G)_{\mathfrak{X}}$ and so $y \in \pi_1((G \times G)_{\mathfrak{X}}) = G_{\mathfrak{X}^*}$. Therefore $Y = G_{\mathfrak{X}^*}$ and so $G_{\mathfrak{X}^*}$ is a maximal $\mathfrak{X}^*$-subgroup of G. Hence $\mathfrak{X}^*$ is normal.

Conversely, suppose that $\mathfrak{X}^*$ is a normal $\mathfrak{R}$-Fitting class. Let $G \in \mathfrak{R}$ and let X be an $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$. Since $G_{\mathfrak{X}^*}/G_{\mathfrak{X}}$ is a central section of G, we have $X \lhd G_{\mathfrak{X}^*}X$. Thus $G_{\mathfrak{X}^*}X$ is a product of normal $\mathfrak{X}^*$-subgroups and so $G_{\mathfrak{X}^*}X \in \mathfrak{X}^*$. But $G_{\mathfrak{X}^*}$ is a maximal $\mathfrak{X}^*$-subgroup of G and so $X \leqq G_{\mathfrak{X}^*}$. But since $G_{\mathfrak{X}^*}/G_{\mathfrak{X}}$ is central we then have $X \lhd G$ and so $X = G_{\mathfrak{X}}$. Thus $G_{\mathfrak{X}}$ is maximal $\mathfrak{X}$-subgroup of G and so $\mathfrak{X}$ is a normal $\mathfrak{R}$-Fitting class.

(ii) If $\mathfrak{X}^* = \mathfrak{R}$ then, for any $G \in \mathfrak{R}$, $G/G_{\mathfrak{X}} = G_{\mathfrak{X}^*}/G_{\mathfrak{X}}$ is abelian and so $\mathfrak{X}$ is an abelian normal $\mathfrak{R}$-Fitting class.

Conversely, suppose that $\mathfrak{X}$ is an abelian normal $\mathfrak{R}$-Fitting class and let $G \in \mathfrak{R}$. Let $\langle t \rangle$ be a cyclic group of order two and form the wreath product $W = G \wr \langle t \rangle$. Since $\mathfrak{R}$ is closed under finite direct products and finite extensions, we have $W \in \mathfrak{R}$. The base group of W is isomorphic to $G \times G$ and, for each $g \in G$, $(g^{-1}, g) = [(g, 1), t] \in W' \leqq W_{\mathfrak{X}}$. Therefore $(g^{-1}, g) \in W_{\mathfrak{X}} \cap (G \times G) = (G \times G)_{\mathfrak{X}}$ and so $(G \times G)_{\mathfrak{X}}$ is subdirect in $G \times G$. Therefore $G \in \mathfrak{X}^*$ and so $\mathfrak{X}^* = \mathfrak{R}$.

It was shown in [3, Lemma 2.8] that if $\mathfrak{X}$ is an s-closed $\mathfrak{R}$-Fitting class, then $\mathfrak{X}^* = \mathfrak{X}$. The normal $\mathfrak{S}_1$-Fitting class $\mathfrak{E}\mathfrak{N}$ discussed in [4] is clearly s-closed and so $(\mathfrak{E}\mathfrak{N})^* = \mathfrak{E}\mathfrak{N} \neq \mathfrak{S}_1$. Further examples of normal Fitting classes which are not abelian are provided by the members of Locksec $(\mathfrak{E}\mathfrak{N})$, the Lockett section of $\mathfrak{E}\mathfrak{N}$ (see [3]). In particular, the results of the following section show that if $\mathfrak{X}$ is an abelian normal $\mathfrak{S}_1$-Fitting class then $\mathfrak{X} \cap \mathfrak{E}\mathfrak{N}$ is a nonabelian normal $\mathfrak{S}_1$-Fitting class.

3. Intersections and products of normal Fitting classes

Before discussing the main results of this section it is convenient to note that any normal $\mathfrak{R}$-Fitting class contains all hypercentral $\mathfrak{R}$-groups.

Lemma 3.1. *Let $\mathfrak{X}$ be a normal $\mathfrak{R}$-Fitting class and let $\mathfrak{H}$ denote the class of hypercentral groups. Then $\mathfrak{H} \cap \mathfrak{R} \subseteq \mathfrak{X}$.*

Proof. If $\mathfrak{X}$ contains the infinite cyclic group, then $\mathfrak{X}$ contains all cyclic groups and so contains $\mathfrak{H} \cap \mathfrak{R}$ [2, Theorem 2.4].

We may therefore assume that $\mathfrak{X}$ does not contain the infinite cyclic group and so $\mathfrak{X}$

consists entirely of Černikov groups [2, Corollary 2.3]. If $\mathfrak{K}$ contains the infinite cyclic group then it contains all finitely generated abelian-by-finite groups and so the example constructed in Theorem 5.1 of [2] shows that there is a $\mathfrak{K}$-group G which does not have $\mathfrak{X}$-injectors. This is contrary to $\mathfrak{X}$ being a normal $\mathfrak{K}$-Fitting class and so we may assume that $\mathfrak{K}$ consists entirely of Černikov groups. But the class of finite $\mathfrak{X}$-groups is a normal Fitting class of finite soluble groups and so contains all finite nilpotent groups [5, Satz 5.1]. But every hypercentral Černikov group and hence every $\mathfrak{H} \cap \mathfrak{K}$-group is generated by ascendant finite nilpotent groups and so is an $\mathfrak{X}$-group.

The following theorem provides a good illustration of some of the differences in working with abelian and nonabelian normal Fitting classes. The first part of the theorem is almost trivial because we can use the fact that $G_{\mathfrak{X}} \geqq G'$. There seems to be no comparable characterization for nonabelian normal Fitting classes so that a direct proof has to be given.

