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# STRUCTURAL PROPERTIES OF ELEMENTARY OPERATORS

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**1. Introduction.** Let $\mathscr{A}$ and $\mathscr{B}$ denote complex Banach algebras and let $\mathscr{M}$ be a left Banach $\mathscr{A}$-module and a right Banach $\mathscr{B}$-module. If

$$A = (A_1, \dots, A_n) \in \mathscr{A}^{(n)} \quad \text{and} \quad B = (B_1, \dots, B_n) \in \mathscr{B}^{(n)},$$

we define the bounded linear *elementary operator* $R(A, B)$, acting on $\mathscr{M}$, by

$$R(A, B)(X) = \sum_{i=1}^{n} A_i X B_i.$$

For the case $\mathscr{M} = \mathscr{A} = \mathscr{B}$, elementary operators were introduced by Lumer and Rosenblum [**19**], who studied their spectral properties. In this setting many authors subsequently studied spectral, algebraic, metric, and structural properties of elementary operators, with particular attention devoted to the inner derivations $\delta_a$ ($\delta_a(x) = ax - xa$) [**25**], generalized derivations $\tau(a, b)$ ($\tau(a, b)(x) = ax - xb$) [**9**, **10**], and elementary multiplications $S(a, b)$ ($S(a, b)(x) = axb$), including left and right multiplications $L_a$ and $R_b$ [**11**]. In the case when $\mathscr{A} = \mathscr{L}(\mathscr{H})$, the algebra of all bounded linear operators on a separable complex Hilbert space $\mathscr{H}$, a fairly complete spectral analysis of elementary operators, including the Fredholm theory of such operators, is given in [**11**, **12**, **13**]. This theory also extends to the case when $\mathscr{A} = \mathscr{B} = \mathscr{L}(\mathscr{H})$ and $\mathscr{M}$ is a norm ideal of $\mathscr{L}(\mathscr{H})$ in the sense of [**24**].

These results show that spectral properties of $R(A, B)$ reflect the joint spectral properties of the elements of $A$ or $B$, and in the sequel we illustrate analogous results concerning structural properties of elementary operators. We shall work in the following settings:

1) $\mathscr{M} = \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$, the space of bounded linear operators from $\mathscr{H}_1$ to $\mathscr{H}_2$, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are separable infinite dimensional complex Hilbert spaces. $\mathscr{M}$ is a left module for $\mathscr{A} = \mathscr{L}(\mathscr{H}_2)$ and a right module for $\mathscr{B} = \mathscr{L}(\mathscr{H}_1)$ (under the usual composition of operators).

2) $\mathscr{M} = \widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)} \equiv \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)/\mathscr{K}(\mathscr{H}_1, \mathscr{H}_2)$, where $\mathscr{K}(\mathscr{H}_1, \mathscr{H}_2)$ is the space of compact operators from $\mathscr{H}_1$ to $\mathscr{H}_2$. $\mathscr{M}$ is a left module for

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$$\mathscr{A} = \widetilde{\mathscr{L}(\mathscr{H}_1)} \equiv \mathscr{L}(\mathscr{H}_1)/\mathscr{K}(\mathscr{H}_1)$$

and a right module for

$$\mathscr{B} = \widetilde{\mathscr{L}(\mathscr{H}_2)} \equiv \mathscr{L}(\mathscr{H}_2)/\mathscr{K}(\mathscr{H}_2).$$

To explain the properties of elementary operators that we will investigate we need some definitions. Let $\mathscr{X}$ be a complex Banach space and let $\mathscr{L}(\mathscr{X})$ denote the algebra of bounded operators on $\mathscr{X}$. For every $T \in \mathscr{L}(\mathscr{X})$, $\lambda \in \mathbf{C}$, define

$$\mathscr{X}_T(\lambda) = \{x \in \mathscr{X}: \lim_{k\to\infty} \|(T - \lambda)^k x\|^{1/k} = 0\},$$

$$\mathscr{X}_T^{\text{alg}} = \{x \in \mathscr{X}: \text{There exists a monic polynomial } p = p_x \text{ such that } p(T)x = 0\}.$$

Clearly, $\mathscr{X}_T(\lambda)$ and $\mathscr{X}_T^{\text{alg}}$ are $T$-hyperinvariant linear manifolds in $\mathscr{X}$. We say that $T$ has the *strong spectral splitting property* (s.s.s.p.) if

$$\mathscr{X} = \underset{\lambda\in\mathbf{C}}{\text{c.l.m.}}\ \mathscr{X}_T(\lambda);$$

this property is stronger than the spectral splitting property studied in [**1**]; note that if $\sigma(T)$, the spectrum of $T$, is finite, say $\sigma(T) = \{\lambda_1, \ldots, \lambda_p\}$, then

$$\mathscr{X} = \mathscr{X}_T(\lambda_1) \dotplus \ldots \dotplus \mathscr{X}_T(\lambda_p) \quad [\mathbf{18}, \mathbf{23}],$$

while if $\mathscr{X} = \mathscr{X}_T(\lambda)$, then $\sigma(T) = \{\lambda\}$ [**6**, p. 28]. We say that $T$ is *pseudoalgebraic* if

$$\mathscr{X} = (\mathscr{X}_T^{\text{alg}})^-;$$

recall that $T$ is algebraic, i.e., $T$ satisfies a monic polynomial, if and only if

$$\mathscr{X}_T^{\text{alg}} = \mathscr{X} \quad [\mathbf{21}, \text{p. } 63].$$

The operator $T$ is *pseudodiagonal* if

$$\mathscr{X} = \underset{\lambda\in\mathbf{C}}{\text{c.l.m.}}\ \ker(T - \lambda).$$

$T$ is *finitely diagonal* if there exist idempotents $E_1, \ldots, E_p \in \mathscr{L}(\mathscr{X})$,

$$\sum_{i=1}^{p} E_i = 1_{\mathscr{X}}, \quad E_iE_j = \delta_{ij}E_i, \quad E_iT = TE_i,$$

such that

$$T = \sum_{i=1}^{p} \lambda_i E_i$$

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for distinct scalars $\lambda_1, \ldots, \lambda_p$. Thus $T$ is finitely diagonal if and only if

$$\mathscr{X} = \ker(T - \lambda_1) \dotplus \ldots \dotplus \ker(T - \lambda_p),$$

and in this case $T$ is clearly pseudodiagonal. Using [**2**, **6**] it is not difficult to verify the following implications:

$T$ is pseudodiagonal $\Rightarrow$ $T$ is pseudoalgebraic $\Rightarrow$ $T$ has the s.s.s.p.

In Section 4 we shall describe the elementary operators $R(\widetilde{A}, \widetilde{B})$ on $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$, having any of the latter three properties, in terms of the norm closed algebras generated by $\widetilde{A}$ and $\widetilde{B}$. In Theorem 4.7 we show that $R(\widetilde{A}, \widetilde{B})$ has the s.s.s.p. if and only if it has finite spectrum and that $R(\widetilde{A}, \widetilde{B})$ is pseudoalgebraic if and only if it is algebraic. In Theorem 4.8 we prove that $R(\widetilde{A}, \widetilde{B})$ is pseudodiagonal if and only if it is finitely diagonal.

Elementary operators on $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ are less tractable, but in Section 5 we present partial analogues of the preceding results for this setting. Complete results are obtained in special cases, e.g., for generalized derivations (Theorem 5.10).

Let $\mathscr{J}$ denote a 2-sided ideal of $\mathscr{L}(\mathscr{H})$. J. G. Stampfli [**25**] proved that if $\mathscr{J} \neq \{0\}$, then $\mathscr{J} \not\subset \operatorname{Ran}(\delta_A)$ $(A \in \mathscr{L}(\mathscr{H}))$. In [**11**, Theorem 2.3] it is proved that Ran $R(A, B)$ contains a nonzero 2-sided ideal if and only if $R(A, B)$ is right invertible in $\mathscr{L}(\mathscr{L}(\mathscr{H}))$. In Section 3 we consider the range inclusion

$$\operatorname{Ran} R(A, B) \subset \mathscr{J}.$$

For the cases $\mathscr{J} = \{0\}$ or $\mathscr{J} = \mathscr{K}(\mathscr{H})$ this inclusion was characterized by Fong and Sourour [**15**]. They proved in [**15**, Theorem 1] that if $\{B_1, \ldots, B_n\}$ is independent, then $R(A, B) = 0$ (i.e., Ran $R(A, B) \subset \{0\}$) if and only if $A_i = 0$ $(1 \leqq i \leqq n)$. Analogously, if $\{B_1, \ldots, B_n\}$ is independent modulo $\mathscr{K}(\mathscr{H})$, then

$$\operatorname{Ran} R(A, B) \subset \mathscr{K}(\mathscr{H})$$

if and only if $A_i \in \mathscr{K}(\mathscr{H})$ $(1 \leqq i \leqq n)$ [**15**, Theorem 3]. In Theorem 3.1 we prove that if $\mathscr{J}$ is a proper 2-sided ideal of $\mathscr{L}(\mathscr{H})$ and $\{B_1, \ldots, B_n\}$ is independent modulo $\mathscr{K}(\mathscr{H})$, then

$$\operatorname{Ran} R(A, B) \subset \mathscr{J}$$

if and only if $A_i \in \mathscr{J}$ $(1 \leqq i \leqq n)$.

The hypothesis that $\{B_1, \ldots, B_n\}$ is independent modulo $\mathscr{K}(\mathscr{H})$ cannot be weakened to independence modulo $\mathscr{J}$. Indeed, if $\mathscr{J} = C_1$ (the trace class) and if $A$ and $B$ belong to the Hilbert-Schmidt ideal $C_2$, then $AXB \in C_1$ for every $X$ in $\mathscr{L}(\mathscr{H})$ whether or not $A$ or $B$ belongs to $C_1$. The range inclusion problem for the case when $\{B_1, \ldots, B_n\}$ is dependent modulo $\mathscr{K}(\mathscr{H})$ has been studied in [**14**], where it is solved for the operators $\tau(A, B)$ and $S(A, B)$; for arbitrary elementary operators, a complete characterization of range inclusion remains unsolved.

In [15] Fong and Sourour conjectured that there is no nonzero compact elementary operator on the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and we verify this conjecture as a consequence of a more general result (Theorem 4.1). In this connection we recall some terminology. For a Banach space $\mathcal{X}$, $\mathcal{K}(\mathcal{X})$ denotes the ideal in $\mathcal{L}(\mathcal{X})$ of all compact operators on $\mathcal{X}$;

$$\pi:\mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{X})/\mathcal{K}(\mathcal{X})$$

denotes the canonical projection onto the quotient Banach algebra. For $T \in \mathcal{L}(\mathcal{X})$, we usually denote $\pi(T)$ by $\tilde{T}$ and we set

$$\|T\|_e = \|\tilde{T}\| \quad \text{and} \quad \sigma_e(T) = \sigma(\tilde{T}).$$

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**2. Banach algebra prerequisites.** In this section we record for future reference several results concerning elements of commutative Banach algebras, particularly commutative subalgebras of the Calkin algebra. We begin by recalling some standard results about the maximal ideal space of a commutative Banach algebra.

Let $\mathcal{A}$ denote a commutative Banach algebra with identity $e$, and let $\mathcal{M}(\mathcal{A})$ denote the maximal ideal space of $\mathcal{A}$. Thus, from the Gelfand-Mazur Theorem [8], $M \in \mathcal{M}(\mathcal{A})$ if and only if there exists a (unique) complex homomorphism

$$f = f_M:\mathcal{A} \to \mathbf{C}$$

such that $M = \ker f_M$ (and we frequently identify $M$ with $f_M$). Note that for each $a \in \mathcal{A}$,

$$a - f_M(a)e \in M.$$

Since $f_M \in \mathcal{A}^*$ (the dual space of $\mathcal{A}$), $\mathcal{M}(\mathcal{A})$ is given the $w^*$-topology relative to the embedding $\mathcal{A} \hookrightarrow \mathcal{A}^{**}$; thus, for $a \in \mathcal{A}$, the function

$$\hat{a}:\mathcal{M}(\mathcal{A}) \to \mathbf{C},$$

defined by $\hat{a}(M_f) = f(a)$, is continuous. For a compact space $X$, $C(X)$ denotes the Banach algebra of continuous complex functions on $X$ under the sup norm. The Gelfand transform

$$\wedge:\mathcal{A} \to C(\mathcal{M}(\mathcal{A}))$$

is defined by $\wedge(a) = \hat{a}$, and we denote its range by $\mathscr{A}^\wedge$. Recall that the Shilov boundary of $\mathscr{M}(\mathscr{A})$, $\Gamma(\mathscr{A})$, is defined by the relation

$$\|\hat{a}\| = \sup_{M \in \Gamma(a)} |\hat{a}(M)|, \quad \forall a \in \mathscr{A} \quad [\mathbf{22}, \text{ p. } 132].$$

For $a \in \mathscr{A}$, let $\sigma(a)$ denote the spectrum of $a$. Denote the radical of $\mathscr{A}$ by Rad $\mathscr{A}$, i.e.,

$$\text{Rad } \mathscr{A} = \bigcap_{M \in \mathscr{M}(\mathscr{A})} M.$$

Since $\mathscr{A}$ is commutative, Rad $\mathscr{A}$ coincides with the ideal of all quasi-nilpotent elements of $\mathscr{A}$ [**22**, p. 57], i.e.,

$$\text{Rad } \mathscr{A} = \{a \in \mathscr{A} : \sigma(a) = \{0\}\};$$

thus $a \in \mathscr{A}$ is quasinilpotent if and only if $\hat{a} = 0$. More generally, a subset $S \subset \mathscr{A}$ is *independent modulo quasinilpotents* (i.e., the image of $S$ in $\mathscr{A}/\text{Rad } \mathscr{A}$ is independent) if and only if $S^\wedge$ is independent.

Let $X$ be a nonempty set and let $\mathscr{F}(X)$ denote the vector space of complex functions on $X$. We record without proof the following elementary fact.

LEMMA 2.1. *$\{f_1, \dots, f_n\} \subset \mathscr{F}(X)$ is independent if and only if there exist distinct points $x_1, \dots, x_n \in X$ such that*

$$\det[f_i(x_j)]_{1 \leqq i,j \leqq n} \neq 0.$$

LEMMA 2.2. *Let $\{T_k\}_{k=1}^p$ denote a sequence in $\mathscr{A}$ such that $\{T_k^\wedge\}_{k=1}^p$ is independent in $\mathscr{A}^\wedge$. Then there exists $\{M_k\}_{k=1}^p \subset \Gamma(\mathscr{A})$ such that*

$$\det[T_j^\wedge(M_k)]_{1 \leqq j,k \leqq p} \neq 0.$$

*Proof.* The isometric linear mapping

$$r : \mathscr{A}^\wedge \to C(\Gamma(\mathscr{A}))$$

defined by

$$r(\hat{a}) = \hat{a}|\Gamma(\mathscr{A})$$

maps $\{T_k^\wedge\}_{k=1}^p$ to an independent set, so the result follows from Lemma 2.1

Recall from [**28**] that a nonempty subset $\mathscr{S} \subset \mathscr{A}$ consists of *joint topological divisors of zero* if for any finite subset $\{s_1, \dots, s_n\} \subset \mathscr{S}$,

$$\inf\left\{\sum_{i=1}^n \|s_i x\| : \|x\| = 1\right\} = 0.$$

LEMMA 2.3. *If $\mathscr{A}$ is a separable commutative Banach algebra and $\mathscr{S} \subset \mathscr{A}$ consists of joint topological divisors of zero, then there exists $\{x_n\}_{n=1}^\infty \subset \mathscr{A}$, $\|x_n\| = 1 \ \forall\, n$, such that*

$$\lim \|x_n s\| = 0 \quad \forall s \in \mathscr{S}.$$

*Proof.* Let $\{s_i\}_{i=1}^{\infty}$ be a dense sequence in $\mathcal{S}$. For $n \geqq 1$, there exists $x_n \in \mathcal{A}$, $\|x_n\| = 1$, such that

$$\sum_{i=1}^{n} \|x_n s_i\| < 1/n;$$

thus

$$\lim_{n\to\infty} \|x_n s_i\| = 0 \quad \text{for each } i,$$

and the result follows.

