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A NECESSARY AND SUFFICIENT CONDITION FOR THE OSCILLATION OF AN EVEN ORDER NONLINEAR DELAY DIFFERENTIAL EQUATION

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1. Introduction. In this paper we study the oscillatory behavior of the even order nonlinear delay differential equation

$$(1)\quad (r(t)y'(t))^{(2n-1)} + \sum_{i=1}^{n} p_i(t)F_i(y_{\tau_i}(t), y_{\sigma_i}'(t), y_{\sigma_i}''(t), \ldots y_{\sigma_i}^{(2n-1)}(t)) = 0,$$

where

$$y_{\tau_i}(t) = y(t - \tau_i(t)), \quad y_{\sigma_i}^{(i)}(t) = y^{(i)}(t - \sigma_i(t)), \quad i = 1, 2, 3, \ldots, 2n - 1;$$

(i) denotes the order of differentiation with respect to t. The delay terms τ_i, σ_i are assumed to be real-valued, continuous, non-negative, non-decreasing and bounded by a common constant M on the half line $(t_0, +\infty)$ for some $t_0 \geqq 0$. It is also assumed throughout this paper that $r(t)$ and $p_i(t)$ are all real valued and continuous in (t_0, ∞). In addition sufficient smoothness of co-efficients for the existence of solutions in $C^{2n}(t_0, \infty)$ will be assumed without mention. A good discussion of these conditions can be found in [**5**] and [**11**].

A solution $y(t)$ of (1) which is continuous and defined on some half line $[t_0, +\infty)$ is said to be oscillatory if it has arbitrarily large zeros, i.e. if $y(t_1) = 0$, $t_1 > t_0$ then there exists $t_2 > t_1$ such that $y(t_2) = 0$; otherwise it is non-oscillatory. Equation (1) is said to be oscillatory if all its non-trivial continuous solutions defined on some half line $[t_0, +\infty)$ are oscillatory; otherwise it is called non-oscillatory.

It will be further assumed throughout this paper that in relation to (1), the following conditions are satisfied.

(i) $p_i(t)$ are eventually positive;

(ii) $r(t) \in C^{2n-1}(t_0, \infty)$, $r(t)$ is bounded and satisfies

$$r(t) > 0, \quad r'(t) > 0, \quad (-1)^{i+1}r^{(i)}(t) \geqq 0 \quad i = 2, 3, \ldots, 2n - 1.$$

Recently, Grollwitzer [**5**] has given necessary and sufficient condition for the delay equation

$$(2)\qquad y''(t) + q(t)y_\tau^\alpha(t) = 0$$

to be oscillatory. Dahiya and Singh [**3**] extended these results to the even order delay equation

$$(3)\qquad y^{(2n)}(t) + q(t)y_\tau^\alpha(t) = 0,$$

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and thus generalized similar other results due to Ličko and Švec [**9**]. In equations (2) and (3) it was assumed that α is the ratio of odd integers and either $\alpha > 1$ or $0 < \alpha < 1$. The case for $\alpha = 1$ was treated by Bradley [**2**] who considered the equation

$$(4) \qquad y''(t) + p(t)y(t - \tau(t)) = 0$$

and proved sufficiency theorems not only for equation (4) but also for the more general equation

$$(5) \qquad [r(t)y'(t)]' + p(t)f(y(t), y(g(t))) = 0.$$

A general situation is presented by the equation

$$(6) \qquad x^{(n)} + p(t)g(x, x', x'', \ldots, x^{n-1}) = 0, \quad (n \text{ even})$$

for which a necessary and sufficient condition is given by Onose [**10**] under one of the assumptions

$$\operatorname*{Lim\,inf}_{|x_1|\to\infty} \frac{|g(x_1, x_2, \ldots, x_n)|}{|x_1|^r} > 0, \quad r > 1.$$

Our results extend Onose's results to a more general situation presented by a nonlinear delay equation (1). The proof for sufficiency part is entirely different.

By proving a necessary and sufficiency type theorem for an equation slightly less general than equation (1), we will generalize the results due to [**1**; **2**; **3**; **5**; **9**] and extend, in part, the results given in [**10**; **12**; **13**].