Theorem 3.2. (i) *If $\mathfrak{X}$ and $\mathfrak{Y}$ are abelian normal $\mathfrak{K}$-Fitting classes, then $\mathfrak{X} \cap \mathfrak{Y}$ is an abelian normal $\mathfrak{K}$-Fitting class.*

(ii) *If $\mathfrak{X}$ and $\mathfrak{Y}$ are normal $\mathfrak{K}$-Fitting classes then $\mathfrak{X} \cap \mathfrak{Y}$ is a normal $\mathfrak{K}$-Fitting class.*

Proof. (i) If $G \in \mathfrak{K}$ then, by the definition, $G_{\mathfrak{X}} \geqq G'$ and $G_{\mathfrak{Y}} \geqq G'$. Therefore $G_{\mathfrak{X} \cap \mathfrak{Y}} = G_{\mathfrak{X}} \cap G_{\mathfrak{Y}} \geqq G'$ and so $\mathfrak{X} \cap \mathfrak{Y}$ is abelian normal.

(ii) Let $G \in \mathfrak{K}$; we show that $G_{\mathfrak{X} \cap \mathfrak{Y}}$ is a maximal $\mathfrak{X} \cap \mathfrak{Y}$-subgroup of G. Note first that, by Lemma 3.1, $\mathfrak{X}$ and $\mathfrak{Y}$ both contain $\mathfrak{H} \cap \mathfrak{K}$ and so $\mathfrak{X} \cap \mathfrak{Y} \supseteq \mathfrak{H} \cap \mathfrak{K}$. It follows by Theorem 3.1 of [2] that $G/G_{\mathfrak{X} \cap \mathfrak{Y}}$ is abelian-by-finite. Let $F/G_{\mathfrak{X} \cap \mathfrak{Y}}$ be the Fitting subgroup of $G/G_{\mathfrak{X} \cap \mathfrak{Y}}$ so that G/F is finite. We prove by induction on $|G/F|$ that $G_{\mathfrak{X} \cap \mathfrak{Y}} (= G_{\mathfrak{X}} \cap G_{\mathfrak{Y}})$ is a maximal $\mathfrak{X} \cap \mathfrak{Y}$-subgroup of G. This is clearly the case if $F = G$ and so we may suppose that $F \lneqq G$.

Suppose that $H \geqq F$. Then $[F, H_{\mathfrak{X} \cap \mathfrak{Y}}] \leqq F \cap H_{\mathfrak{X} \cap \mathfrak{Y}}$. Since $F \cap H_{\mathfrak{X} \cap \mathfrak{Y}} \lhd H_{\mathfrak{X} \cap \mathfrak{Y}}$, $F \cap H_{\mathfrak{X} \cap \mathfrak{Y}}$ is an $\mathfrak{X} \cap \mathfrak{Y}$-group. Also $F \cap H_{\mathfrak{X} \cap \mathfrak{Y}} \operatorname{sn} F \lhd G$ so that $F \cap H_{\mathfrak{X} \cap \mathfrak{Y}} = G_{\mathfrak{X} \cap \mathfrak{Y}}$. Therefore $[F, H_{\mathfrak{X} \cap \mathfrak{Y}}] \leqq G_{\mathfrak{X} \cap \mathfrak{Y}}$ so that $H_{\mathfrak{X} \cap \mathfrak{Y}} \leqq C_G(F/G_{\mathfrak{X} \cap \mathfrak{Y}}) \leqq F$. It follows that $G_{\mathfrak{X} \cap \mathfrak{Y}} = F \cap H_{\mathfrak{X} \cap \mathfrak{Y}} = H_{\mathfrak{X} \cap \mathfrak{Y}}$. By the induction hypothesis we have:

(*) If $F \leqq H \lneqq G$, then $G_{\mathfrak{X} \cap \mathfrak{Y}}$ is a maximal $\mathfrak{X} \cap \mathfrak{Y}$-subgroup of H.

Now let V be an $\mathfrak{X} \cap \mathfrak{Y}$-subgroup of G containing $G_{\mathfrak{X} \cap \mathfrak{Y}}$ and suppose that $V \gneqq G_{\mathfrak{X} \cap \mathfrak{Y}}$. There is a maximal normal subgroup M/F of G/F with G/M having prime order p. By (*), $G_{\mathfrak{X} \cap \mathfrak{Y}}$ is a maximal $\mathfrak{X} \cap \mathfrak{Y}$-subgroup of M. Also $V \cap M \lhd V$ and so $V \cap M$ is an $\mathfrak{X} \cap \mathfrak{Y}$-subgroup of M containing $G_{\mathfrak{X} \cap \mathfrak{Y}}$. Therefore $V \cap M = G_{\mathfrak{X} \cap \mathfrak{Y}}$ and so $|V/G_{\mathfrak{X} \cap \mathfrak{Y}}| = p$. If $FV \neq G$ then, by (*), $G_{\mathfrak{X} \cap \mathfrak{Y}}$ would be a maximal $\mathfrak{X} \cap \mathfrak{Y}$-subgroup of FV, contrary to $G_{\mathfrak{X} \cap \mathfrak{Y}} \lneqq V \leqq FV$. Therefore we must have $FV = G$ so that $|G/F| = p$.