LEMMA 2.4. *Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$, $\|\tilde{T}\| = 1$. Let $\mathcal{M}$ be a finite dimensional subspace of $\mathcal{H}$. If $\{a_k\}_{k=1}^{\infty} \subset \mathbf{R}^+$ satisfies $a_k \to 0$, then there exists an orthogonal sequence $\{x_k\}_{k=1}^{\infty} \subset \mathcal{H} \ominus \mathcal{M}$ such that*

$$\|x_k\| \leqq 1 + a_k \quad \textit{and} \quad \|Tx_k\| = 1, \forall k \geqq 1.$$

*Proof.* Having chosen $x_1, \ldots, x_{k-1}$ satisfying the above requirements, let

$$\mathcal{M}_k = \langle x_1, \ldots, x_{k-1} \rangle \oplus \mathcal{M}.$$

Then

$$\|T|\mathcal{M}_k^{\perp}\| \geqq \|\tilde{T}\| \geqq 1,$$

so there exists $x_k' \in \mathcal{M}_k^{\perp}$, $\|x_k'\| = 1$, such that

$$\|Tx_k'\| \geqq 1/(1 + a_k);$$

let

$$x_k = (1/\|Tx_k'\|)x_k'.$$

LEMMA 2.5. *If $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ satisfies*

$$x_n \xrightarrow{w} 0 \quad \textit{and} \quad \overline{\lim} \|x_n\| \leqq 1,$$

*then for $T \in \mathcal{L}(\mathcal{H})$,*

$$\overline{\lim} \|Tx_n\| \leqq \|\tilde{T}\|.$$

*Proof.* For each compact operator $K$,

$$\|Kx_n\| \to 0 \quad \text{and}$$

$$\|(T + K)x_n\| - \|Kx_n\| \leqq \|Tx_n\| \leqq \|(T + K)x_n\| + \|Kx_n\|,$$

whence

$$\overline{\lim} \|Tx_n\| = \overline{\lim} \|(T + K)x_n\| \leqq \overline{\lim} \|T + K\| \|x_n\| \leqq \|T + K\|,$$

so the result follows.

LEMMA 2.6. *Suppose* $\mathscr{A}$ *is a commutative separable subalgebra of* $\widetilde{\mathscr{L}(\mathscr{H})}$, *where* $\mathscr{H}$ *is a complex separable infinite dimensional Hilbert space. Then for each* $M \in \Gamma(\mathscr{A})$, *there exists a self-adjoint projection* $P_M \in \mathscr{L}(\mathscr{H})$ *such that*

$$\widetilde{P}_M \neq 0 \quad \text{and} \quad \widetilde{T}\widetilde{P}_M = \widetilde{T}^\wedge(M)\widetilde{P}_M \quad \forall T \in \mathscr{A}.$$

*Proof.* Since $M \in \Gamma(\mathscr{A})$ consists of joint topological divisors of zero [**28**], Lemma 2.3 implies that there exists a sequence $\{\widetilde{R}_n\}_{n=1}^{\infty} \subset \mathscr{A}$, $\|\widetilde{R}_n\| = 1 \ \forall\, n$, such that

$$\lim \|\widetilde{R}_n\widetilde{S}\| = 0 \quad \forall \widetilde{S} \in M.$$

Let $\{\widetilde{S}_j\}_{j=1}^{\infty}$ be a dense sequence in $\mathscr{A}$. For $k \geqq 1$, there exists $n_k \in \mathbf{N}$, $n_k > n_{k-1}$, and $T_k \in \mathscr{L}(\mathscr{H})$, such that $\widetilde{T}_k = \widetilde{R}_{n_k}$ and

$$\|(\widetilde{S}_j - \widetilde{S}_j^\wedge(M))\widetilde{T}_k\| \leqq 3^{-k} \quad \text{for } 1 \leqq j \leqq k.$$

We claim that there exists a sequence $\{h'_k\}_{k=1}^{\infty} \subset \mathscr{H}$ such that

$$\|h'_k\| \leqq 1 + 1/k,$$

$\{T_k h'_k\}_{k=1}^{\infty}$ is an orthonormal sequence, and

$$\|(S_j - \widetilde{S}_j^\wedge(M))T_k h'_k\| \leqq 2^{-k} \quad \text{for } 1 \leqq j \leqq k.$$

Suppose that $h'_1, \ldots, h'_p$ have been chosen to satisfy the above requirements for all $j$, $k$ with $1 \leqq j \leqq k \leqq p$. Let

$$\mathscr{M} = \langle T_{p+1}{}^* T_j h'_j \rangle_{j=1}^{p}$$

and let $\mathscr{N} = \mathscr{H} \ominus \mathscr{M}$. Since $\dim \mathscr{M} < \infty$, Lemma 2.4 implies that there is an orthogonal sequence

$$\{x_k^{(p)}\}_{k=1}^{\infty} \subset \mathscr{N}$$

such that for each $k \geqq 1$,

$$\|x_k^{(p)}\| \leqq 1 + 1/pk \quad \text{and} \quad \|T_{p+1}x_k^{(p)}\| = 1.$$

Clearly,

$$x_k^{(p)} \xrightarrow{w} 0 \quad \text{and} \quad \varlimsup_{k\to\infty} \|x_k^{(p)}\| \leqq 1,$$

so Lemma 2.5 implies that for $j \leqq p + 1$,

$$\varlimsup_{k\to\infty} \|(S_j - \widetilde{S}_j^\wedge(M))T_{p+1}x_k^{(p)}\| \leqq \|(\widetilde{S}_j - \widetilde{S}_j^\wedge(M))\widetilde{T}_{p+1}\|$$
$$\leqq 3^{-(p+1)}.$$

It follows that for some fixed $k$ (sufficiently large) and $h'_{p+1} = x_k^{(p)}$,

$$\|(S_j - \widetilde{S}_j^\wedge(M))T_{p+1}h'_{p+1}\| < 2^{-(p+1)}, \quad 1 \leqq j \leqq p + 1.$$

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Let $P_M$ denote the orthogonal projection of $\mathscr{H}$ onto the closed subspace spanned by $\{T_k h'_k\}_{k=1}^{\infty}$. Clearly, $\widetilde{P}_M \neq 0$, and since

$$\sum_{k=1}^{\infty} \|(S_j - \widetilde{S}_j^{\wedge}(M))T_k h'_k\| < \infty,$$

it follows that $(S_j - \widetilde{S}_j^{\wedge}(M))P_M$ is trace class. Thus

$$\widetilde{S}_j \widetilde{P}_M = \widetilde{S}_j^{\wedge}(M)\widetilde{P}_M \quad \forall j;$$

the result now follows from the density of $\{\widetilde{S}_j\}$ and the continuity of $f_M$.

We conclude this section by recalling two results of D. Voiculescu [**27**] concerning subalgebras of the Calkin algebra

$$\mathscr{Q}(\mathscr{H}) = \mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H}).$$

Let $\mathscr{H}$ be a separable Hilbert space and let $T$ be in $\mathscr{L}(\mathscr{H})$. A closed subspace $\mathscr{M} \subset \mathscr{H}$ is *essentially T-invariant* if

$$(1 - P_{\mathscr{M}})TP_{\mathscr{M}} \in \mathscr{K}(\mathscr{H}),$$

where $P_{\mathscr{M}}$ denotes the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. For a norm closed subalgebra $\mathscr{A} \subset \mathscr{Q}(\mathscr{H})$, let Lat $\mathscr{A}$ denote the set of all self-adjoint projections $p \in \mathscr{Q}(\mathscr{H})$ such that

$$(1 - p)xp = 0 \quad \forall x \in \mathscr{A}.$$

Since projections in $\mathscr{Q}(\mathscr{H})$ lift to projections in $\mathscr{L}(\mathscr{H})$ [**5**, Theorem 2.4], it follows that

Lat $\mathscr{A} = \{\widetilde{P}_{\mathscr{M}}: \mathscr{M}$ is an essentially $T$-invariant subspace of $\mathscr{H}$ for every

$T \in \mathscr{L}(\mathscr{H})$ such that $\widetilde{T} \in \mathscr{A}\}$.

Following [**27**], we let

$$\text{Alg(Lat } \mathscr{A}) = \{y \in \mathscr{Q}(\mathscr{H}):(1 - p)yp = 0 \quad \forall p \in \text{Lat } \mathscr{A}\}.$$

We will have occasion to use a reflexivity theorem of D. Voiculescu [**27**, Theorem 1.8] which states that if $\mathscr{A}$ is a separable norm-closed subalgebra of $\mathscr{Q}(\mathscr{H})$ containing the identity, then

$$\text{Alg Lat } \mathscr{A} = \mathscr{A}.$$

We also require the following variant of D. Voiculescu's non-commutative Weyl-von Neumann Theorem [**27**, Theorem 1.5]: Let $\mathscr{H}$, $\mathscr{H}'$ be complex separable infinite dimensional Hilbert spaces, let $\mathscr{C} \supset \mathscr{K}(\mathscr{H})$ be a separable $C^*$-subalgebra of $\mathscr{L}(\mathscr{H})$, and let

$$\rho:\widetilde{\mathscr{C}} \to \mathscr{L}(\mathscr{H}')$$

be a faithful *-representation. Then there exists a unitary operator $U:\mathcal{H} \to \mathcal{H} \oplus \mathcal{H}'$ such that

$$T - U^*(T \oplus \rho(\tilde{T}))U \in \mathcal{K}(\mathcal{H}), \quad \forall T \in \mathcal{C}.$$

**3. On range inclusion and ideals.** In this section we present our characterization of the range inclusion Ran $R(A, B) \subset \mathcal{J}$. Throughout this section $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ denote arbitrary $n$-tuples of operators in $\mathcal{L}(\mathcal{H})$. Let $\mathcal{J}$ denote a proper 2-sided ideal in $\mathcal{L}(\mathcal{H})$. A subset $\mathcal{S} \subset \mathcal{L}(\mathcal{H})$ is *independent* (resp., *dependent*) mod $\mathcal{J}$ if the image of $\mathcal{S}$ in $\mathcal{L}(\mathcal{H})/\mathcal{J}$ is independent (resp., dependent). Given a linear manifold $\mathcal{V} \subset \mathcal{L}(\mathcal{H})$, a subset $\mathcal{S} \subset \mathcal{V}$ *spans* $\mathcal{V}$ mod $\mathcal{J}$ if

$$\mathcal{V} \subset \langle \mathcal{S} \rangle + \mathcal{J} \equiv \{S + J : S \in \langle \mathcal{S} \rangle, J \in \mathcal{J}\}$$

(where $\langle \mathcal{S} \rangle$ denotes the (not necessarily closed) linear span of $\mathcal{S}$).

Let $J$ denote the ideal set corresponding to $\mathcal{J}$. Let $\mathcal{H}_i$, $i = 1, 2, 3$, denote separable infinite dimensional complex Hilbert spaces, and let $T \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$. Recall from [**3**] that $T$ is *affiliated* with $\mathcal{J}$ if, when the eigenvalues of $(T^*T)^{1/2}$ are arranged in a sequence (counting multiplicities), that sequence belongs to $J$. Recall also that if $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is affiliated with $\mathcal{J}$,

$$Q \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3), \quad \text{and} \quad S \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_1),$$

then $QT$ and $TS$ are also affiliated with $\mathcal{J}$ [**3**].

The main result of this section is the following characterization of Ran $R \subset \mathcal{J}$ for $\mathcal{J} \neq \mathcal{F}$ (the ideal of all finite rank operators in $\mathcal{L}(\mathcal{H})$). The extension of this characterization to $\mathcal{J} = \mathcal{F}$ will be obtained as a corollary.

THEOREM 3.1. *Let $\mathcal{J}$ denote a proper 2-sided ideal of $\mathcal{L}(\mathcal{H})$, $\mathcal{J} \neq \mathcal{F}$. If $\{B_1, \dots, B_n\}$ is independent* mod $\mathcal{K}(\mathcal{H})$, *then*

$$\text{Ran } R(A, B) \subset \mathcal{J}$$

*if and only if $A_i \in \mathcal{J}$ $(1 \leqq i \leqq n)$.*

Before proving Theorem 3.1, we require several preliminary lemmas. In the sequel, for $T$ in $\mathcal{L}(\mathcal{H})$, $P(T)$ denotes the orthogonal projection onto $\mathcal{H} \ominus \ker(T)$, the initial space of $T$.

LEMMA 3.2. *Let $L$ in $\mathcal{L}(\mathcal{H})$ be an operator that is not compact. For $1 \leqq n < \infty$, let $\{K_i\}_{i=1}^n$ denote a sequence of compact operators in $\mathcal{L}(\mathcal{H})$. Let $\mathcal{J} \neq \mathcal{F}$ denote a proper 2-sided ideal in $\mathcal{L}(\mathcal{H})$. Then there exists an orthogonal projection $P \leqq P(L)$ such that $LP$ is not compact and $K_iP$ is in $\mathcal{J}$ $(1 \leqq i \leqq n)$.*

*Proof.* The proof is by induction on $n$. Let $n = 1$ and let $P_1 = P(L)$. Since $L|P_1\mathcal{H}$ is not compact, there exists an infinite rank projection

$Q \leqq P_1$ such that $L|Q\mathscr{H}$ is bounded below. Let $\mathscr{M} = Q\mathscr{H}$. Since $K_1|\mathscr{M}$ is compact, there exists an infinite dimensional subspace $\mathscr{L} \subset \mathscr{M}$ such that

$$K_0 = K_1|\mathscr{L}:\mathscr{L} \to \mathscr{H}$$

is affiliated with $\mathscr{J}$ [**3**, Theorem 2.1]. Let $P$ denote the orthogonal projection of $\mathscr{H}$ onto $\mathscr{L}$. Then $K_1P:\mathscr{H} \to \mathscr{H}$ is also affiliated with $\mathscr{J}$, hence $K_1P$ is in $\mathscr{J}$. Since $P \leqq Q \leqq P_1 = P(L)$, then $L|P\mathscr{H}$ is bounded below, so $LP$ is not compact and the proof is complete in this case.

Assume the result for $n - 1$. Since $L$ is not compact, the induction hypothesis implies that there exists an orthogonal projection $P_1 \leqq P(L)$ such that $LP_1$ is not compact and $K_iP_1$ is in $\mathscr{J}$ $(1 \leqq i \leqq n - 1)$. Let

$$P_0 = P(LP_1) \leqq P(P_1) = P_1 \leqq P(L).$$

Let $\mathscr{M} = P_0\mathscr{H}$. Since $LP_1|\mathscr{M}$ is not compact, there exists an infinite rank orthogonal projection $Q \leqq P_0$ such that $LP_1|Q\mathscr{H}$ is bounded below. Since $K_n|Q\mathscr{H}$ is compact, [**3**, Theorem 2.1] implies that there exists an infinite rank projection $P \leqq Q$ such that

$$K_n|P\mathscr{H}:P\mathscr{H} \to \mathscr{H}$$

is affiliated with $\mathscr{J}$, so that $K_nP$ is in $\mathscr{J}$. Note also that

$$K_iP = (K_iP_1)P \in \mathscr{J} \quad (1 \leqq i \leqq n - 1).$$

Since $P \leqq Q$, then $L|P\mathscr{H}$ is bounded below; thus $LP$ is not compact and the proof is complete.