Now consider $VG_{\mathfrak{X}}$ and let $W = (VG_{\mathfrak{X}})_{\mathfrak{Y}}$. Since $W \lhd VG_{\mathfrak{X}}$ and $G_{\mathfrak{X}} \leqq VG_{\mathfrak{X}}$, we have $W \cap G_{\mathfrak{X}}$ is a normal $\mathfrak{X} \cap \mathfrak{Y}$-subgroup of $G_{\mathfrak{X}}$. Thus $W \cap G_{\mathfrak{X}}$ is a subnormal $\mathfrak{X} \cap \mathfrak{Y}$-subgroup of G and it clearly contains $G_{\mathfrak{X} \cap \mathfrak{Y}}$; therefore $W \cap G_{\mathfrak{X}} = G_{\mathfrak{X} \cap \mathfrak{Y}}$. But $\mathfrak{Y}$ is a normal Fitting class and so W is a maximal $\mathfrak{Y}$-subgroup of $VG_{\mathfrak{X}}$. Since $G_{\mathfrak{X} \cap \mathfrak{Y}} \lneqq V \leqq VG_{\mathfrak{X}}$, we must have $G_{\mathfrak{X}\cap\mathfrak{Y}} \lneqq W$. Since $|VG_{\mathfrak{X}}/G_{\mathfrak{X}}| = p$ and $W \cap G_{\mathfrak{X}} = G_{\mathfrak{X}\cap\mathfrak{Y}}$ it now follows that $|W/G_{\mathfrak{X}\cap\mathfrak{Y}}| = p$. If $W = V$, then $VG_{\mathfrak{X}}$ would be a product of two normal $\mathfrak{X}$-subgroups V and $G_{\mathfrak{X}}$ and so $VG_{\mathfrak{X}} \in \mathfrak{X}$ contrary to $G_{\mathfrak{X}}$ being a maximal $\mathfrak{X}$-subgroup of G. Therefore $W \neq V$ and so $WV/G_{\mathfrak{X}\cap\mathfrak{Y}}$ is a group of order p^2 and so is abelian. Therefore WV is the product of two normal $\mathfrak{Y}$-subgroups and so $WV \in \mathfrak{Y}$.

Hence as W is a maximal $\mathfrak{Y}$-subgroup of $VG_{\mathfrak{X}}$ it follows that $V \leqq W$. Then

$$p^2 = |WV : G_{\mathfrak{X}\cap\mathfrak{Y}}| = |W : G_{\mathfrak{X}\cap\mathfrak{Y}}| = p,$$

a contradiction. Hence $V = G_{\mathfrak{X}\cap\mathfrak{Y}}$.

A second construction which enables us to give new examples of (abelian) normal Fitting classes is provided by considering products of Fitting classes. This is discussed in [**2**, Theorems 3.5 and 5.10] where the additional assumption is made that either (a) $\mathfrak{K}$ is Q-closed or (b) $\mathfrak{K}$ contains the class $\mathfrak{P}$ of polycyclic groups. If $\mathfrak{X}$ is a $\mathfrak{K}$-Fitting class containing $\mathfrak{H} \cap \mathfrak{K}$ and $\mathfrak{Y}$ is a $\mathfrak{K}$-Fitting class, then

$$\mathfrak{X}\mathfrak{Y} = \{G \in \mathfrak{K} : G/G_{\mathfrak{X}} \in \mathfrak{Y}\}$$

is a $\mathfrak{K}$-Fitting class. The restriction on $\mathfrak{K}$ is to ensure that the factor group $G/G_{\mathfrak{X}}$ is a $\mathfrak{K}$-group.

Theorem 3.3. *Suppose that either* (a) *$\mathfrak{K}$ is Q-closed or* (b) *$\mathfrak{K} \supseteq \mathfrak{P}$. Let $\mathfrak{X}$ be a $\mathfrak{K}$-Fitting class containing $\mathfrak{H} \cap \mathfrak{K}$. If $\mathfrak{Y}$ is an* (*abelian*) *normal $\mathfrak{K}$-Fitting class, then $\mathfrak{X}\mathfrak{Y}$ is an* (*abelian*) *normal $\mathfrak{K}$-Fitting class.*

Proof. (i) Let $G \in \mathfrak{K}$ and suppose that $\mathfrak{Y}$ is an abelian normal $\mathfrak{K}$-Fitting class. Then $(G/G_{\mathfrak{X}})_{\mathfrak{Y}} \geqq (G/G_{\mathfrak{X}})'$ and so $G/G_{\mathfrak{X}\mathfrak{Y}}$ is abelian.

(ii) Let $G \in \mathfrak{K}$ and let $\mathfrak{Y}$ be a normal $\mathfrak{K}$-Fitting class. Then $G_{\mathfrak{X}\mathfrak{Y}}/G_{\mathfrak{X}}$ is the unique $\mathfrak{Y}$-injector of $G/G_{\mathfrak{X}}$. By [**2**, Theorem 5.10], $G_{\mathfrak{X}\mathfrak{Y}}$ is an $\mathfrak{X}\mathfrak{Y}$-injector of G and so $\mathfrak{X}\mathfrak{Y}$ is a normal $\mathfrak{K}$-Fitting class.

Theorem 3.4. *Suppose that either* (a) *$\mathfrak{K}$ is Q-closed or* (b) *$\mathfrak{K} \supseteq \mathfrak{P}$. Let $\mathfrak{X}$ be an abelian normal $\mathfrak{K}$-Fitting class and $\mathfrak{Y}$ any $\mathfrak{K}$-Fitting class. Then $\mathfrak{X}\mathfrak{Y}$ is an abelian normal $\mathfrak{K}$-Fitting class.*

Proof. If $G \in \mathfrak{K}$, then $G_{\mathfrak{X}\mathfrak{Y}} \geqq G_{\mathfrak{X}} \geqq G'$.

However, this last result does not hold for nonabelian normal Fitting classes. For, consider the $\mathfrak{S}_1$-Fitting classes $\mathfrak{X} = \mathfrak{E}\mathfrak{N}$ and $\mathfrak{Y} = \mathfrak{H}$; then $\mathfrak{X}\mathfrak{Y} = \mathfrak{E}\mathfrak{N}\mathfrak{H}$. Let T be the symmetric group of degree 4 and note that the $\mathfrak{H}$-injectors of T are its Sylow 2-subgroups. Let $G = C_\infty \wr T$, taking T in its usual permutation representation. Then $G_{\mathfrak{E}\mathfrak{N}} = B$, the base group. If V is a Sylow 2-subgroup of T, then $BV \in \mathrm{Inj}_{\mathfrak{X}\mathfrak{Y}}(G)$. Clearly BV is not normal in G and so $\mathfrak{X}\mathfrak{Y}$ is not a normal Fitting class.

It is perhaps worth noting from Theorem 3.3 that $\mathfrak{H}\mathfrak{E}\mathfrak{N}$ is a normal $\mathfrak{S}_1$-Fitting class.

One point of interest in this example is that for every $\mathfrak{S}_1$-group G we have $G/G_{\mathfrak{H}\mathfrak{C}\mathfrak{N}}$ finite but this factor group may still be nonabelian. For example, let $H = C_\infty \wr S_3$, where S_3 is taken in its usual permutation representation, and let $A = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle$ be the base group of H. The free abelian group of rank 6 can be made into a $\mathbb{Z}A$-module $M = M_1 \oplus M_2 \oplus M_3$ by letting a_i act on $M_i = \mathbb{Z} \oplus \mathbb{Z}$ according to the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and a_i act trivially on M_j if $j \neq i$. Now consider the induced representation of H on $M_0 = M \otimes_{\mathbb{Z}A} \mathbb{Z}H \cong \mathbb{Z}^{36}$ and form the split extension G of M_0 by H. Then $G_{\mathfrak{H}} = M_0$ and $G_{\mathfrak{H}\mathfrak{C}\mathfrak{N}} = M_0 A$ so that $G/G_{\mathfrak{H}\mathfrak{C}\mathfrak{N}} \cong S_3$.

Having introduced products we can give further necessary and sufficient conditions for a $\mathfrak{K}$-Fitting class to be normal or abelian normal. These conditions are clearly related to Theorem 2.1 but do provide a further illustration of the distinction between abelian and nonabelian normal Fitting class.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are $\mathfrak{K}$-Fitting classes then $\mathfrak{X}$ is *strictly normal* in $\mathfrak{Y}$ (written $\mathfrak{X} \lhd < \mathfrak{Y}$) if each $G \in \mathfrak{K}$ possess both $\mathfrak{X}$-injectors and $\mathfrak{Y}$-injectors and each $\mathfrak{X}$-injector of G is a normal subgroup of some $\mathfrak{Y}$-injector of G. Strict normality in finite soluble groups was introduced in [1].

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be $\mathfrak{K}$-Fitting classes such that $\mathfrak{X} \lhd < \mathfrak{Y}$. If $G \in \mathfrak{K}$ and V is a $\mathfrak{Y}$-injector of G, then it follows from the conjugacy of injectors [7, Theorem B] that $V_{\mathfrak{X}}$ is an $\mathfrak{X}$-injector of G.

Let $\mathfrak{Y}$ be a $\mathfrak{K}$-Lockett class and assume that every $\mathfrak{K}$-group has a unique conjugacy class of $\mathfrak{Y}$-injectors. Let $\mathfrak{X} \in \text{Locksec}(\mathfrak{Y})$. It will be shown in 3.7 that every $\mathfrak{K}$-group has a unique conjugacy class of $\mathfrak{X}$-injectors and $\mathfrak{X} \lhd < \mathfrak{Y}$.

Theorem 3.5. *Suppose that either* (a) $\mathfrak{K}$ *is Q-closed or* (b) $\mathfrak{K} \supseteq \mathfrak{P}$. *Let* $\mathfrak{X}$ *be a* $\mathfrak{K}$*-Fitting class. Then*

(i) $\mathfrak{X}$ *is an abelian normal* $\mathfrak{K}$*-Fitting class if and only if* $\mathfrak{X}\mathfrak{H} \cap \mathfrak{K} = \mathfrak{K}$,

(ii) $\mathfrak{X}$ *is a normal* $\mathfrak{K}$*-Fitting class if and only if* $\mathfrak{X} \lhd < \mathfrak{X}\mathfrak{H} \cap \mathfrak{K}$.

Proof. (i) Let $\mathfrak{X}$ be an abelian normal $\mathfrak{K}$-Fitting class and let $G \in \mathfrak{K}$. Then $G_{\mathfrak{X}} \geqq G'$ so that $G/G_{\mathfrak{X}} \in \mathfrak{H}$ and $G \in \mathfrak{X}\mathfrak{H} \cap \mathfrak{K}$.

Conversely, suppose that $\mathfrak{K} = \mathfrak{X}\mathfrak{H} \cap \mathfrak{K}$ and let $G \in \mathfrak{K}$. We show that $G \in \mathfrak{X}^*$ so that $\mathfrak{K} = \mathfrak{X}^*$ and so, by Theorem 2.1, $\mathfrak{X}$ is abelian normal. Suppose that $G \notin \mathfrak{X}^*$ so that $G/G_{\mathfrak{X}^*}$ is a non-trivial hypercentral group. There is a prime p such that $G/G_{\mathfrak{X}^*}$ is not a p-group and so $(G/G_{\mathfrak{X}^*}) \wr C_p$ is not hypercentral. Consider $H = G \wr C_p$ and let B be the base group of H. It is clear that $H_{\mathfrak{X}^*} = B_{\mathfrak{X}^*}$ and that $H/H_{\mathfrak{X}^*} \cong (G/G_{\mathfrak{X}^*}) \wr C_p \notin \mathfrak{H}$. This is contrary to every $\mathfrak{K}$-group being in the class $\mathfrak{X}\mathfrak{H}$ and so we must have $G \in \mathfrak{X}^*$ and $\mathfrak{K} = \mathfrak{X}^*$, as required.

(ii) Let $\mathfrak{X}$ be a normal $\mathfrak{K}$-Fitting class. Then, by Lemma 3.1, $\mathfrak{X} \supseteq \mathfrak{H} \cap \mathfrak{K}$ and so $\mathfrak{X}\mathfrak{H} \cap \mathfrak{K} \supseteq \mathfrak{H}^2 \cap \mathfrak{K}$. Therefore the $\mathfrak{X}\mathfrak{H}$-radical of any $\mathfrak{K}$-group G has finite index in G and so, by [2, Theorem 4.3], G has $\mathfrak{X}\mathfrak{H}$-injectors. Let Y be an $\mathfrak{X}\mathfrak{H}$-injector of G; then $Y \geqq G_{\mathfrak{X}}$

and since $G_{\mathfrak{X}}$ is a maximal $\mathfrak{X}$-subgroup of G, $G_{\mathfrak{X}}$ is the unique $\mathfrak{X}$-injector of Y. Thus $\mathfrak{X} \lhd < \mathfrak{X}\mathfrak{H} \cap \mathfrak{K}$.