LEMMA 3.3. *Let $\mathscr{J}$ denote a proper 2-sided ideal in $\mathscr{L}(\mathscr{H})$, $\mathscr{J} \neq \mathscr{F}$. If $n \geqq 1$ and $L_1, \ldots, L_n$ is a sequence of operators in $\mathscr{L}(\mathscr{H})$ with $L_1$ not compact, then there exists an orthogonal projection $P \leqq P(L_1)$ such that $\{L_1P, \ldots, L_nP\}$ contains a subset $\mathscr{S}$ that is linearly independent* mod $\mathscr{K}(\mathscr{H})$ *and which spans* $\langle L_1P, \ldots, L_nP\rangle$ mod $\mathscr{J}$.

*Proof.* The proof is by induction on $n \geqq 1$. Let $n = 1$. Let $P = P(L_1)$; since $L_1$ is not compact, then $L_1P$ is not compact. Thus $\mathscr{S} = \{L_1P\}$ is independent mod $\mathscr{K}(\mathscr{H})$ and spans $\langle L_1P\rangle$ mod $\mathscr{J}$.

Assume the result is true for $n - 1$. Let $P_1 = P(L_1)$ and note that $L_1P_1$ is not compact. If $\{L_1P_1, \ldots, L_nP_1\}$ is independent mod $\mathscr{K}(\mathscr{H})$, then the result follows by setting

$$P = P_1 \quad \text{and} \quad \mathscr{S} = \{L_1P, \ldots, L_nP\}.$$

If $\{L_1P_1, \ldots, L_nP_1\}$ is dependent mod $\mathscr{K}(\mathscr{H})$, then since $L_1P_1$ is not compact, we may assume (by relabelling $L_2, \ldots, L_n$ if necessary) that there exist scalars $c_1, \ldots, c_{n-1}$ and $K \in \mathscr{K}(\mathscr{H})$ such that

$$L_nP_1 = c_1L_1P_1 + \ldots + c_{n-1}L_{n-1}P_1 + K.$$

Since $L_1P_1$ is not compact and $K$ is compact, Lemma 3.2 implies that there exists an orthogonal projection $R \leqq P(L_1P_1)$ $(= P_1)$ such that $L_1P_1R$ is not compact and $KR$ is in $\mathscr{J}$. Since $P_1R = R$, it follows that $L_1R$ is not compact and

$$L_nR = c_1L_1R + \ldots + c_{n-1}L_{n-1}R + KR.$$

The induction hypothesis, applied to $L_1R, \ldots, L_{n-1}R$, implies that there exists an orthogonal projection $Q \leqq P(L_1R)$ and a subset

$$\mathscr{S} \subset \{L_1RQ, \ldots, L_{n-1}RQ\}$$

such that $\mathscr{S}$ is independent mod $\mathscr{K}(\mathscr{H})$ and spans

$$\langle L_1RQ, \ldots, L_{n-1}RQ \rangle \text{ mod } \mathscr{J}.$$

Note that since $\ker(R) \subset \ker(L_1R)$, then

$$Q \leqq P(L_1R) \leqq P(R) = R,$$

so that $RQ = Q$. If we set $P = Q$, then

$$\mathscr{S} \subset \{L_1P, \ldots, L_{n-1}P\}$$

is independent mod $\mathscr{K}(\mathscr{H})$ and spans

$$\langle L_1P, \ldots, L_{n-1}P \rangle \text{ mod } \mathscr{J}.$$

Moreover,

$$L_nP = L_nRP = c_1L_1P + \ldots + c_{n-1}L_{n-1}P + (KR)P,$$

so $\mathscr{S}$ spans

$$\langle L_1P, \ldots, L_nP \rangle \text{ mod } \mathscr{J}.$$

Since $P \leqq R \leqq P(L_1)$, the proof is complete.

Let $\mathscr{J}$ denote a proper 2-sided ideal in $\mathscr{L}(\mathscr{H})$, $\mathscr{J} \neq \mathscr{F}$, and let $\mathscr{M}$ denote a proper infinite dimensional subspace of $\mathscr{H}$.

LEMMA 3.4. *If $S \in \mathscr{L}(\mathscr{M})$ is affiliated with $\mathscr{J}$, then $T = S \oplus 0_{\mathscr{M}^\perp}$ is in $\mathscr{J}$.*

*Proof.* Since $S$ is affiliated with $\mathscr{J}$, each characteristic sequence for $S$ belongs to $J$, the ideal set of $\mathscr{J}$. It follows from [**5**, Lemma 1.2] that each characteristic sequence for $T = S \oplus 0_{\mathscr{M}^\perp}$ also belongs to $J$ and thus $T$ is in $\mathscr{J}$.

*Proof of Theorem* 3.1. Assume that $\text{Ran}(R(A, B)) \subset \mathscr{J}$ and that $\{B_1, \ldots, B_n\}$ is independent mod $\mathscr{K}(\mathscr{H})$. We seek to prove that $A_i \in \mathscr{J}$ $(1 \leqq i \leqq n)$. The proof is by induction on $n \geqq 1$.

For the case $n = 1$, suppose that $A_1XB_1 \in \mathscr{J}$ for every $X$ in $\mathscr{L}(\mathscr{H})$ and suppose that $\{B_1\}$ is independent mod $\mathscr{K}(\mathscr{H})$. Since $B_1$ is not compact, there exists an infinite dimensional subspace $\mathscr{M} \subset \mathscr{H}$ such that $B_1|\mathscr{M}$ is bounded below. Thus there exists $L \in \mathscr{L}(\mathscr{H})$ such that $LB_1P = P$, where $P$ denotes the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. Let $V$ denote an isometry in $\mathscr{L}(\mathscr{H})$ whose range is $\mathscr{M}$. Then

$$V^*LB_1V = V^*V = 1_{\mathscr{H}},$$

so $A_1 = [A_1(V^*L)B_1]V$ is in $\mathscr{J}$ and the proof of this case is complete.

Assume that the theorem is valid for $n - 1$, and consider the case when

$$\mathrm{Ran}(R(A, B)) \subset \mathscr{J},$$

where $A = (A_1, \dots, A_n)$, $B = (B_1, \dots, B_n)$ and $\{B_1, \dots, B_n\}$ is independent mod $\mathscr{K}(\mathscr{H})$. Suppose first that $\{A_1, \dots, A_n\}$ is dependent mod $\mathscr{J}$. Then there exist scalars $c_1, \dots, c_n$, not all zero, such that

$$c_1A_1 + \dots + c_nA_n = J \in \mathscr{J}.$$

We may assume (by reordering the $A_i$'s and $B_i$'s) that $c_1 \neq 0$, and thus we may further assume that $c_1 = 1$, i.e.,

$$A_1 = J - c_2A_2 - \dots - c_nA_n.$$

For $X$ in $\mathscr{L}(\mathscr{H})$,

$$(J - c_2A_2 - \dots - c_nA_n)XB_1 + A_2XB_2 + \dots + A_nXB_n$$

is in $\mathscr{J}$, and since $J \in \mathscr{J}$, it follows that

$$(*) \qquad A_2X(B_2 - c_2B_1) + A_3X(B_3 - c_3B_1) + \dots + A_nX(B_n - c_nB_1) \in \mathscr{J}.$$

Since $\{B_1, \dots, B_n\}$ is independent mod $\mathscr{K}(\mathscr{H})$, it follows readily that $\{B_i - c_iB_1\}_{i=2}^n$ is independent mod $\mathscr{K}(\mathscr{H})$, so (*) and the induction hypothesis imply that $A_i \in \mathscr{J}$ $(2 \leqq i \leqq n)$. Since

$$\mathrm{Ran}(R(A, B)) \subset \mathscr{J}$$

it now follows that

$$A_1XB_1 \in \mathscr{J} \quad \text{for every } X \in \mathscr{L}(\mathscr{H}),$$

and since $B_1$ is not compact, the case $n = 1$ (above) implies that $A_1 \in \mathscr{J}$. Thus $A_i \in \mathscr{J}$ $(1 \leqq i \leqq n)$ and the proof is complete in this case.

To complete the induction it thus suffices to assume the result for $n - 1$ and to prove that if

$$\mathrm{Ran}(R(A, B)) \subset \mathscr{J}$$

and $\{B_1, \dots, B_n\}$ is independent mod $\mathscr{K}(\mathscr{H})$, then $\{A_1, \dots, A_n\}$ is dependent mod $\mathscr{J}$. We have

$$(1) \qquad A_1XB_1 + \dots + A_nXB_n \in \mathscr{J} \quad \text{for every } X \text{ in } \mathscr{L}(\mathscr{H}).$$

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Moreover, $B_n = BU$, where $B_n^* = U^*B$ denotes the polar decomposition of $B_n^*$; thus $B \geqq 0$ and $U$ is a partial isometry such that $BUU^* = B$. Let

$$C_i = B_iU^* \quad (1 \leqq i \leqq n - 1);$$

(1) implies that

(2) $$A_1XC_1 + \ldots + A_{n-1}XC_{n-1} + A_nXB \in \mathscr{J} \text{ for each } X \text{ in } \mathscr{L}(\mathscr{H}).$$

We distinguish two cases.

*Case* 1. Suppose there exists an essentially $B$-invariant subspace $\mathscr{M}$ that is not essentially invariant for all of $C_1, \ldots, C_{n-1}$. We may assume without loss of generality that $\mathscr{M}$ is not essentially invariant for $C_1$; in particular, we may assume that $\mathscr{M}$ and $\mathscr{M}^\perp$ are infinite dimensional, hence isomorphic. Relative to the decomposition

(3) $$\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp,$$

the operator matrix of $B$ is of the form

$$\begin{pmatrix} B_{11} & B_{12} \\ K & B_{22} \end{pmatrix},$$

with $K \in \mathscr{K}(\mathscr{M}, \mathscr{M}^\perp)$, and the matrix of $C_1$ is of the form

$$\begin{pmatrix} C_{11} & C_{12} \\ D_1 & C_{22} \end{pmatrix},$$

with $D_1 \in \mathscr{L}(\mathscr{M}, \mathscr{M}^\perp)$ and $D_1$ not compact.

Let $V$ denote an isomorphism of $\mathscr{M}^\perp$ onto $\mathscr{M}$; thus $VD_1 \in \mathscr{L}(\mathscr{M})$ is not compact and $VK \in \mathscr{L}(\mathscr{M})$ is compact. It follows from Lemma 2.2 (applied with $\mathscr{H}$ replaced by $\mathscr{M}$) that there exists an orthogonal projection $Q \in \mathscr{L}(\mathscr{M})$ such that $VD_1Q$ is not compact and $VKQ \in \mathscr{L}(\mathscr{M})$ is affiliated with $\mathscr{J}$. Relative to the decomposition (3), let

$$R = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}$$

and let

$$T = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}.$$

Let

$$L_i = TC_iR \quad (1 \leqq i \leqq n - 1)$$

and note that

$$L_1 = TC_1R = \begin{pmatrix} VD_1Q & 0 \\ 0 & 0 \end{pmatrix},$$

which is not compact. Another matrix calculation shows that

$$TBR = \begin{pmatrix} VKQ & 0 \\ 0 & 0 \end{pmatrix};$$

since $VKQ$ is affiliated with $\mathcal{J}$, Lemma 2.3 implies that $TBR = VKQ \oplus 0_{\mathcal{M}^\perp}$ belongs to $\mathcal{J}$. It thus follows from (2) that for each $X$ in $\mathcal{L}(\mathcal{H})$,

$$(4) \qquad A_1XL_1 + \ldots + A_{n-1}XL_{n-1} = [A_1(XT)C_1 + \ldots + A_{n-1}(XT)C_{n-1} + A_n(XT)B]R - A_nX(TBR) \in \mathcal{J}.$$

Since $L_1$ is not compact, Lemma 3.3 implies that there exists an orthogonal projection $P \leqq P(L_1)$ and an integer $k$, $1 \leqq k \leqq n - 1$, such that (after perhaps reordering $L_2, \ldots, L_{n-1}$ (and $A_2, \ldots, A_{n-1}$ correspondingly) )

$$\mathcal{S} \equiv \{L_1P, \ldots, L_kP\}$$

is independent mod $\mathcal{K}(\mathcal{H})$ and spans

$$\langle L_1P, \ldots, L_{n-1}P\rangle \text{ mod } \mathcal{J}.$$

Thus if $k < n - 1$, then for $1 \leqq i \leqq n - 1 - k$, there exist scalars $c_{i1}, \ldots, c_{ik}$ and there exists $J_i \in \mathcal{J}$ such that

$$(5) \qquad L_{k+i}P = c_{i1}L_1P + \ldots + c_{ik}L_kP + J_i.$$

Since, for each $X$ in $\mathcal{L}(\mathcal{H})$,

$$A_1XL_1P + \ldots + A_kXL_kP + A_{k+1}XL_{k+1}P + \ldots + A_{n-1}XL_{n-1}P \in \mathcal{J}.$$

then (5) implies that

$$A_1XL_1P + \ldots + A_kXL_kP + A_{k+1}X(c_{11}L_1P + \ldots + c_{1k}L_kP) + \ldots + A_{n-1}X(c_{n-1-k,1}L_1P + \ldots + c_{n-1-k,k}L_kP) \in \mathcal{J},$$

and thus

$$(6) \qquad (A_1 + c_{11}A_{k+1} + \ldots + c_{n-1-k,1}A_{n-1})XL_1P + \ldots + (A_k + c_{1k}A_{k+1} + \ldots + c_{n-1-k,k}A_{n-1})XL_kP \in \mathcal{J}.$$

Note that the validity of the theorem for $n - 1$ readily implies its validity for $j$ whenever $1 \leqq j \leqq n - 1$. In particular, since $\{L_1P, \ldots, L_kP\}$ is independent mod $\mathcal{K}(\mathcal{H})$ and $k \leqq n - 1$, the induction hypothesis and (6) imply that

$$A_1 + c_{11}A_{k+1} + \ldots + c_{n-1-k,1}A_{n-1}$$

belongs to $\mathcal{J}$, i.e., $\{A_1, \ldots, A_n\}$ is dependent mod $\mathcal{J}$.

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*Case* 2. In the remaining case, every essentially $B$-invariant subspace is essentially invariant for $C_1, \ldots, C_{n-1}$. Let $\mathscr{A} = C^*(\widetilde{B})$. If $\widetilde{P}$ is in Lat($\mathscr{A}$), then

$$(1 - \widetilde{P})\widetilde{B}\widetilde{P} = 0,$$

whence

$$(1 - \widetilde{P})\widetilde{C}_i\widetilde{P} = 0 \quad (1 \leqq i \leqq n - 1).$$

Thus each $C_i$ is in Alg(Lat($\mathscr{A}$) ). The reflexivity theorem of D. Voiculescu [**27**, Theorem 1.8] implies that

$$\text{Alg}(\text{Lat}(\mathscr{A})) = \mathscr{A},$$

and thus $\widetilde{C}_i \in \mathscr{A}$ $(1 \leqq i \leqq n - 1)$. Thus, since $B \geqq 0$, there exist continuous functions

$$\varphi_i:\sigma_e(B) \rightarrow \mathbf{C}$$

such that

$$\widetilde{C}_i = \varphi_i(\widetilde{B}) \quad (1 \leqq i \leqq n - 1).$$

(Recall that $C^*(\widetilde{B}) \approx C(\sigma_e(B))$ [**11**].)

Since $B_n = BU$ and $B_n$ is not compact, then $\widetilde{B} \neq 0$, so there exists $\lambda \in \sigma_e(B)$ such that $\lambda \neq 0$. Since

$$\lambda \in \sigma_e(B) = \sigma_{1e}(B),$$

there exists an infinite rank projection $P$ such that

$$(\widetilde{B} - \lambda)\widetilde{P} = 0,$$

i.e., $\widetilde{B}\widetilde{P} = \lambda\widetilde{P}$, whence

$$\varphi_i(\widetilde{B})\widetilde{P} = \varphi_i(\lambda)\widetilde{P} \quad (1 \leqq i \leqq n - 1).$$

Thus there exists $K_i \in \mathscr{L}(\mathscr{H})$ such that

$$C_iP = \varphi_i(\lambda)P + K_i \quad (1 \leqq i \leqq n - 1).$$

Similarly, there exists $K_n \in \mathscr{K}(\mathscr{H})$ such that

$$BP = \lambda P + K_n.$$

Since, for each $X$ in $\mathscr{L}(\mathscr{H})$,

$$A_1XC_1 + \ldots + A_{n-1}XC_{n-1} + A_nXB \in \mathscr{J},$$

then

$$A_1XC_1P + \ldots + A_{n-1}XC_{n-1}P + A_nXBP \in \mathscr{J},$$

whence

$$(7) \qquad A_1X(\varphi_1(\lambda)P + K_1) + \ldots + A_{n-1}X(\varphi_{n-1}(\lambda)P + K_{n-1}) + A_nX(\lambda P + K_n) \in \mathscr{J}.$$

Lemma 3.2 implies that there exists an infinite rank orthogonal projection $Q \leqq P$ such that $K_iQ \in \mathscr{J}$ $(1 \leqq i \leqq n)$. Thus (7) implies that for each $X$ in $\mathscr{L}(\mathscr{H})$,

$$A_1X\varphi_1(\lambda)Q + \ldots + A_{n-1}X\varphi_{n-1}(\lambda)Q + A_nX\lambda Q \in \mathscr{J}.$$

Thus

$$(\varphi_1(\lambda)A_1 + \ldots + \varphi_{n-1}(\lambda)A_{n-1} + \lambda A_n)XQ \in \mathscr{J},$$

for every operator $X$ in $\mathscr{L}(\mathscr{H})$. Since $Q$ is not compact, it follows from the case $n = 1$ that

$$\varphi_1(\lambda)A_1 + \ldots + \varphi_{n-1}(\lambda)A_{n-1} + \lambda A_n \in \mathscr{J};$$

since $\lambda \neq 0$, then $\{A_1, \ldots, A_n\}$ is dependent mod $\mathscr{J}$. The proof of Theorem 3.1 is now complete.

COROLLARY 3.5. *Let $\mathscr{J}$ denote a proper 2-sided ideal of $\mathscr{L}(\mathscr{H})$. If $\{B_1, \ldots, B_n\}$ is independent* mod $\mathscr{K}(\mathscr{H})$, *then*

$$\text{Ran}(R(A, B)) \subset \mathscr{J}$$

*if and only if* $A_i \in \mathscr{J}$ $(1 \leqq i \leqq n)$.

*Proof.* The result follows from Theorem 3.1 if $\mathscr{J} \neq \mathscr{F}$, so it remains to consider the case when

$$\text{Ran}(R(A, B)) \subset \mathscr{F}.$$

In this case, since $\mathscr{F} \subset \mathscr{J}$ for every proper ideal $\mathscr{J}$, Theorem 3.1 implies that $A_i \in \mathscr{J}$ for each proper ideal $\mathscr{J} \neq \mathscr{F}$. Since the intersection of all such ideals is equal to $\mathscr{F}$ [**4**, Corollary 4.7], it follows that $A_i \in \mathscr{F}$ $(1 \leqq i \leqq n)$.

**4. Elementary operators on $\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}$.** In this section we analyze the structure of elementary operators on the space $\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}$, particularly with respect to compactness, pseudoalgebraicity, the strong spectral splitting property, and pseudodiagonalizability.

THEOREM 4.1. *The canonical quotient map*

$$\mathscr{L}(\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}) \rightarrow \mathscr{L}(\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)})/\mathscr{K}(\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)})$$

*is isometric on elementary operators.*

*Proof.* Invoking unitary equivalence, we may suppose that

$$\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}.$$

Let $\widetilde{R} = R(\widetilde{A}, \widetilde{B})$, $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n)$. Let $\mathscr{C}$ denote a separable $C^*$-subalgebra of $\mathscr{L}(\mathscr{H})$ such that $\mathscr{K}(\mathscr{H}) \subset \mathscr{C}$, $A_i$, $B_i \in \mathscr{C}$ $(1 \leqq i \leqq n)$, and such that

$$\|\widetilde{R}\| = \sup\{ \|\widetilde{R}(\widetilde{X})\| : \|\widetilde{X}\| = 1, \widetilde{X} \in \mathscr{C}\}.$$

Let $\rho:\widetilde{\mathscr{C}} \rightarrow \mathscr{L}(\mathscr{H})$ be a faithful representation of infinite multiplicity. By Voiculescu's Theorem, we can find a unitary operator

$$U:\mathscr{H} \rightarrow \bigoplus_{j=1}^{\infty} \mathscr{H}$$

such that

$$T - U^*(T \oplus \rho(\widetilde{T}) \oplus \ldots \oplus \rho(\widetilde{T}) \oplus \ldots)U \in \mathscr{K}(\mathscr{H}) \quad \forall T \in \mathscr{C}.$$

Since $\widetilde{R}$ is unitarily equivalent to the operator $\widetilde{R}'$ obtained by replacing $A_j$, $B_j$ by

$$A_j' = A_j \oplus \rho(\widetilde{A}_j) \oplus \ldots \oplus \rho(\widetilde{A}_j) \oplus \ldots,$$
$$B_j' = B_j \oplus \rho(B_j) \oplus \ldots \oplus \rho(B_j) \oplus \ldots,$$

it suffices to prove that

$$\|\widetilde{R}'\| = \inf\left\{ \|\widetilde{R}' - \widetilde{R}_0'\| : \widetilde{R}_0' \in \mathscr{K}\left(\mathscr{Q}\left(\bigoplus_{j=1}^{\infty} \mathscr{H}\right)\right)\right\}.$$

Let $\epsilon > 0$ and let

$$\widetilde{R}_0' \in \mathscr{K}\left(\mathscr{Q}\left(\bigoplus_{j=1}^{\infty} \mathscr{H}\right)\right).$$

For $X \in \mathscr{C}$, $\|\widetilde{X}\| = 1$, let

$$\mathscr{X}_{\widetilde{X}} \subset \mathscr{Q}\left(\bigoplus_{j=1}^{\infty} \mathscr{H}\right)$$

be the linear manifold

$$\{\pi(0 \oplus \alpha_2\rho(\widetilde{X}) \oplus \alpha_3\rho(\widetilde{X}) \oplus \ldots)\},$$

where only a finite number of the $\alpha_k$'s are nonzero. Since $\rho$ has infinite multiplicity, if $\widetilde{X} \neq 0$, then

$$\widetilde{\rho(\widetilde{X})} \neq 0,$$

and thus

$$\dim \mathscr{X}_{\widetilde{X}} = \infty.$$

Since $\widetilde{R}_0'$ is compact, it follows that $\widetilde{R}_0'|\mathscr{X}_{\widetilde{X}}$ is not bounded below; thus there exists $\widetilde{X}' \in \mathscr{X}_{\widetilde{X}}$ such that

$$\|\widetilde{X}'\| = \|\widetilde{X}\| \quad \text{and} \quad \|\widetilde{R}_0'(\widetilde{X}')\| < \epsilon.$$

We have

$$\|\widetilde{R}'(\widetilde{X}')\| = \left\|\sum_{j=2}^{\infty} \oplus \left[\alpha_j \sum_{i=1}^{n} \rho(\widetilde{A}_i)\rho(\widetilde{X})\rho(\widetilde{B}_i)\right]\right\|_e$$

$$= \left\|\sum_{j=2}^{\infty} \oplus \alpha_j \rho(\widetilde{R}(\widetilde{X}))\right\|_e$$

$$= (\sup_j |\alpha_j|)\,\|\rho(\widetilde{R}(\widetilde{X}))\|_e$$

$$= \|\rho(\widetilde{R}(\widetilde{X}))\|_e$$

$$= \|\widetilde{R}(\widetilde{X})\|$$

(since $\rho$ has infinite multiplicity). Now

$$\|\widetilde{R}(\widetilde{X})\| = \|\widetilde{R}'(\widetilde{X}')\| \leqq \|(\widetilde{R}' - \widetilde{R}'_0)(\widetilde{X}')\| + \epsilon \leqq \|\widetilde{R}' - \widetilde{R}'_0\| + \epsilon,$$

so

$$\|\widetilde{R}'\| = \|\widetilde{R}\| = \sup_{\substack{X \in \mathscr{C} \\ \|\widetilde{X}\|=1}} \|\widetilde{R}(\widetilde{X})\| \leqq \|\widetilde{R}' - \widetilde{R}'_0\| + \epsilon.$$

Since $\epsilon$ and $\widetilde{R}'_0$ are arbitrary, then

$$\|\widetilde{R}'\| \leqq \|\widetilde{R}'\|_e$$

and the proof is complete.

*Remark.* The preceding result shows that there are no nonzero compact elementary operators on

$$\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)};$$

this answers a question of Fong and Sourour [**15**].

We next begin the spectral analysis of elementary operators on $\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}$.

PROPOSITION 4.2. *Let* $\{R_k\}_{k=1}^{\infty}$ *be a sequence of nonzero elementary operators acting in* $\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}$. *Then for every* $a$, $0 < a < 1$, *there exists*

$$\widetilde{X}_0 \in \widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}$$

*such that*

$$\|\widetilde{X}_0\| = 1 \quad \text{and} \quad \text{dist}(\widetilde{X}_0, \ker \widetilde{R}_k) \geqq a \quad \forall k \geqq 1.$$

*Proof.* We may assume that

$$\widetilde{R}_k = R(\widetilde{A}^{(k)}, \widetilde{B}^{(k)}),$$

where

$$A^{(k)} = \{A_j^{(k)}\}_{j=1}^{n_k} \subset \mathcal{L}(\mathcal{H}_2),$$
$$B^{(k)} = \{B_j^{(k)}\}_{j=1}^{n_k} \subset \mathcal{L}(\mathcal{H}_1),$$

and $\widetilde{A}^{(k)}$ and $\widetilde{B}^{(k)}$ are each linearly independent $n_k$-tuples. Let $\mathcal{C}_2$, $\mathcal{C}_1$ denote the $C^*$-algebras generated by $\{A_j^{(k)}\}_{j,k}$, $\mathcal{K}(\mathcal{H}_2)$, resp. by $\{B_j^{(k)}\}_{j,k}$, $\mathcal{K}(\mathcal{H}_1)$, and let

$$\rho_1:\mathcal{C}_1 \to \mathcal{L}(\mathcal{H}_1) \quad \text{and} \quad \rho_2:\mathcal{C}_2 \to \mathcal{L}(\mathcal{H}_2)$$

be faithful representations of infinite multiplicity. By Voiculescu's Theorem (Section 1), there exist unitary operators

$$U_1:\mathcal{H}_1 \to \mathcal{H}_1' \equiv \sum_{j=1}^{\infty} \oplus \mathcal{H}_1,$$

$$U_2:\mathcal{H}_2 \to \mathcal{H}_2' \equiv \sum_{j=1}^{\infty} \oplus \mathcal{H}_2,$$

such that

$$T - U_1^*(T \oplus \rho_1(\widetilde{T}) \oplus \ldots \oplus \rho_1(\widetilde{T}) \oplus \ldots)U_1 \in \mathcal{K}(\mathcal{H}_1), \quad \forall T \in \mathcal{C}_1,$$
$$S - U_2^*(S \oplus \rho_2(\widetilde{S}) \oplus \ldots \oplus \rho_2(\widetilde{S}) \ldots)U_2 \in \mathcal{K}(\mathcal{H}_2), \quad \forall S \in \mathcal{C}_2.$$

Let

$$\rho_1'(T) = T \oplus \rho_1(\widetilde{T}) \oplus \ldots \oplus \rho_1(\widetilde{T}) \oplus \ldots, \; T \in \mathcal{C}_1$$

and

$$\rho_2'(S) = S \oplus \rho_2(\widetilde{S}) \oplus \ldots \oplus \rho_2(\widetilde{S}) \oplus \ldots, \; S \in \mathcal{C}_2.$$

Thus

$$\widetilde{R}_k(\widetilde{X}) = \sum_{j=1}^{n_k} \widetilde{U_2{}^*\rho_2'(A_j^{(k)})U_2}\widetilde{X}\widetilde{U_1{}^*\rho_1'(B_j^{(k)})U_1}.$$

Define the elementary operator $R_k'$ on $\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')$ by

$$R_k'(Y) = \sum_{j=1}^{n_k} \rho_2'(A_j^{(k)})Y\rho_1'(B_j^{(k)});$$

thus $Y \in \ker \widetilde{R}_k'$ if and only if

$$\widetilde{U_2{}^*YU_1} \in \ker \widetilde{R}_k,$$

so it suffices to prove that there exists

$$\widetilde{Y}_0 \in \widetilde{\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')}, \quad \|\widetilde{Y}_0\| = 1,$$

such that

$$\operatorname{dist}(\widetilde{Y}_0, \ker(\widetilde{R}'_k)) \geqq a \quad \forall k \geqq 1.$$

For $k \geqq 1$, define the elementary operator $S_k$ on $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ by

$$S_k(X) = \sum_{j=1}^{n_k} \rho_2(\widetilde{A}_j^{(k)})X\rho_1(\widetilde{B}_j^{(k)}).$$

Since $\rho_1$ and $\rho_2$ have infinite multiplicity and $\widetilde{A}^{(k)}$ and $\widetilde{B}^{(k)}$ are each linearly independent, the sequences

$$\{\widetilde{\rho_1(\widetilde{A}_j^{(k)})}\}_{j=1}^{n_k}, \quad \{\widetilde{\rho_2(\widetilde{B}_j^{(k)})}\}_{j=1}^{n_k}$$

are each independent, so [**15**, Theorem 3] implies that $\widetilde{S}_k$ is a nonzero elementary operator on

$$\widetilde{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)}.$$

Riesz' Lemma implies that there exists $Y_k \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, $\|\widetilde{Y}_k\| = 1$, such that

$$\operatorname{dist}(\widetilde{Y}_k, \ker \widetilde{S}_k) \geqq a.$$

Let $Q_j$ denote the projection of $\mathcal{H}_1'$ onto the $j$-th coordinate space. Let $P_j$ denote an isometry of $\mathcal{H}_2$ onto the $j$-th coordinate space of $\mathcal{H}_2'$; thus

$$\mathcal{H}_1' = \sum_{j=1}^{\infty} \oplus Q_j\mathcal{H}_1' \quad \text{and} \quad \mathcal{H}_2' = \sum_{j=1}^{\infty} P_j\mathcal{H}_2.$$

Let

$$Y_0 = \sum_{j=1}^{\infty} P_{j+1}Y_jQ_{j+1} \in \mathcal{L}(\mathcal{H}_1', \mathcal{H}_2').$$

For $Y \in \mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')$, let $(Y_{ij})$ denote the operator matrix of $Y$ with respect to the preceding decompositions of $\mathcal{H}_1'$ and $\mathcal{H}_2'$. Note that for each $m \geqq 1$, the $m + 1$, $m + 1$ entry of $R'_k(Y)$ is equal to $S_k(Y_{mm})$. If $\widetilde{Y} \in \ker \widetilde{R}'_k$, then $R'_k(Y)$ is compact, and thus

$$S_k(Y_{kk}) \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2).$$

If $K \in \mathcal{K}(\mathcal{H}_1', \mathcal{H}_2')$, then

$$\|Y_0 - Y + K\| \geqq \|Y_k - Y_{kk} + K_{kk}\| \geqq \|\widetilde{Y}_k - \widetilde{Y}_{kk}\|$$
$$\geqq \operatorname{dist}(\widetilde{Y}_k, \ker \widetilde{S}_k) \geqq a.$$

Thus $\|\widetilde{Y}_0 - \widetilde{Y}\| \geqq a$ and it follows that

$$\operatorname{dist}(\widetilde{Y}_0, \ker \widetilde{R}'_k) \geqq a \quad \forall k.$$

Since clearly $\|\widetilde{Y}_0\| = 1$, the proof is complete.