Conversely, suppose that $\mathfrak{X} \lhd < \mathfrak{X}\mathfrak{H} \cap \mathfrak{K}$. Let Y be an $\mathfrak{X}\mathfrak{H}$-injector of the $\mathfrak{K}$-group G and let $X = Y_{\mathfrak{X}}$ so that X is an $\mathfrak{X}$-injector of G. It is clear that $Y \geqq G_{\mathfrak{X}\mathfrak{H}}$ and part (I) of the proof of Theorem 5.10 in [2] asserts that $Y_{\mathfrak{X}} = G_{\mathfrak{X}}$. Therefore $G_{\mathfrak{X}}$ is an $\mathfrak{X}$-injector of G and so $\mathfrak{X}$ is a normal $\mathfrak{K}$-Fitting class.

Again we note how this result applies to the nonabelian $\mathfrak{S}_1$-Fitting class $\mathfrak{E}\mathfrak{N}$. If $\mathfrak{X} = \mathfrak{E}\mathfrak{N}$, then $\mathfrak{X}\mathfrak{H} = \mathfrak{E}\mathfrak{H}^2 \neq \mathfrak{S}_1$. The example given earlier of an $\mathfrak{S}_1$-group G in which $G/G_{\mathfrak{H}\mathfrak{E}\mathfrak{N}} \cong S_3$ is an example of an $\mathfrak{S}_1$-group not in the class $\mathfrak{E}\mathfrak{H}^2$.

One of our objectives in studying normal and abelian normal Fitting classes is to construct examples of Fitting classes $\mathfrak{X}$ which always yield $\mathfrak{X}$-injectors. A normal Fitting class automatically provides a unique $\mathfrak{X}$-injector $G_{\mathfrak{X}}$ but we turn our attention now to combining abelian normal Fitting classes with other known Fitting classes.

The main result which we give here (Theorem 3.9) follows easily from results given in [3] on the relationship between $\mathfrak{X}$-injectors and $\mathfrak{X}^*$-injectors. The results we require were proved under the assumption that $\mathfrak{K} \supseteq \mathfrak{P}$. The need for this assumption can be traced back to our proof of Corollary 6.2 in [2]. However the result of Menegazzo and Newell [7] referred to in the introduction allows us to give a direct proof of this without further restrictions on $\mathfrak{K}$.

Theorem 3.6. *Let $\mathfrak{X}$ be a $\mathfrak{K}$-Fitting class such that every $\mathfrak{K}$-group has $\mathfrak{X}$-injectors. Let V be an $\mathfrak{X}$-injector of G where $G \in \mathfrak{K}$ and let $V \leqq H \leqq G$. Then $V \in \mathrm{Inj}_{\mathfrak{X}}(H)$.*

Proof. By Theorem A of [7] and Theorems 2.4 and 3.1 of [2], there is a normal subgroup M of G such that $M/G_{\mathfrak{X}}$ is finite and every $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$ is contained in M. Since $G_{\mathfrak{X}} \leqq V \leqq H$, we have $G_{\mathfrak{X}} \leqq H_{\mathfrak{X}}$ and so any $\mathfrak{X}$-subgroup of H containing $H_{\mathfrak{X}}$ is contained in M and hence in $M \cap H$. It is therefore sufficient to show that $V \in \mathrm{Inj}_{\mathfrak{X}}(M \cap H)$.

It is known that $V \in \mathrm{Inj}_{\mathfrak{X}}(M)$ and that there is an $\mathfrak{X}$-injector U of $M \cap H$. We prove by induction on $|M/G_{\mathfrak{X}}|$ that V and U are conjugate in H. Let $N/G_{\mathfrak{X}}$ be a maximal normal subgroup of $M/G_{\mathfrak{X}}$ so that M/N is nilpotent.

Now $V \cap N \in \mathrm{Inj}_{\mathfrak{X}}(N)$, $V \cap N \leqq H \cap N \leqq N$ and $|N/G_{\mathfrak{X}}| < |G/G_{\mathfrak{X}}|$. Therefore, by induction, $V \cap N \in \mathrm{Inj}_{\mathfrak{X}}(N \cap H)$. Also $U \cap N \in \mathrm{Inj}_{\mathfrak{X}}(H \cap N)$ and so $V \cap N$ and $U \cap N$ are conjugate in $H \cap N$ and we may assume that $V \cap N = U \cap N = W$, say. Then W is a maximal $\mathfrak{X}$-subgroup of $H \cap N$, $H \cap M/H \cap N$ is nilpotent and U, V are maximal $\mathfrak{X}$-subgroups of $H \cap M$ containing W. By Hartley's Lemma [2, Lemma 4.1], U and V are conjugate in $H \cap M$ and, in particular, $V \in \mathrm{Inj}_{\mathfrak{X}}(H \cap M)$.

We state the results required from [3] as corollaries; the proofs are essentially the same as the proofs of Theorems 4.5 and 4.6. in [3].

Corollary 3.7. *Let $\mathfrak{X}$ be a $\mathfrak{K}$-Fitting class such that every $\mathfrak{K}$-group has $\mathfrak{X}^*$-injectors and let $G \in \mathfrak{K}$. Then*

(i) *G has a unique conjugacy class of $\mathfrak{X}$-injectors,*

(ii) *if* $V \in \mathrm{Inj}_{\mathfrak{X}^*}(G)$, *then* $V_{\mathfrak{X}} \in \mathrm{Inj}_{\mathfrak{X}}(G)$.