Observe that if $\mathscr{X}$ is a $B$-space, $T \in \mathscr{L}(\mathscr{X})$ is pseudoalgebraic, and $\sigma(T)$ is countable, then there exists a sequence of monic polynomials $\{p_k\}_{k=1}^{\infty}$ such that

$$\mathscr{X}_T^{\text{alg}} = \bigcup_{k=1}^{\infty} \ker p_k(T) \quad \text{and}$$

$$\ker p_k(T) \subset \ker p_{k+1}(T) \quad \forall k \geqq 1;$$

indeed, it is not difficult to check that the polynomials

$$p_k(x) = (x - \lambda_1)^k \dots (x - \lambda_k)^k$$

satisfy the requirements, where

$$\sigma(T) = \{\lambda_i\}_{i=1}^{\infty}.$$

THEOREM 4.3. *If* $\widetilde{R}$ *is a pseudoalgebraic elementary operator on* $\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}$ *with countable spectrum, then* $\widetilde{R}$ *is algebraic.*

*Proof.* Let $\{p_k\}_{k=1}^{\infty}$ be a sequence of monic polynomials such that

$$\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}_{\widetilde{R}}^{\text{alg}} = \bigcup_{k=1}^{\infty} \ker p_k(\widetilde{R}) \quad \text{and}$$

$$\ker p_k(\widetilde{R}) \subset \ker p_{k+1}(\widetilde{R}) \quad \forall k \geqq 1.$$

Clearly $p_k(\widetilde{R})$ is an elementary operator, so if $p_k(\widetilde{R}) \neq 0 \ \forall \ k \geqq 1$, then Proposition 4.2 implies that

$$\bigcup_{k=1}^{\infty} \ker p_k(\widetilde{R})$$

is not dense, contradicting the hypothesis that $\widetilde{R}$ is pseudoalgebraic. Thus for some $k \geqq 1$, $p_k(\widetilde{R}) = 0$, whence, $\widetilde{R}$ is algebraic.

In the remainder of this section $\widetilde{A}$ and $\widetilde{B}$ will be commutative $n$-tuples. Let $P \in \mathscr{L}(\mathscr{H}_2)$ and $Q \in \mathscr{L}(\mathscr{H}_1)$ be such that $\widetilde{P}^* \neq 0$ is a jointly invariant idempotent for $\widetilde{A}^*$ (i.e., $\widetilde{P}^2 = \widetilde{P}$ and $\widetilde{P}\widetilde{A}_i\widetilde{P} = \widetilde{P}\widetilde{A}_i(1 \leqq i \leqq n)$) and $\widetilde{Q} \neq 0$ is a jointly invariant idempotent for $\widetilde{B}$ (i.e., $\widetilde{Q}^2 = \widetilde{Q}$ and $\widetilde{B}_i\widetilde{Q} = \widetilde{Q}\widetilde{B}_i\widetilde{Q}(1 \leqq i \leqq n)$). Let $\widetilde{P}\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}\widetilde{Q}$ denote the space

$$\{\widetilde{P}\widetilde{X}\widetilde{Q} : \widetilde{X} \in \widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}\},$$

a closed subspace of $\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}$. Define the operator $R({}_{\widetilde{P}}\widetilde{A}, \widetilde{B}_{\widetilde{Q}})$ on $\widetilde{P}\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}\widetilde{Q}$ by

$$R({}_{\widetilde{P}}\widetilde{A}, \widetilde{B}_{\widetilde{Q}})(\widetilde{X}) = \sum_{k=1}^{n} \widetilde{P}\widetilde{A}_k\widetilde{X}\widetilde{B}_k\widetilde{Q}.$$

LEMMA 4.4. 1) $R({}_{\tilde{P}}\tilde{A}, \tilde{B}_{\tilde{Q}})$ *is equivalent (via an isometric isomorphism of B-spaces) to*

$$R(\tilde{A}', \tilde{B}') \in \mathscr{L}(\widetilde{\mathscr{L}(Q\mathscr{H}_1, P\mathscr{H}_2)}),$$

*where*

$$A'_k = PA_k|P\mathscr{H}_2, B'_k = QB_k|Q\mathscr{H}_1.$$

2) $R(\tilde{A}, \tilde{B})$ *has the* s.s.s.p. $\Rightarrow R({}_{\tilde{P}}\tilde{A}, \tilde{B}_{\tilde{Q}})$ *has the* s.s.s.p.
3) $R(\tilde{A}, \tilde{B})$ *is pseudoalgebraic* $\Rightarrow R({}_{\tilde{P}}\tilde{A}, \tilde{B}_{\tilde{Q}})$ *is pseudoalgebraic.*
4) $R(\tilde{A}, \tilde{B})$ *is pseudodiagonal* $\Rightarrow R({}_{\tilde{P}}\tilde{A}, \tilde{B}_{\tilde{Q}})$ *is pseudodiagonal.*

*Proof.* 1) the equivalence is implemented by

$$U:\widetilde{\mathscr{L}(Q\mathscr{H}_1, P\mathscr{H}_2)} \to \tilde{P}\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}\tilde{Q}$$

defined by

$$U(\tilde{T}) = \tilde{P}\tilde{T}\tilde{Q}\tilde{Q} \quad (T:Q\mathscr{H}_1 \to P\mathscr{H}_2).$$

Parts 2), 3), 4) follow from the fact that if $p(z)$ is a polynomial, then

$$p(R({}_{\tilde{P}}\tilde{A}, \tilde{B}_{\tilde{Q}}))(\tilde{P}\tilde{X}\tilde{Q}) = \tilde{P}p(R(\tilde{A}, \tilde{B}))(\tilde{X})\tilde{Q}.$$

LEMMA 4.5. *Let $Q \in \mathscr{L}(\mathscr{H}_1)$ satisfy $Q^2 = Q$ and $\tilde{Q} \neq 0$. Let $T \in \mathscr{L}(\mathscr{H}_2)$ be polynomially compact and let $p(z)$ be the monic minimal polynomial of $\tilde{T}$. Define $\tilde{R}$ on $\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}\tilde{Q}$ by $\tilde{R}(\tilde{X}) = \tilde{T}\tilde{X}$. If $\tilde{R}$ is pseudodiagonal, then $p$ has simple roots.*

*Proof.* Let $T'$ be a compact perturbation of $T$ with minimal polynomial $p(z)$ [**20**]. Suppose $\lambda$ is a multiple root of $p(z)$; we may assume $\lambda = 0$. It follows from [**10**, Lemma 2.14] that there is an invertible operator $J \in \mathscr{L}(\mathscr{H}_2)$ and an orthogonal decomposition

$$(4.1) \qquad \mathscr{H}_2 = \mathscr{L}_1 \oplus \mathscr{L}_2$$

such that the matrix of $J^{-1}T'J$ relative to (4.1) is of the form

$$\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

where $T_1$ is nilpotent and $T_2$ is invertible.
Consider the decomposition

$$\mathscr{L}_1 = \ker T_1 \oplus (\mathscr{L}_1 \ominus \ker T_1);$$

the matrix of $J^{-1}T'J$ relative to

$$(4.2) \qquad \mathscr{H}_2 = \ker T_1 \oplus (\mathscr{L}_1 \ominus \ker T_1) \oplus \mathscr{L}_2$$

is of the form

$$\begin{pmatrix} 0 & A & 0 \\ 0 & C & 0 \\ 0 & 0 & T_2 \end{pmatrix}$$

where $C$ is nilpotent, and it follows readily that $\begin{pmatrix} A \\ C \end{pmatrix}$ is not compact. Thus $P = A^*A + C^*C \geqq 0$ is not compact, so there is an orthogonal decomposition of $\mathscr{L}_1' \equiv \mathscr{L}_1 \ominus \ker T_1$,

$$\mathscr{L}_1' = \mathscr{K}_1 \oplus \mathscr{K}_2, \ \dim \mathscr{K}_1 = \infty \tag{4.3}$$

such that relative to (4.3), $P = P_1 \oplus P_2$ with $P_1$ invertible.

Consider the orthogonal decompositions

$$\begin{aligned} \mathscr{H}_1 &= Q\mathscr{H}_1 \oplus (Q\mathscr{H}_1)^{\perp} \text{ and} \\ \mathscr{H}_2 &= \ker T_1 \oplus \mathscr{L}_1' \oplus \mathscr{L}_2. \end{aligned} \tag{4.4}$$

Suppose $X:\mathscr{H}_1 \to \mathscr{H}_2$ satisfies

$$(\widetilde{R} - \alpha)(\widetilde{X}) = 0 \quad \text{for some } \alpha.$$

The matrix of $J^{-1}X$ relative to (4.4) is of the form

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \\ X_{31} & X_{32} \end{pmatrix}$$

and since $\widetilde{T}\widetilde{X} = \alpha\widetilde{X}$, then

$$\widetilde{J}^{-1}\widetilde{T}'\widetilde{J}\widetilde{J}^{-1}\widetilde{X} = \alpha\widetilde{J}^{-1}\widetilde{X},$$

whence

$$\widetilde{A}\widetilde{X}_{21} = \alpha\widetilde{X}_{11}, \ \widetilde{C}\widetilde{X}_{21} = \alpha\widetilde{X}_{21}. \tag{4.5}$$

If $\alpha \neq 0$, then since $C$ is nilpotent, it follows that $\widetilde{X}_{21} = 0$. If $\alpha = 0$, then (4.5) implies that $PX_{21}$ is compact. Relative to (4.3), $P = P_1 \oplus P_2$ and

$$X_{21} = \begin{pmatrix} X_{21}^{(1)} \\ X_{21}^{(2)} \end{pmatrix}$$

and since $P_1$ is invertible, $X_{21}^{(1)}$ is compact. Thus, in either case, the matrix of $J^{-1}X$ relative to

$$\mathscr{H}_1 = Q\mathscr{H}_1 \oplus (Q\mathscr{H}_1)^{\perp} \quad \text{and } \mathscr{H}_2 = \ker T_1 \oplus \mathscr{K}_1 \oplus \mathscr{K}_2 \oplus \mathscr{L}_2 \tag{4.6}$$

is of the form

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21}^{(1)} & X_{22}^{(1)} \\ X_{21}^{(2)} & X_{22}^{(2)} \\ X_{31} & X_{32} \end{pmatrix} \tag{4.7}$$

with $X_{21}^{(1)}$ compact.

Let $V:Q\mathscr{H}_1 \to \mathscr{K}_1$ denote an (infinite rank) isometric mapping. Now $Q \in \mathscr{L}(\mathscr{H}_1)$ has a matrix of the form

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$$\begin{pmatrix} 1 & D \\ 0 & 0 \end{pmatrix}$$

relative to

$$\mathscr{H}_1 = Q\mathscr{H}_1 \oplus (Q\mathscr{H}_1)^{\perp}.$$

Let $Y:\mathscr{H}_1 \to \mathscr{H}_2$ have the matrix

$$\begin{pmatrix} 0 & 0 \\ V & VD \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ V & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & D \\ 0 & 0 \end{pmatrix}$$

relative to (4.6); thus

$$\widetilde{Y} \in \widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}\widetilde{Q}.$$

If

$$X_0 \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2) \quad \text{and} \quad \widetilde{X} = \widetilde{X}_0\widetilde{Q} \in \bigvee_{\alpha} \ker(\widetilde{R} - \alpha),$$

then $J^{-1}X$ has a matrix of the form (4.7),

$$\|\widetilde{J}\widetilde{Y} - \widetilde{X}\| = \|\widetilde{J}(\widetilde{Y} - \widetilde{J}^{-1}\widetilde{X})\| \geqq (1/\|\widetilde{J}^{-1}\|)\|\widetilde{Y} - \widetilde{J}^{-1}\widetilde{X}\|$$
$$\geqq (1/\|\widetilde{J}^{-1}\|)\|\widetilde{V} - \widetilde{X}_{21}^{(1)}\| = 1/\|\widetilde{J}^{-1}\| \quad \text{(using (4.7))}.$$

Thus

$$\operatorname{dist}[\widetilde{J}\widetilde{Y}, \bigvee_{\alpha} \ker(\widetilde{R} - \alpha)] \geqq 1/\|\widetilde{J}^{-1}\| > 0$$

and so $\widetilde{R}$ is not pseudodiagonal; the proof is complete.

LEMMA 4.6. *Let $Q \in \mathscr{L}(\mathscr{H}_1)$ satisfy $Q^2 = Q$, $\widetilde{Q} \neq 0$. For $T \in \mathscr{L}(\mathscr{H}_2)$, define $\widetilde{R}$ on $\widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}\widetilde{Q}$ by $\widetilde{R}(\widetilde{X}) = \widetilde{T}\widetilde{X}$. Then we have*

1) *$\widetilde{R}$ has the* s.s.s.p. *$\Rightarrow \sigma(\widetilde{T})$ is finite.*

2) *$\widetilde{R}$ is pseudoalgebraic $\Rightarrow \widetilde{T}$ is algebraic.*

3) *$\widetilde{R}$ is pseudodiagonal $\Rightarrow \widetilde{T}$ is a linear combination of commuting idempotents that are polynomials in $\widetilde{T}$.*

*Proof.* 1) Let $W \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ be a partial isometry with initial space $Q\mathscr{H}_1$ that maps onto $\mathscr{H}_2$. Let $V = WQ$; since $V$ is right invertible, there exists $\delta > 0$ such that if

$$X \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2) \quad \text{and} \quad \|\widetilde{X} - \widetilde{V}\| < \delta,$$

then $\widetilde{X}$ is right invertible and $\widetilde{X}\widetilde{X}^* \in \widetilde{\mathscr{L}(\mathscr{H}_2)}$ is invertible. Since $\widetilde{R}$ has the s.s.s.p., there exists $\widetilde{X} \in \widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}\widetilde{Q}$ such that

$$\|\widetilde{X} - \widetilde{V}\| < \delta \quad \text{and} \quad \widetilde{X} = \sum_{j=1}^{m} \widetilde{X}_j,$$

where

$$\lim_{k\to\infty} \|(\widetilde{T} - \lambda_j)^k \widetilde{X}_j\|^{1/k} = 0 \quad \text{for some } \lambda_j.$$

Thus

$$\lim_{k\to\infty} \left\|\prod_{j=1}^{m} (\widetilde{T} - \lambda_j)^k \widetilde{X}\right\|^{1/k} = 0,$$

and since $\widetilde{X}\widetilde{X}^*$ is invertible, it follows that

$$\lim_{k\to\infty} \left\|\prod_{j=1}^{m} (\widetilde{T} - \lambda_j)^k\right\|^{1/k} = 0.$$

Since

$$\prod_{j=1}^{m} (\widetilde{T} - \lambda_j)$$

is quasinilpotent, the spectral mapping theorem implies that $\sigma(\widetilde{T})$ is finite.

2) If $\widetilde{R}$ is pseudoalgebraic, we define $V$ and

$$\widetilde{X} = \sum_{j=1}^{m} \widetilde{X}_j$$

as above, except now $p_j(\widetilde{T})\widetilde{X}_j = 0$ for a monic polynomial $p_j$. Then $p = p_1 p_2 \dots p_m$ satisfies $p(\widetilde{T})\widetilde{X} = 0$, whence

$$p(\widetilde{T}) = p(\widetilde{T})\widetilde{X}(\widetilde{X}^*(\widetilde{X}\widetilde{X}^*)^{-1}) = 0;$$

thus $\widetilde{T}$ is algebraic.

3) If $\widetilde{R}$ is pseudodiagonal, then 2) implies that $\widetilde{T}$ is algebraic, and Lemma 4.5 shows that the minimal polynomial of $\widetilde{T}$ has simple roots. Thus the operator $T'$ in the proof of Lemma 4.5 is similar to a normal operator with finite spectrum and so is a linear combination of commuting idempotents that are polynomials in $T'$. Since $\widetilde{T}' = \widetilde{T}$, the result follows.

THEOREM 4.7. *Suppose* $\{\widetilde{B}_k\}_{k=1}^p$ *is independent modulo quasinilpotents and* $\sigma(\widetilde{B}_k) = \{0\}$, $k > p$. *Let* $\mathcal{A}$ *denote the closed algebra generated by* $\widetilde{1}, \widetilde{A}_1, \dots, \widetilde{A}_p$. *Then we have*

1) $R(\widetilde{A}, \widetilde{B})$ *has the* s.s.s.p. $\Rightarrow \dim \mathcal{A}^\wedge < \infty$;

2) $R(\widetilde{A}, \widetilde{B})$ *is pseudoalgebraic* $\Rightarrow \dim \mathcal{A} < \infty$;

3) $R(\widetilde{A}, \widetilde{B})$ *is pseudodiagonal* $\Rightarrow \dim \mathcal{A} < \infty$ *and* $\mathcal{A}$ *is semi-simple.*

*Proof.* Let $\mathcal{B}$ denote the closed algebra generated by $\widetilde{1}, \widetilde{B}_1, \dots, \widetilde{B}_n$. The hypothesis implies that $\{\widetilde{B}_1^\wedge, \dots, \widetilde{B}_p^\wedge\}$ is independent. Lemma 2.2 implies that there exists $\{M_k\}_{k=1}^p \subset \Gamma(\mathscr{B})$ such that

(4.8) $\det[\widetilde{B}_j^\wedge(M_k)]_{1 \leqq j,k \leqq p} \neq 0.$

Lemma 2.6 implies that for each $k$, $1 \leqq k \leqq p$, there exists an orthogonal projection $P_k$, $\widetilde{P}_k \neq 0$, such that

(4.9) $\widetilde{B}_j\widetilde{P}_k = \widetilde{B}_j^\wedge(M_k)\widetilde{P}_k, \quad j = 1, \ldots, n.$

A calculation using (4.9) shows that if $\widetilde{Y} \in \widetilde{\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)}$ and $\widetilde{X} = \widetilde{Y}\widetilde{P}_k$, then

(4.10) $$R(\widetilde{A}, \widetilde{B}_{\widetilde{P}_k})(\widetilde{X}) = \left(\sum_{j=1}^p \widetilde{B}_j^\wedge(M_k)A_j\right)\widetilde{X}$$

(since $\widetilde{B}_j^\wedge = 0$ for $j > p$).

(4.11) Let $$\widetilde{A}_k' = \sum_{j=1}^p \widetilde{B}_j^\wedge(M_k)\widetilde{A}_j \quad (1 \leqq k \leqq p).$$

1) If $R(\widetilde{A}, \widetilde{B})$ has the s.s.s.p., then so does $R(\widetilde{A}, \widetilde{B}_{\widetilde{P}_k})$ $(1 \leqq k \leqq p)$ (Lemma 4.4 (2)). Lemma 4.6 (1), (4.10), and (4.11) imply that $\sigma(\widetilde{A}_k')$ is finite, so (4.8) implies that $\sigma(\widetilde{A}_k)$ is finite $(1 \leqq k \leqq p)$. Thus there exists a polynomial $p_k$ such that

$$\sigma(p_k(\widetilde{A}_k)) = \{0\},$$

whence

$$p_k(\widetilde{A}_k^\wedge) = p_k(\widetilde{A}_k)^\wedge = 0 \quad (1 \leqq k \leqq p);$$

it follows that $\mathscr{A}^\wedge$ is finite dimensional.

2) If $R(\widetilde{A}, \widetilde{B})$ is pseudoalgebraic, then we may proceed as above (using Lemma 4.4 (3), Lemma 4.6 (2), (4.10), and (4.11)) to conclude that $\widetilde{A}_k'$ is algebraic $(1 \leqq k \leqq p)$; thus (4.8) and (4.11) imply that $\widetilde{A}_k$ is algebraic $(1 \leqq k \leqq p)$ and so $\dim \mathscr{A} < \infty$.

3) If $R(\widetilde{A}, \widetilde{B})$ is pseudodiagonal, then Lemmas 4.4 (4) and 4.6 (3) imply that $\widetilde{A}_k'$ is a linear combination of commuting idempotents that are polynomials in $\widetilde{A}_k'$ $(1 \leqq k \leqq p)$. It follows from (4.8) and (4.11) that each $\widetilde{A}_k$ is a linear combination of idempotents in $\mathscr{A}$, and the result follows.

We recall that if $S$ and $T$ are commuting Banach space operators then

$$\sigma(S + T) \subset \sigma(S) + \sigma(T) \text{ [7, 22]};$$

in particular, if $\sigma(T) = \{0\}$, then

$$\sigma(S + T) = \sigma(S).$$

If $A = \{A_i\}_{i=1}^n$ and $B = \{B_i\}_{i=1}^n$ are commutative $n$-tuples in $\mathscr{L}(\mathscr{H})$ and

$$R_i(\widetilde{X}) = \widetilde{A}_i\widetilde{X}\widetilde{B}_i \quad (\widetilde{X} \in \widetilde{\mathscr{L}(\mathscr{H})}),$$

then

$$\sigma(R_i) = \sigma(\widetilde{A}_i)\sigma(\widetilde{B}_i) \quad \textbf{[17]} \text{ (cf. } \textbf{[9]}\text{)},$$

and thus

$$\sigma(R(\widetilde{A}, \widetilde{B})) \subset \sum_{i=1}^{n} \sigma(\widetilde{A}_i)\sigma(\widetilde{B}_i) \quad \text{(cf. } \textbf{[7]}\text{)}.$$

THEOREM 4.8. 1) $R(\widetilde{A}, \widetilde{B})$ *has the* s.s.s.p. *if and only if* $\sigma(R(\widetilde{A}, \widetilde{B}))$ *is finite.*

2) $R(\widetilde{A}, \widetilde{B})$ *is pseudoalgebraic if and only if* $R(\widetilde{A}, \widetilde{B})$ *is algebraic.*

*Proof.* We may assume that $\{\widetilde{B}_k\}_{k=1}^{p}$ is linearly independent modulo quasinilpotents, $\{\widetilde{A}_k\}_{k=1}^{p}$ is independent modulo quasinilpotents, and for $k > p$, either $\widetilde{A}_k$ or $\widetilde{B}_k$ is quasinilpotent. Let

$$\widetilde{A}' = (\widetilde{A}_1, \dots, \widetilde{A}_p), \quad \widetilde{A}'' = (\widetilde{A}_{p+1}, \dots, \widetilde{A}_n),$$
$$\widetilde{B}' = (\widetilde{B}_1, \dots, \widetilde{B}_p), \quad \text{and} \quad \widetilde{B}'' = (\widetilde{B}_{p+1}, \dots, \widetilde{B}_n);$$

then

$$\sigma(R(\widetilde{A}'', \widetilde{B}'')) = \{0\}$$

(see the preceding remarks).

1) Assume that $R(\widetilde{A}, \widetilde{B})$ has the s.s.s.p. Since $R(\widetilde{A}'', \widetilde{B}'')$ is quasinilpotent and commutes with $R(\widetilde{A}, \widetilde{B})$, it follows that $R(\widetilde{A}', \widetilde{B}')$ has the s.s.s.p., so the proof of Theorem 4.7 (1) shows that $\sigma(\widetilde{A}_i)$ is finite $(1 \leqq i \leqq p)$. Similarly, since $R(\widetilde{B}'^*, \widetilde{A}'^*)$ has the s.s.s.p., then $\sigma(\widetilde{B}_i)$ is finite $(1 \leqq i \leqq p)$; thus

$$\sigma(R(\widetilde{A}, \widetilde{B})) = \sigma(R(\widetilde{A}', \widetilde{B}')) \subset \sum_{i=1}^{p} \sigma(\widetilde{A}_i)\sigma(\widetilde{B}_i)$$

is finite. The converse in 1) is true in general.

2) If $R(\widetilde{A}, \widetilde{B})$ is pseudoalgebraic, then $R(\widetilde{A}, \widetilde{B})$ has the s.s.s.p. Thus by 1), $\sigma(R(\widetilde{A}, \widetilde{B}))$ is finite, and Theorem 4.3 implies that $R(\widetilde{A}, \widetilde{B})$ is algebraic. The converse is clear.

In the sequel let $T$ be a finitely diagonal operator on a $B$-space $\mathscr{X}$. Let

$$\{E_i\}_{i=1}^{n} \subset \mathscr{L}(\mathscr{X})$$

be a family of commuting idempotents such that

$$\sum_{i=1}^{n} E_i = 1_{\mathscr{X}},$$

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$$E_iE_j = 0 \quad \text{for } i \neq j, \quad \text{and}$$

$$T = \sum_{i=1}^{n} \alpha_i E_i \quad \text{for certain scalers } \{\alpha_i\}_{i=1}^{n}.$$

We record without proof the following elementary result.

LEMMA 4.9. 1) *If* $S \in \mathcal{L}(\mathcal{X})$ *is pseudodiagonal,* $T$ *(as above) is finitely diagonal, and* $SE_i = E_iS (1 \leqq i \leqq n)$, *then* $T + S$ *is pseudodiagonal.*

2) *If* $S \in \mathcal{L}(\mathcal{X})$ *is a linear combination of commuting idempotents, then* $S$ *is finitely diagonal.*

THEOREM 4.10. *The following are equivalent:*

1) $R(\widetilde{A}, \widetilde{B})$ *is pseudodiagonal;*

2) $R(\widetilde{A}, \widetilde{B})$ *is finitely diagonal;*

3) $R(\widetilde{A}, \widetilde{B}) = R(\widetilde{A}', \widetilde{B}')$ *where* $\widetilde{A}' = (\widetilde{A}'_1, \dots, \widetilde{A}'_p)$ *and the* $A'_i$*'s are simultaneously finitely diagonal (i.e., finitely diagonal with respect to the same family of idempotents), and* $\widetilde{B}' = (\widetilde{B}'_1, \dots, \widetilde{B}'_p)$ *and the* $\widetilde{B}'_i$*'s are simultaneously finitely diagonal.*

*Proof.* 2) $\Rightarrow$ 1) is clear, and 3) $\Rightarrow$ 2) follows from Lemma 4.9 (2), since in this case $R(\widetilde{A}, \widetilde{B})$ may be expressed as a linear combination of $p^2$ idempotents (each of the form $\widetilde{X} \to \widetilde{E}\widetilde{X}\widetilde{F}$, where $\widetilde{E}$ and $\widetilde{F}$ are idempotents).

Assume now that $R(\widetilde{A}, \widetilde{B})$ is pseudodiagonal; we may rewrite $R$ so that $\{\widetilde{A}_1, \dots, \widetilde{A}_n\}$ is independent, $\{\widetilde{B}_1, \dots, \widetilde{B}_n\}$ is independent, $\{\widetilde{B}_1, \dots, \widetilde{B}_p\}$ is independent modulo quasinilpotents, and $\sigma(\widetilde{B}_i) = \{0\}$ for $i > p$. The proof of Theorem 4.7 (3) shows that each $\widetilde{A}_k$ $(1 \leqq k \leqq p)$ is a linear combination of idempotents in the algebra $\mathcal{A}$ generated by $\widetilde{1}, \widetilde{A}_1, \dots, \widetilde{A}_p$, from which it follows that $\widetilde{A}_1, \dots, \widetilde{A}_p$ are simultaneously finitely diagonal.

Moreover, Theorem 4.7 (3) shows that $\mathcal{A}$ is semi-simple, so $\{\widetilde{A}_1, \dots, \widetilde{A}_p\}$ is independent modulo quasinilpotents. Thus, by taking adjoints and applying the preceding argument, we conclude that $\widetilde{B}_1, \dots, \widetilde{B}_p$ are simultaneously finitely diagonal using idempotents generated by $\widetilde{B}_1, \dots, \widetilde{B}_p$. It follows readily that

$$\sum_{k=1}^{p} L_{\widetilde{A}_k} R_{\widetilde{B}_k}$$

is finitely diagonal, so Lemma 4.9 (1) implies that

$$S \equiv R(\widetilde{A}, \widetilde{B}) - \sum_{k=1}^{p} L_{\widetilde{A}_k} R_{\widetilde{B}_k}$$

is pseudodiagonal. Since $S$ is also quasinilpotent, Theorem 4.8 (2) implies that $S$ is pseudodiagonal and nilpotent; hence $S = 0$ and 3) follows. (Moreover, [**15**, Theorem 3] implies that $p = n$.) The proof is now complete.