Corollary 3.8. *Let $\mathfrak{X}$ be a $\mathfrak{R}$-Fitting class such that every $\mathfrak{R}$-group has $\mathfrak{X}$-injectors and let $G \in \mathfrak{R}$. Then G has a unique conjugacy class of $\mathfrak{X}^*$-injectors.*

Theorem 3.9. *Let $\mathfrak{Z}$ be an abelian normal $\mathfrak{R}$-Fitting class and let $\mathfrak{X}$ be a $\mathfrak{R}$-Fitting class such that every $\mathfrak{R}$-group has $\mathfrak{X}$-injectors. Then every $\mathfrak{R}$-group has $\mathfrak{X} \cap \mathfrak{Z}$-injectors.*

Proof. By Corollary 3.8, every $\mathfrak{R}$-group has $\mathfrak{X}^*$-injectors. But, by [3, Lemma 2.4] and Theorem 2.1, $(\mathfrak{X} \cap \mathfrak{Z})^* = \mathfrak{X}^* \cap \mathfrak{Z}^* = \mathfrak{X}^* \cap \mathfrak{R} = \mathfrak{X}^*$ and so every $\mathfrak{R}$-group has $(\mathfrak{X} \cap \mathfrak{Z})^*$-injectors. By Corollary 3.7, every $\mathfrak{R}$-group has $\mathfrak{X} \cap \mathfrak{Z}$-injectors.

This result means that we can now construct further examples of $\mathfrak{R}$-Fitting classes which always yield injectors. However they can not give examples which do not have property (A) unless the Fitting class $\mathfrak{X}$ which begins the construction already fails to have property (A).

Theorem 3.10. *Let $\mathfrak{Z}$ be an abelian normal $\mathfrak{R}$-Fitting class and let $\mathfrak{X}$ be a $\mathfrak{R}$-Fitting class with property* (A). *Then $\mathfrak{X} \cap \mathfrak{Z}$ also has property* (A).

Proof. Let $G \in \mathfrak{R}$; since $\mathfrak{X}$ satisfies property (A), there is a normal $\mathfrak{R}$-Fitting class $\mathfrak{Y}$ such that $G_{\mathfrak{Y}}/G_{\mathfrak{X}}$ is finite and each $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$ is contained in $G_{\mathfrak{Y}}$. Since $\mathfrak{Z}$ is normal, $\mathfrak{Z} \supseteq \mathfrak{H} \cap \mathfrak{R}$ (Lemma 3.1) and so $G/G_{\mathfrak{Z}}$ is finitely generated. Also since $\mathfrak{Z}$ is abelian normal $G/G_{\mathfrak{Z}}$ is abelian and so has finite torsion subgroup $G_{\mathfrak{Z}\mathfrak{E}}/G_{\mathfrak{Z}}$. Thus $G_{(\mathfrak{Z}\mathfrak{E}) \cap \mathfrak{Y}} = G_{\mathfrak{Z}\mathfrak{E}} \cap G_{\mathfrak{Y}}$ is a finite extension of $G_{\mathfrak{Z}} \cap G_{\mathfrak{X}} = G_{\mathfrak{Z} \cap \mathfrak{X}}$. Let V be an $\mathfrak{X} \cap \mathfrak{Z}$ subgroup of G containing $G_{\mathfrak{X} \cap \mathfrak{Z}}$. Since $G' \leqq G_{\mathfrak{Z}}$ we have $[V, G_{\mathfrak{X}}] \leqq G_{\mathfrak{Z}} \cap G_{\mathfrak{X}} = G_{\mathfrak{X} \cap \mathfrak{Z}}$ and so $VG_{\mathfrak{X}}$ is a product of its two normal $\mathfrak{X}$-subgroups V and $G_{\mathfrak{X}}$. Hence $VG_{\mathfrak{X}} \in \mathfrak{X}$ and so $VG_{\mathfrak{X}} \leqq G_{\mathfrak{Y}}$. Also $V/G_{\mathfrak{X} \cap \mathfrak{Z}}$ is finite and so $VG_{\mathfrak{Z}} \leqq G_{\mathfrak{Z}\mathfrak{E}}$. Therefore $V \leqq G_{\mathfrak{Z}\mathfrak{E}} \cap G_{\mathfrak{Y}} = G_{(\mathfrak{Z}\mathfrak{E}) \cap \mathfrak{Y}}$. The $\mathfrak{R}$-Fitting class $\mathfrak{Z}\mathfrak{E} \cap \mathfrak{Y}$ is normal, by Theorem 3.2, $G_{(\mathfrak{Z}\mathfrak{E}) \cap \mathfrak{Y}}/G_{\mathfrak{Z} \cap \mathfrak{X}}$ is finite and each $\mathfrak{X} \cap \mathfrak{Z}$-subgroup of G containing $G_{\mathfrak{X} \cap \mathfrak{Z}}$ is contained in $G_{(\mathfrak{Z}\mathfrak{E}) \cap \mathfrak{Y}}$.

An interesting question related to Theorem 3.9 which we have been unable to answer is the following:

Question 2. Let $\mathfrak{Z}$ be a normal $\mathfrak{R}$-Fitting class and let $\mathfrak{X}$ be a $\mathfrak{R}$-Fitting class such that every $\mathfrak{R}$-group has $\mathfrak{X}$-injectors. Does every $\mathfrak{R}$-group necessarily have $\mathfrak{X} \cap \mathfrak{Z}$-injectors?

The construction of Theorem 3.9 is, of course, only of interest if we have interesting examples of abelian normal Fitting classes. This is the purpose of the final section.

4. Examples of abelian normal Fitting classes

Our construction follows that of Blessenohl and Gaschütz [5] with one significant change to give examples which have an infinite character.

Let A be an abelian group and, for each $G \in \mathfrak{K}$, let $f_G: G \to A$ be a homomorphism. The pair (A, f) is called an *abelian normal $\mathfrak{K}$-Fitting pair* if

(1) if H asc G and $G \in \mathfrak{K}$, *then* $f_H = f_G|_H$,

(2) $A = \langle g f_G : g \in G, G \in \mathfrak{K} \rangle$.