**5. Elementary operators on $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$.** In this section we study pseudoalgebraic and pseudodiagonal elementary operators on $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. We record for future use the following elementary observation: if $\mathcal{M}$ and $\mathcal{F}$ are (not necessarily closed) linear subspaces of a Hilbert space $\mathcal{H}$, with $\dim \mathcal{F} < \infty$ and $\dim \mathcal{M} = \infty$, then

$$\dim \mathcal{M} \cap (\mathcal{F}^{\perp}) = \infty.$$

LEMMA 5.1. *Let $\mathcal{H}$ be a complex separable infinite dimensional Hilbert space. Let*

$$\{T_{j,k}\}_{k=1}^{n_j} \subset \mathcal{L}(\mathcal{H}) \quad (j = 1, 2, \dots)$$

*be given, with $n_j < \infty$ and* rank $T_{j,1} = \infty$ *for each $j \geqq 1$. Then there exists an orthonormal sequence $\{e_j\}_{j=1}^{\infty}$ such that*

$$T_{j,1}e_j \neq 0 \quad \text{and} \quad \mathcal{H}_j \perp \mathcal{H}_k \quad \text{for } j \neq k,$$

*where*

$$\mathcal{H}_j = \text{c.l.m.}\{T_{j,k}e_j\}_{k=1}^{n_j}.$$

*Proof.* We can determine the $e_j$'s successively. Indeed, if $\{e_j\}_{j=1}^{m}$ satisfy the requirements for some $m \geqq 1$, we can choose

$$e_{m+1} \in \text{Ran}(T_{m+1,1}{}^*), \quad \|e_{m+1}\| = 1,$$

such that $e_{m+1}$ is orthogonal to

$$\mathcal{F} = \text{c.l.m.}\left\{ \{e_j\}_{j=1}^{m} \cup \bigcup_{\substack{1 \leqq j \leqq m \\ 1 \leqq k \leqq n_{m+1}}} T_{m+1,k}{}^*\mathcal{H}_j \right\};$$

this is possible since

$$\dim \mathcal{F} < \infty \quad \text{and} \quad \dim \text{Ran } R_{m+1,1}{}^* = \infty.$$

PROPOSITION 5.2. *Let $R(A, B) \in \mathcal{L}(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ be such that both $\{A_1, \dots, A_n\}$ and $\{B_1, \dots, B_n\}$ are linearly independent modulo the ideal of finite rank operators. Then the range of $R(A, B)$ is not contained in the set of finite rank operators in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$.*

*Proof.* Apply Lemma 5.1 to the system $\{T_{j,k}\}_{k=1}^{m_j}$, $j = 1, 2, \dots$, where $m_j = n$ and $T_{j,k} = B_k$ $(1 \leqq k \leqq n)$. (Note that $B_1$ is not a finite rank operator.) Thus there exists an orthonormal sequence $\{e_j\}_{j=1}^{\infty}$ such that $B_1e_j \neq 0$ and $\mathcal{M}_j \perp \mathcal{M}_h$ for $j \neq h$, where

$$\mathcal{M}_j = \text{c.l.m.}\{B_ie_j\}_{i=1}^{n}.$$

Let $P_j$ denote the orthogonal projection onto $\langle e_j \rangle$. Since

$$B_1P_je_j = B_1e_j \neq 0,$$

there exists a maximal linearly independent subset of $\{B_1P_j, \ldots, B_nP_j\}$ of the form $\{B_1P_j, B_{j_2}P_j, \ldots, B_{j_{n_j}}P_j\}$; equivalently, $\{B_1e_j, B_{j_2}e_j, \ldots, B_{j_{n_j}}e_j\}$ is a maximal independent subset of $\{B_ie_j\}_{i=1}^n$. Let $j_1 = 1$; then for $i \geqq 1$,

$$B_iP_j = \sum_{k=1}^{n_j} c_{ijk}B_{j_k}P_j,$$

where

$$c_{1jk} = \delta_{k1}.$$

For $X \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$,

$$\sum_{i=1}^{n} A_iXB_iP_j = \sum_{k=1}^{n_j}\left(\sum_{i=1}^{n} c_{ijk}A_i\right)XB_{j_k}P_j.$$

Let

$$A_{jk} = \sum_{i=1}^{n} c_{ijk}A_i.$$

Note that since $c_{1j1} = 1$ and $\{A_1, \ldots, A_n\}$ is independent modulo finite rank operators, then

$$A_{j1} = \sum_{i=1}^{n} c_{ij1}A_i$$

has infinite rank for each $j \geqq 1$.

We may now apply Lemma 5.1 to the system $\{A_{jk}\}_{k=1}^{n_j}$, $j = 1, 2, \ldots$; thus there exists an orthonormal sequence

$$\{f_j\}_{j=1}^{\infty} \subset \mathscr{H}_2$$

such that $A_{j_1}f_j \neq 0$ and $\mathscr{N}_j \perp \mathscr{N}_h$ for $j \neq h$, where

$$\mathscr{N}_j = \text{c.l.m.}\{A_{jk}f_j\}_{k=1}^{n_j}.$$

We define $X \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ as follows. For $j \geqq 1$, there is a unit vector

$$u_j \in \mathscr{M}_j \ominus \langle B_{j_k}P_je_j\rangle_{k=2}^{n_j}$$

such that $(B_1e_j, u_j) \neq 0$. We define

$$Xu_j = f_j \quad \text{and} \quad X|\mathscr{H} \ominus \langle \{u_j\}_{j=1}^{\infty}\rangle = 0;$$

thus

$$X(B_1e_j) = \alpha_jf_j \quad \text{with } \alpha_j \neq 0.$$

Now

$$R(A, B)(X)e_j = \left(\sum_{i=1}^{n} A_iXB_iP_j\right)e_j = \sum_{k=1}^{n_j} A_{j_k}XB_{j_k}P_je_j = \alpha_jA_{j1}f_j,$$

and it follows that $\{R(A, B)(X)e_j\}_{j=1}^{\infty}$ is an orthogonal sequence. Since $R(A, B)(X)$ has infinite rank, the proof is complete.

Let $J$ be an ideal set. For a separable infinite dimensional Hilbert space $\mathscr{H}$, let $\mathscr{J}_{\mathscr{H}}$ denote the unique ideal in $\mathscr{L}(\mathscr{H})$ with ideal set $J$. For Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}_1 \approx \mathscr{H}_2 \approx \mathscr{H}$, let $\mathscr{J}_{\mathscr{H}_1\mathscr{H}_2}$ denote the operators in $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ affiliated with $\mathscr{J}_{\mathscr{H}}$. We use $\mathscr{J}$ to denote any of the sets $\mathscr{J}_{\mathscr{H}}, \mathscr{J}_{\mathscr{H}_i}, \mathscr{J}_{\mathscr{H}_1\mathscr{H}_2}$. We omit the proof of the following elementary result.

LEMMA 5.3. *Let $R(A, B)$ be an elementary operator on $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$, where $A = (A_1, \dots, A_n)$, $B = (B_1, \dots, B_n)$. Then for a given ideal $\mathscr{J}$ of $\mathscr{L}(\mathscr{H})$,*

$$R = \sum_{i=1}^{p} L_{A_i'}R_{B_i'} + \sum_{i=p+1}^{r} L_{F_i}R_{B_i} + \sum_{i=r+1}^{n} L_{A_i}R_{G_i},$$

*where $\{A_i'\}_{i=1}^{p}$ ($\subset \langle A_1, \dots, A_n\rangle$) is independent modulo $\mathscr{J}$, $\{B_i'\}_{i=1}^{p}$ ($\subset \langle B_1, \dots, B_n\rangle$) is independent modulo $\mathscr{J}$,*

$$F_i \in \mathscr{J} \cap \langle A_1, \dots, A_n\rangle \ (p + 1 \leqq i \leqq r), \quad \text{and}$$

$$G_i \in \mathscr{J} \cap \langle B_1, \dots, B_n\rangle \quad (r + 1 \leqq i \leqq n).$$

*(One or more of the three sums may be absent.)*

We refer to any such decomposition of $R$ as a *standard form of $R$ relative to $\mathscr{J}$*. Note that if $\operatorname{Ran} R \subset \mathscr{J}$ and $p \geqq 1$, then

$$\operatorname{Ran} \sum_{i=1}^{p} L_{A_i'}R_{B_i'} \subset \mathscr{J}.$$

Proposition 5.2 shows this cannot occur when $\mathscr{J} = \mathscr{F}$, the ideal of finite rank operators, and thus we have the following result.

COROLLARY 5.4. *Every elementary operator $R = R(A, B)$ on $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ whose range is contained in the finite rank operators is of the form*

$$R = \sum_{k=1}^{p} L_{A_k'}R_{B_k'},$$

*where for each $k$ at least one of $A_k'$ or $B_k'$ is a finite rank operator; moreover, $A_k' \in \mathscr{A}$ ($1 \leqq k \leqq p$) and $B_k' \in \mathscr{B}$ ($1 \leqq k \leqq p$).*

For an elementary operator $R$ on $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ and projections $P \in \mathscr{L}(\mathscr{H}_1)$, $Q \in \mathscr{L}(\mathscr{H}_2)$, define

$${}_PR_Q \in \mathscr{L}(\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2))$$

by

$$_PR_Q(X) = PR(X)Q.$$

LEMMA 5.5. *Let $R = R(A, B)$ be an elementary operator on $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$. Suppose $X \in \operatorname{Ran} R$ and $\mathscr{H}'$ is an infinite dimensional linear manifold contained in $\mathscr{H}_1 \ominus \ker X$. Let $\mathscr{M}_i$ be a finite dimensional subspace of $\mathscr{H}_i$, $i = 1, 2$. Then there exist rank one projections $P \in \mathscr{L}(\mathscr{H}_1)$, $Q \in \mathscr{L}(\mathscr{H}_2)$ and finite dimensional subspaces $\mathscr{H}_i' \subset \mathscr{H}_i \ominus \mathscr{M}_i$, $i = 1, 2$, such that*

1) $_QR_P \neq 0$, *i.e., there exists $Y \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ such that $QR(Y)P \neq 0$.*

2) $QR(Y)P = QR(P_{\mathscr{H}_2'}YP_{\mathscr{H}_1'})P \ \forall Y \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$;

3) *For $0 < a < 1$, there exists $X \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ such that*

$$X = P_{\mathscr{H}_2'}XP_{\mathscr{H}_1'}, \quad \|X\| = 1, \quad \operatorname{dist}(X, \ker {}_QR_P) \geqq a, \quad \textit{and}$$

$$\|_QR_P(X)\| \geqq a\|_QR_P\|.$$

*Proof.* Let

$$\mathscr{M} = \text{c.l.m.}\{B_k^*h : 1 \leqq k \leqq n,\ h \in \mathscr{M}_1\}$$

and let

$$\mathscr{N} = \text{c.l.m.}\{X^*A_kh : 1 \leqq k \leqq n,\ h \in \mathscr{M}_2\}.$$

Let $\mathscr{P} = \mathscr{M} \vee \mathscr{N} \subset \mathscr{H}_1$; since $\dim \mathscr{P} < \infty$, there exists a unit vector $x \in \mathscr{P}^\perp \cap \mathscr{H}'$. Let

$$\mathscr{H}_1' = \text{c.l.m.}\{B_1x, \dots, B_nx\}, \quad P = P_{\langle x \rangle},$$

$$\mathscr{H}_2' = \langle A_1^*Xx, \dots, A_n^*Xx \rangle, \quad Q = P_{\langle Xx \rangle};$$

clearly $\mathscr{H}_1' \perp \mathscr{M}_1$ and $\mathscr{H}_2' \perp \mathscr{M}_2$.

1) Let $Y \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ be such that $R(Y) = X$; then

$$(_QR_P)(Y)x = QR(Y)Px = QXPx = Xx \neq 0.$$

2) For each $h \in \mathscr{H}_1$, $Y \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ we have

$$QR(P_{\mathscr{H}_2'}YP_{\mathscr{H}_1'})Ph = \sum_{i=1}^n QA_iP_{\mathscr{H}_2'}YP_{\mathscr{H}_1'}(h, x)x$$

$$= \sum_{i=1}^n QA_iP_{\mathscr{H}_2'}YB_iPh.$$

Now

$$0 = (1 - P_{\mathscr{H}_2'})A_i^*Xx = (1 - P_{\mathscr{H}_2'})A_i^*QXx,$$

so

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$$QA_iP_{\mathcal{H}_2'} = QA_i \quad \text{and}$$

$$\sum_{i=1}^{n} QA_iP_{\mathcal{H}_2'}YB_iPh = \sum_{i=1}^{n} QA_iYB_iPh = ({}_QR_P)(Y)h;$$

thus

$$QR(Y)P = QR(P_{\mathcal{H}_2'}YP_{\mathcal{H}_1'})P.$$

3) Let

$$W = \{X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2): X = P_{\mathcal{H}_2'}XP_{\mathcal{H}_1'}\},$$

a closed subspace of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Since

$${}_QR_P|_{W:W} \rightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$$

is nonzero (by 1) and 2) ), for $0 < a < 1$ there exists $X \in W$, $\|X\| = 1$, such that

$$\text{dist}(X, \ker {}_QR_P|W) \geqq a \quad \text{and} \quad \|({}_QR_P|W)(X)\| \geqq a\|{}_QR_P|W\|;$$

thus 2) implies

$$\|{}_QR_P(X)\| \geqq a\|{}_QR_P\|.$$

Relative to the decompositions

$$\mathcal{H}_1 = \mathcal{H}_1' \oplus (\mathcal{H}_1')^{\perp} \quad \text{and} \quad \mathcal{H}_2 = \mathcal{H}_2' \oplus (\mathcal{H}_2')^{\perp},$$

the operator matrix of $X$ is of the form

$$X = \begin{pmatrix} X' & 0 \\ 0 & 0 \end{pmatrix}$$

$X' \in \mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')$, $\|X'\| = 1$. If $Y \in \ker {}_QR_P$ and

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

then since

$$Y' \equiv \begin{pmatrix} 0 & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

is in $\ker {}_QR_P$, then $Y - Y' \in \ker {}_QR_P|W$. Thus

$$\|X - Y\| \geqq \|X' - Y_{11}\| = \|X - (Y - Y')\| \geqq a$$

and it follows that

$$\text{dist}(X, \ker {}_QR_P) \geqq a.$$

PROPOSITION 5.6. *Let* $\{R_j\}_{j=1}^{\infty}$ *be a sequence of elementary operators on* $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ *such that*

$$\text{Ran } R_j \not\subset \mathcal{F}(\mathcal{H}_1, \mathcal{H}_2) \quad \forall j.$$

*Then for* $0 < a < 1$, *there exists* $X_0 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, $\|X_0\| = 1$, *such that*

$$\text{dist}(X_0, \ker R_j) \geqq a, \quad \forall j.$$

*Proof.* Let $0 < a < 1$. We seek to produce sequences $\{M_1^{(j)'}\}_{j=1}^{\infty}$, $\{M_2^{(j)'}\}_{j=1}^{\infty}$ consisting of finite dimensional subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and to produce rank one projections $Q_j \in \mathcal{L}(\mathcal{H}_2)$, $P_j \in \mathcal{L}(\mathcal{H}_1)$, and $X_j \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that for each $j \geqq 1$,

1) $\quad {}_{Q_j}R_{jP_j} \neq 0,$

2) $\quad {}_{Q_j}R_{jP_j}(Y) = Q_jR_j(P_{\mathcal{M}_2}(j)'YP_{\mathcal{M}_1}(j)')P_j \ \forall Y \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2).$

3) $\quad X_j = P_{\mathcal{M}_2}(j)'X_jP_{\mathcal{M}_1}(j)', \quad \|X_j\| = 1, \text{dist}(X_j, \ker {}_{Q_j}R_{jP_j}) \geqq a,$

and

$$\|{}_{Q_j}R_{jP_j}(X_j)\| \geqq a\|{}_{Q_j}R_{jP_j}\|.$$

Let $Y_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be such that $W_1 \equiv R_1(Y_1)$ has infinite rank. Let $\mathcal{M}^{(1)'} = \ker(W_1^{\perp})$ and let

$$\mathcal{M}_1^{(1)} = \{0\} \subset \mathcal{H}_1, \quad \mathcal{M}_2^{(1)} = \{0\} \subset \mathcal{H}_2.$$

Lemma 5.5 implies that there exist rank one projections $P_1 \in \mathcal{L}(\mathcal{H}_1)$, $Q_1 \in \mathcal{L}(\mathcal{H}_2)$, finite dimensional subspaces

$$\mathcal{M}_i^{(1)'} \subset \mathcal{H}_i \ominus \mathcal{M}_i^{(1)}, \quad i = 1, 2, \quad \text{and}$$

$$X_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$$

such that 1)-3) hold for $j = 1$. Suppose we have chosen $\mathcal{M}_1^{(k)'}$, $\mathcal{M}_2^{(k)'}$, $Q_k$, $P_k$, and $X_k$ satisfying 1)-3) for $1 \leqq k \leqq j - 1$. Choose $Y_j \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that $W_j \equiv R_j(Y_j)$ has infinite rank, and let

$$\mathcal{M}^{(j)'} = \mathcal{H}_1 \ominus (\ker W_j).$$

Let

$$\mathcal{M}_1^{(j)} = \bigvee_{k=1}^{j-1} \mathcal{M}_1^{(k)'} \quad \text{and} \quad \mathcal{M}_2^{(j)} = \bigvee_{k=1}^{j-1} \mathcal{M}_2^{(k)'}.$$

Lemma 5.5 implies that there exist rank one projections $P_j \in \mathcal{L}(\mathcal{H}_1)$, $Q_j \in \mathcal{L}(\mathcal{H}_2)$, finite dimensional subspaces

$$\mathcal{M}_i^{(j)'} \subset \mathcal{H}_i \ominus \mathcal{M}_i^{(j)} \quad (i = 1, 2), \quad \text{and}$$

$$X_j \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2),$$

such that 1)-3) hold for $j$, so 1)-3) hold for all $j \geqq 1$.