Lemma 4.1. *Let (A, f) be an abelian normal $\mathfrak{K}$-Fitting pair and let $\mathfrak{X}(A, f) = \{G \in \mathfrak{K} : G f_G = 1\}$. Then $\mathfrak{X}(A, f)$ is an abelian normal $\mathfrak{K}$-Fitting class and, for each $G \in K$, $G_{\mathfrak{X}(A,f)} = \mathrm{Ker}\,(f_G)$.*

Proof. (a) Let $G \in \mathfrak{X}(A, f)$ and H asc G.
By condition (1) above, $H f_H = H f_G \leqq G f_G = 1$ and so $H \in \mathfrak{X}(A, f)$.
(b) Let $G \in \mathfrak{K}$ and let G be generated by ascendant $\mathfrak{X}(A, f)$-subgroups M_i, $i \in I$.

If $g \in G$, then $g = g_1 \ldots g_n$ with $g_j \in M_{i(j)}$ and so

$$\begin{aligned} g f_G &= (g_1 f_G) \ldots (g_n f_G) \\ &= (g_1 f_{M_{i(1)}}) \ldots (g_n f_{M_{i(n)}}), \text{ since } M_{i(j)} \text{ asc } G, \\ &= 1, \text{ since } M_{i(j)} \in \mathfrak{X}(A, f). \end{aligned}$$

Thus $G \in \mathfrak{X}(A, f)$ and parts (a) and (b) show that $\mathfrak{X}(A, f)$ is a $\mathfrak{K}$-Fitting class.

(c) Let $G \in \mathfrak{K}$ and $K = \mathrm{Ker}(f_G) \geqq G'$. Then $K f_K = K f_G = 1$ and so $K \in \mathfrak{X}(A, f)$. Hence the $\mathfrak{X}(A, f)$-radical of G contains K. Conversely, if $G_{\mathfrak{X}}$ is the $\mathfrak{X}(A, f)$-radical of G, then $G_{\mathfrak{X}} f_G = G_{\mathfrak{X}} f_{G_{\mathfrak{X}}} = 1$ and so $G_{\mathfrak{X}} \leqq K$. Thus $\mathrm{Ker}(f_G)$ is the $\mathfrak{X}(A, f)$-radical of G and since $K \geqq G'$, $\mathfrak{X}(A, f)$ is an abelian normal $\mathfrak{K}$-Fitting class.

For finite soluble groups, Lausch [6] showed that every normal Fitting class is obtained in this way. By making the appropriate changes as in the above Lemma, this result can also be generalized as follows:

Lemma 4.2. *Let $\mathfrak{X}$ be an abelian normal $\mathfrak{K}$-Fitting class. Then there is an abelian normal $\mathfrak{K}$-Fitting pair (A, f) such that $\mathfrak{X} = \mathfrak{X}(A, f)$.*

The examples given by Blessenohl and Gashütz can also be used for $\mathfrak{K}$-groups but lead to examples in which $G/G_{\mathfrak{X}(A,f)}$ is always finite. More interesting examples are provided by considering a variation on the construction given in their Satz 3.3.

A $\mathfrak{K}$-group has a normal series

$$G = G_0 \trianglerighteq \cdots \trianglerighteq G_n = 1 \tag{$*$}$$

in which the factors G_{i-1}/G_i are either periodic or torsion-free abelian and rationally irreducible as $\mathbb{Z}G$-modules. Let $M_1, \ldots, M_k$ be the torsion-free rationally irreducible modules in the series (*). Then $M_i \otimes_{\mathbb{Z}} \mathbb{Q}$ is irreducible as a $\mathbb{Q}G$-module and determines a representation ρ_i of G over $\mathbb{Q}$; that is, a homomorphism ρ_i: $G \to GL(n_i, \mathbb{Q})$, where $n_i = \dim_{\mathbb{Q}}(M_i \otimes_Z \mathbb{Q}) = \dim_Z(M_i)$.

For each $g \in G$ and $i = 1, \ldots, k$, let $d_i(g) = \det(\rho_i(g)) \in \mathbb{Q}^*$, the multiplicative group of nonzero rationals, and let $d_G(g) = \prod_{i=1}^k d_i(g) \in \mathbb{Q}^*$. Then

(I) The definition of d_G is independent of the normal series ($*$).

This follows from a form of the Jordan–Hölder Theorem.

(II) If $H \lhd G$, then $d_G|_H = d_H$.

We take the normal series (*) passing through H. Let $M_1, \ldots, M_l$ be the rationally irreducible factors which occur below H; then, by Clifford's Theorem, $M_i \otimes_Z \mathbb{Q}$ is completely reducible as a $\mathbb{Q}H$-module. For $i = 1, \ldots, l$, write $M_i \otimes_Z \mathbb{Q} = V_1 \oplus \cdots \oplus V_{n(i)}$, where the V_j are irreducible $\mathbb{Q}H$-modules and let σ_j be the representation of H corresponding to V_j with $\bar{d}_j(h) = \det(\sigma_j(h))$. Then $d_i(h) = \prod_{j=1}^{n(i)} \bar{d}_j(h)$ and $d_H(h) = \prod_{i=1}^{l} \prod_{j=1}^{n(i)} \bar{d}_j(h)$.

For each $i = l+1, \ldots, k$, we have $d_i(h) = 1$ and so

$$d_G(h) = \prod_{i=1}^{k} d_i(h) = \prod_{i=1}^{l} d_i(h) = \prod_{i=1}^{l} \prod_{j=1}^{n(i)} \bar{d}_j(h) = d_H(h).$$

(III) If H asc G then $d_G|_H = d_H$.

There is an ascending series

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_\beta \lhd \cdots \lhd H_\alpha = G.$$

Write d_β for the homomorphism $d_{H_\beta}: H_\beta \to \mathbb{Q}^*$; then by induction we may assume that $d_\beta|_H = d_H$, for each $\beta < \alpha$.

If $\alpha - 1$ exists, then $H_{\alpha-1} \lhd G$. By (II), $d_G|_{H_{\alpha-1}} = d_{\alpha-1}$ and, by induction, $d_G|_H = d_{\alpha-1}|_H = d_H$.