Note that the construction shows that

$$\mathcal{M}_1^{(j)'} \perp \mathcal{M}_1^{(k)'} \quad \text{for } j \neq k \quad \text{and}$$

$$\mathcal{M}_2^{(j)'} \perp \mathcal{M}_2^{(k)'} \quad \text{for } j \neq k;$$

we may thus define

$$X_0 = \sum_{j=1}^{\infty} X_j$$

(convergence in the strong operator topology), and clearly $\|X_0\| = 1$. If $Y \in \ker R_j$, then 2) shows that

$$P_{\mathscr{H}_2^{(j)\prime}} Y P_{\mathscr{H}_1^{(j)\prime}} \in \ker {}_{Q_j}R_{jP_j},$$

so from 3) we have

$$\begin{aligned}\|X_0 - Y\| &\geqq \|P_{\mathscr{H}_2^{(j)\prime}} X_0 P_{\mathscr{H}_1^{(j)\prime}} - P_{\mathscr{H}_2^{(j)\prime}} Y P_{\mathscr{H}_1^{(j)\prime}}\| \\ &= \|X_j - P_{\mathscr{H}_2^{(j)\prime}} Y P_{\mathscr{H}_1^{(j)\prime}}\| \geqq a;\end{aligned}$$

the proof is complete.

THEOREM 5.7. *If* $R(A, B)$ *is pseudoalgebraic, then there exist projections* $P \in \mathscr{B}$, $Q \in \mathscr{A}$ *such that* $\operatorname{rank}(1 - P) < \infty$, $\operatorname{rank}(1 - Q) < \infty$, *and* $R({}_QA, B_P)$ *is algebraic.*

*Proof.* Since $R(\widetilde{A}, \widetilde{B})$ is also pseudoalgebraic, we may use Theorems 4.7 and 4.8 and a spectral decomposition to reduce to the case when $R(\widetilde{A}, \widetilde{B})$ is nilpotent.

Suppose

$$\operatorname{Ran}(R(A, B)^j) \not\subset \mathscr{F}(\mathscr{H}_1, \mathscr{H}_2) \quad \forall j \geqq 1$$

and let $0 < a < 1$. Apply Proposition 5.6 and its proof to the sequence $\{R'_j\}_{j=1}^{\infty}$, where $R'_j = R(A, B)^j$: thus there exist projections $Q_j$, $P_j$ and $X_j \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ satisfying 1)-3) of Proposition 5.6. We have

$$\|X_j\| = 1, \quad \operatorname{dist}(X_j, \ker {}_{Q_j}R'_{jP_j}) \geqq a, \quad \text{and}$$
$$\|{}_{Q_j}R'_{jP_j}(X_j)\| \geqq a\|{}_{Q_j}R'_{jP_j}\|,$$

so if

$$R_j = (1/\|{}_{Q_j}R'_{jP_j}\|)\,{}_{Q_j}R'_{jP_j},$$

then $\|R_j(X_j)\| \geqq a$.

Since $R(A, B)$ is pseudoalgebraic, there exist operators

$$X'_h \in \ker(R(A, B) - \lambda_h)^{m_h}$$

(for certain scalars $\lambda_h$ and natural numbers $m_h$), $h = 0, 1, \dots, r$, such that $\lambda_0 = 0$, $\lambda_h \neq 0$ for $h > 0$, and

$$\left\|X_0 - \sum_{h=0}^{r} X'_h\right\| \leqq a/2.$$

Since $R(\widetilde{A}, \widetilde{B})$ is nilpotent, the defining properties of $X'_h$ and $\lambda_h$ imply that $X_h'$ is compact for $h > 0$. Since the sequence $\{P_{\mathscr{M}_1}^{(j)\prime}\}_{j=1}^{\infty}$ of Proposition 5.6 satisfies

$$P_{\mathscr{M}_1}^{(j)\prime} \xrightarrow{w} 0,$$

it follows that

$$\|X_h' P_{\mathscr{M}_1}^{(j)\prime}\| \to 0,$$

whence

$$\|R_j(X_h')\| \to 0 \quad (h > 0).$$

Since $R_j(X_0') = 0$ for $j$ large and $R_j(X_0) = R_j(X_j)$ (from (2) of Proposition 5.6), we have

$$a/2 \geqq \overline{\lim_{j\to\infty}} \left\| R_j\left( X_0 - \sum_{h=0}^{r} X_h' \right) \right\|$$
$$= \overline{\lim_{j\to\infty}} \|R_j(X_0)\| = \overline{\lim_{j\to\infty}} \|R_j(X_j)\| \geqq a,$$

which is impossible.

We may thus assume that

$$\text{Ran } R(A, B)^m \subset \mathscr{F}(\mathscr{H}_1, \mathscr{H}_2) \quad \text{for some } m \geqq 1.$$

From Corollary 5.4 we may write

$$R^m = \sum_{i=1}^{p} L_{A_i'} R_{B_i'}, \quad A_i' \in \mathscr{A}, \quad B_i' \in \mathscr{B},$$

and $A_i'$ or $B_i'$ has finite rank ($1 \leqq i \leqq p$). For $1 \leqq i \leqq p$, if $A_i'$ has finite rank, let $P_i$ denote the spectral idempotent of $A_i'$ corresponding to $\{0\} \subset \sigma(A_i')$; if $A_i'$ has infinite rank, let $P_i = 1$; in either case,

$$P_i \in \text{Alg}(A_i') \subset \mathscr{A}$$

and $1 - P_i$ has finite rank. Thus

$$P = \prod_{i=1}^{p} P_i$$

is an idempotent in $\mathscr{A}$, $1 - P$ has finite rank, and $PA_i' = A_i'P = PA_i'P$ is nilpotent ($1 \leqq i \leqq p$). We may similarly construct an idempotent $Q \in \mathscr{B}$, $\text{rank}(1 - Q) < \infty$, such that $QB_i' = B_i'Q = QB_i'Q$ is nilpotent ($1 \leqq i \leqq p$). Clearly, ${}_PR^m{}_Q$ is nilpotent, so it follows that ${}_PR_Q$ is nilpotent (equivalently, $R({}_PA, B_Q)$ is nilpotent).

THEOREM 5.8. *Suppose $\{B_1, \ldots, B_p\}$ is linearly independent modulo essentially quasinilpotent operators with finite spectra and $B_k$, $k > p$, is*

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*an essentially quasinilpotent operator with finite spectrum. Then we have*

1) $R(A, B)$ *is pseudoalgebraic* $\Rightarrow A_k$ *is algebraic,* $1 \leqq k \leqq p$.

2) $R(A, B)$ *is pseudodiagonal and algebraic* $\Rightarrow A_k$ *is similar to a normal operator with finite spectrum,* $1 \leqq k \leqq p$.

*Proof.* Since $R(A, B)$ is pseudoalgebraic, there exist projections $P \in \mathcal{B}$, $Q \in \mathcal{A}$, $\operatorname{rank}(1 - P) < \infty$, $\operatorname{rank}(1 - Q) < \infty$, such that $R(_QA, B_P)$ is algebraic. Since $P$ commutes with each $B_i$ and $\operatorname{rank}(1 - P) < \infty$, it follows that $\{B_iP\}_{i=1}^p$ is independent modulo essentially quasinilpotent operators and $B_kP$ is essentially quasinilpotent with finite spectrum $(k > p)$. Thus there exists a projection $P' \in \mathcal{B}$, $\operatorname{rank}(1 - P') < \infty$, such that $B_kPP'$ is quasinilpotent $\forall k > p$, and clearly $\{B_iPP'\}_{i=1}^p$ is independent modulo quasinilpotents. If we can prove the conclusions of the theorem for $QA_1, \dots, QA_p$ (using $R(_QA, B_{PP'})$), then the same conclusions will hold for $A_1, \dots, A_p$ (since $\operatorname{rank}(1 - Q) < \infty$, $Q \in \mathcal{A}$).

We may thus assume that $\{B_1, \dots, B_p\}$ is independent modulo quasinilpotents, $B_k$ is quasinilpotent for $k > p$, and $R(A, B)$ is algebraic. Since $\{B_1^\wedge, \dots, B_p^\wedge\}$ is independent, Lemma 2.2 implies that there exists

$$\{N_k\}_{k=1}^p \subset \Gamma(\mathcal{B})$$

such that

$$(5.1) \qquad \det[B_j^\wedge(N_k)]_{1 \leqq j,k \leqq p} \neq 0.$$

Let $f$ be a monic polynomial such that $f(R(A, B)) = 0$. For $1 \leqq k \leqq p$, let

$$B'_{jk} = B_j^\wedge(N_k) \quad \text{and} \quad B''_{jk} = B_j - B_j^\wedge(N_k) \in N_k.$$

Since $B_j = B'_{jk} + B''_{jk}$, it follows that

$$f(R(A, B)) = f\left(\sum_{j=1}^p L_{A_j}R_{B'_{jk}}\right) + R(T, S),$$

where $S \subset N_k$ (since its elements are generated by the $B''_{jk}$'s and the $B'_k$'s for $k > p$). Thus we have

$$0 = f(R(A, B))(X) = f\left(\sum_{j=1}^p B_j^\wedge(N_k)A_j\right)X + \sum_{j=1}^m T_jXS_j,$$

$$S_j \in N_k.$$

Lemma 2.3 implies that there exists $\{X_h\}_{h=1}^\infty \subset \mathcal{B}$, $\|X_h\| = 1 \ \forall h$, such that

$$\lim_{h \to \infty} \|S_jX_h\| = 0.$$

If we chose $y_h \in \mathcal{H}$, $\|y_h\| \leqq 2$, such that $x_h \equiv X_hy_h$ is a unit vector, then

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$$\lim_{h\to\infty} \|S_j x_h\| = 0.$$

If

$$A'_k \equiv \sum_{j=1}^{p} B_j^{\wedge}(N_k)A_j \ (\neq 0),$$

then there exists $X \in \mathcal{L}(\mathcal{H})$ such that

$$\overline{\lim_{h\to\infty}} \|A'_k X x_h\| > 0,$$

whence

$$\overline{\lim_{h\to\infty}} \|f(R(A, B))Xx_h\| > 0.$$

This contradiction implies that $f(A'_k) = 0$ and so (5.1) implies that $A_k$ is algebraic ($1 \leqq k \leqq p$). In the case when $R(A, B)$ is algebraic and pseudodiagonal, the minimal polynomial $f$ of $R$ must have simple roots, so each $A'_k$, and thus each $A_k$, will be similar to a normal operator with finite spectrum.

*Remark.* The hypothesis "$R(A, B)$ is algebraic" cannot be discarded in Theorem 5.8 (2). Indeed, let $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and let $\{e_n\}_{n\in\mathbf{N}}$ be an orthonormal basis of $\mathcal{H}$. Let $T \in \mathcal{L}(\mathcal{H})$ be the rank-two operator defined by

$$Te_1 = e_1, \ Te_2 = e_1 + e_2, \ Te_k = 0, \ k \geqq 2,$$

and let $V \in \mathcal{L}(\mathcal{H})$ denote the unilateral shift,

$$Ve_k = e_{k+1}, \quad k \geqq 1.$$

It will be shown below that $S(T, V)(X \to TXV)$ is pseudodiagonal; however, $S(T, V)$ is not algebraic and $T$ is not similar to a normal operator with finite spectrum. Let

$$S_0 \in \mathcal{L}(\mathcal{L}(\mathcal{H}, \mathbf{C}^2))$$

be defined by

$$S_0(Y) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} YV \quad (Y \in \mathcal{L}(\mathcal{H}, \mathbf{C}^2)).$$

To show that $S(T, V)$ is pseudodiagonal, it actually suffices to verify that $S_0$ is pseudodiagonal. Let $\{f_1, f_2\}$ be the canonical basis for $\mathbf{C}^2$. For $|\lambda| < 1$, let

$$Y_1(\lambda) = f_1 \otimes \sum_{n=0}^{\infty} e_n \lambda^n$$

and

$$Y_2(\lambda) = f_1 \otimes \sum_{n=0}^{\infty} ne_n\lambda^n - f_2 \otimes \sum_{n=0}^{\infty} e_n\lambda^n$$

(elements of $\mathscr{L}(\mathscr{H}, \mathbf{C}^2)$ ). It is now straightforward to check that

$$R_0((Y_1)(\lambda)) = \lambda Y_1(\lambda),\ R_0(Y_2(\lambda)) = \lambda Y_2(\lambda), \quad \text{and}$$
$$\text{c.l.m.}\{Y_1(\lambda),\ Y_2(\lambda)\}_{|\lambda|<1} = \mathscr{L}(\mathscr{H}, \mathbf{C}^2).$$

The following is a direct consequence of the last theorem.

COROLLARY 5.9. *If both $A$ and $B$ are linearly independent modulo essentially quasinilpotent operators with finite spectra, then*:

1) *$R(A, B)$ is pseudoalgebraic if and only if $\mathscr{A}$ and $\mathscr{B}$ are finite dimensional.*

2) *$R(A, B)$ is pseudodiagonal and algebraic if and only if both $\mathscr{A}$ and $\mathscr{B}$ are semi-simple and finite dimensional.*

*Remark.* If $T$ is a normal, diagonal operator with infinite spectrum and $S$ is a rank-one projection, then $R = L_T R_S$ is pseudodiagonal, but the algebra generated by $T$ is infinite dimensional.

We conclude with a characterization of the pseudoalgebraic and pseudodiagonal generalized derivations.

THEOREM 5.10. 1) *$\tau(T, S)$ is pseudoalgebraic if and only if $\tau(T, S)$ is algebraic.*

2) *$\tau(T, S)$ is pseudodiagonal if and only if $T$ and $S$ are similar to normal operators with finite spectra.*

*Proof.* 1) Suppose $\tau(T, S)$ is pseudoalgebraic. If $1_{\mathscr{H}}$ and $S$ are linearly independent modulo essentially quasinilpotent operators with finite spectra, then $T$ is algebraic (Theorem 5.8). If $1_{\mathscr{H}}$ and $S$ are not as above, then $S = \alpha 1_{\mathscr{H}} + Q$, where $Q$ is essentially quasinilpotent and has finite spectrum. Since

$$\tau(T, S) = \tau(T, Q) - \alpha L_{1_{\mathscr{H}}},$$

$\tau(T, Q)$ is clearly pseudoalgebraic. Again, by Theorem 5.8, we deduce that $T$ is algebraic. Passing to adjoints we also derive that $S$ is algebraic, and thus $\tau(T, S)$ is algebraic. The converse is trivial.

2) If $\tau(T, S)$ is pseudodiagonal, then the above proof implies that $\tau(T, S)$ is algebraic. Theorem 5.8 (2) and the preceding analysis show that both $T$ and $S$ are similar to normal operators with finite spectra; the converse is clear.

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