If α is a limit ordinal then $G = \bigcup_{\beta<\alpha} H_\beta$. Let $M_1, \ldots, M_k$ be the torsion-free rationally irreducible factors in a normal series (*) for G. Each irreducible $\mathbb{Q}G$-module $M_i \otimes_Z \mathbb{Q}$ may be considered as a $\mathbb{Q}H_\beta$-module for any $\beta < \alpha$. Since $M_i \otimes_Z \mathbb{Q}$ has finite $\mathbb{Q}$-dimension there is some $\beta(i)$ such that $M_i \otimes_Z \mathbb{Q}$ is irreducible as a $\mathbb{Q}H_{\beta(i)}$-module. If we let $\beta = \max\{\beta(1), \ldots, \beta(k)\}$, then each $M_i \otimes_Z \mathbb{Q}$ is irreducible as a $\mathbb{Q}H_\beta$-module. For each $i = 1, \ldots, k$, there is a $j = j(i)$ such that $M_i = G_{j-1}/G_j$ and a $\gamma(i)$ such that $H_{\gamma(i)} \cap G_{j-1} \neq H_{\gamma(i)} \cap G_j$. Taking $\gamma = \max\{\beta, \gamma(1), \ldots, \gamma(k)\}$, we have $H_\gamma \cap G_{j-1}/H_\gamma \cap G_j$ is non-trivial for each $j(i)$ and, since $G_{j-1}/G_j \otimes_Z \mathbb{Q}$ is $\mathbb{Q}H_\gamma$-irreducible, we have $(H_\gamma \cap G_{j-1}/H_\gamma \cap G_j) \otimes_Z \mathbb{Q} \cong M_i \times_Z \mathbb{Q}$ as a $\mathbb{Q}H_\gamma$-module. It follows that $d_G|_{H_\gamma} = d_\gamma$. By induction, $d_\gamma|_H = d_H$ and so $d_G|_H = d_H$.

Now let B be any subgroup of $\mathbb{Q}^*$ and let $\pi: \mathbb{Q}^* \to \mathbb{Q}^*/B = A$ be the natural projection map. If we define $f_G = \pi d_G$: $G \to A$ then it follows from the above that (A, f) is an abelian normal $\mathfrak{R}$-Fitting pair.

The group $\mathbb{Q}^*$ is the direct product of the cyclic group $B_0=\{-1,1\}$ of order two and the infinite cyclic groups $B_p=\{p^n: n\in\mathbb{Z}\}$. Interesting examples arise by taking the group B above to be one of these cyclic subgroups and so obtaining an abelian normal $\mathfrak{R}$-Fitting class $\mathfrak{X}(A_0,f)$ or $\mathfrak{X}(A_p,f)$. Variations on these basic examples can also be obtained by letting π be a set of primes and possibly 0 and letting $B=\mathrm{Dr}\{B_p: p\in\pi\}$.

Let $\mathfrak{R}=\mathfrak{S}_1$ and, for a fixed prime q, consider the group G_q defined as follows. Let $X=\langle x_n: n\in\mathbb{Z}, x_n^q=x_{n-1}\rangle$ and let $Y=\langle y\rangle$ be infinite cyclic; form the split extension G_q of X by Y such that $y^{-1}x_ny=x_{n-1}$. The rationally irreducible factors in a normal series of G_q are $M_1=X$ and $M_2=G_q/X$. For any integer m, $y^{-m}x_ny^m=x_{n-m}$ and so $d_G(y^m)=q^m\in B_q$. Therefore $G_q\in\mathfrak{X}(A_q,f)$ but for $p=0$ or $p\neq q$, we have $G_{\mathfrak{X}(A_p,f)}=X$ so that $G/G_{\mathfrak{X}(A_p,f)}$ is infinite cyclic.

We have therefore constructed an abelian normal $\mathfrak{S}_1$-Fitting class $\mathfrak{X}$ and an $\mathfrak{S}_1$-group G with $G/G_{\mathfrak{X}}$ infinite cyclic. Now let $\mathfrak{Y}=\mathfrak{H}^2$ and let H be an $\mathfrak{S}_1$-group such that $H_{\mathfrak{Y}}/H_{\mathfrak{H}}$ is infinite and the $\mathfrak{Y}$-injectors of H are not normal. In the group $K=G\times H$ we have $K/K_{\mathfrak{X}}$ infinite and $K_{\mathfrak{X}\cap\mathfrak{Y}}/K_{\mathfrak{H}}$ infinite. By Theorem 3.9, K has $\mathfrak{X}\cap\mathfrak{Y}$-injectors and the proof of Theorem 3.10 shows that they are the $\mathfrak{X}\cap\mathfrak{Y}$-injectors of $K_{(\mathfrak{X}\mathfrak{C})\cap\mathfrak{Y}}$. Thus we have an example of an $\mathfrak{S}_1$-Fitting class which always yields injectors and a group K in which the radical is distant from both $K_{\mathfrak{H}}$ and K itself. All our previous examples either had finite index in the group or were closely associated with the hypercentral radical.

One failure of the example given above is that it does not give a polycyclic group G with $G/G_{\mathfrak{X}(A,f)}$ infinite. The reason is that in a polycyclic group an element g acts on a torsion-free rationally irreducible M_i like an invertible integer matrix and so $d_i(g)=\det(\rho_i(g))=\pm 1$. Hence $d_G(g)$ can only take the values 1 and -1 and so $G/G_{\mathfrak{X}(A,f)}=G/\mathrm{Ker}\, f_G\cong\mathrm{Im}\, f_G=\mathrm{Im}\,\pi d_G$ is finite. In fact, we have been unable to construct a corresponding example for polycyclic groups and leave this as a final open question.

Question 3. Is there an abelian normal $\mathfrak{P}$-Fitting class $\mathfrak{X}$ and a polycyclic group G such that $G/G_{\mathfrak{X}}$ is infinite?

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