2. Main results. This section is given to proving necessity and sufficiency theorems. We will need the following two lemmas.

LEMMA 1 (Kiguradze [**8**]). *If $y(t) > 0$, $y'(t) > 0$, $y''(t) < 0$ and $y(t)$ is real, then for sufficiently large t, there exists a constant $L > 0$ such that*

$$\frac{y'(t)}{y(t)} \leqq \frac{L}{t}.$$

LEMMA 2 [**2**, p. 398; **12**]. *Under the hypothesis of Lemma 1, there exist constants $R_i > 0$, $i = 1, 2, \ldots, n$ such that*

$$\frac{y(t - \tau_i(t))}{y(t))} \geqq R_i$$

and

$$\operatorname*{Lim}_{t\to\infty} \frac{y(t - \tau_i(t))}{(y(t))} = 1.$$

THEOREM 1. *Suppose the following additional conditions are satisfied:*
(a) *$F_i : R^{2n} \to R$ is continuous,* $\operatorname{sgn} F_i(x_0, x_1, \ldots, x_{2n-1}) = \operatorname{sgn} x_0$ *and*
(b) $F_i(-x_0, -x_1, \ldots, x_{2n-1}) = -F_i(x_0, x_1, \ldots, x_{2n-1})$ *for all i,*

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(b$_1$) *there exists an index* j *such that*

$$F_j(\lambda x_0, \lambda x_1, \ldots, \lambda x_{2n-1}) = \lambda^{2\beta+1} F_j(x_0, x_1, \ldots, x_{2n-1})$$

for all $(x_0, x_1, \ldots, x_{2n-1}) \in R^{2n}$, *real* $\lambda \neq 0$ *and some integer* $\beta \geqq 0$,

(b$_2$) $F_j \to \infty$ *as* $x_0 \to \infty$ *and* $\int^\infty t^{2n-1} p_j(t) = \infty$.

Then all bounded continuous solutions of equation (1) *are oscillatory. If, however,* β *in assumption* (b$_1$) *is such that* $\beta \geqq 1$, *then all continuous solutions of equation* (1) *are oscillatory.*

Remark. This theorem generalizes Theorem 1 of [**13**].

Proof of Theorem 1. We assume the existence of a non-oscillatory solution $y(t) \not\equiv 0$ of equation (1). Conditions of the theorem imply that $-y(t)$ is also a solution. Therefore, without any loss, we can assume that $y(t) > 0$ eventually. Suppose for $t \geqq t_1 \geqq 0$, $y(t)$ and $y(t - \tau_i(t))$ are positive for all i. Choose t_2 so large that $y(t)$, $y(t - \tau_i(t))$ and $p_i(t)$ are all positive in $[t_2, \infty]$. Due to sign condition on F_i, it follows now from equation (1) that

$$(r(t)y'(t))^{(2n-1)} + p_j(t) F_j(y_{\tau_j}(t), y_{\sigma_j}'(t), y_{\sigma_j}''(t), \ldots, y_{\sigma_j}^{(2n-1)}(t)) < 0. \tag{8}$$

Thus

$$(r(t)y'(t))^{(2n-1)} < 0, \quad t \in [t_2, \infty]. \tag{9}$$

This, in turn, implies that $(r(t)y'(t))^{(2n-2)}$ is decreasing and must eventually have a constant sign. Proceeding this way we find that $r(t)y'(t)$ must eventually have a constant sign and since $r(t) > 0$, it implies $y'(t)$ must eventually have a constant sign. Hence there exists a conveniently large $t_3 \geqq t_2$ such that for $t \geqq t_3$, $y'(t)$ is either positive or negative.

Case 1. $y(t) > 0$, $y'(t) < 0$, $t \in [t_3, \infty]$: Since

$$[r(t)y'(t)]^{(2n-1)} < 0 \quad \text{and} \quad ry'(t) < 0$$

we claim that

$$(r(t)y'(t))' \leqq 0 \quad \text{for} \quad t \in [t_4, \infty), \ t_4 \geqq t_3. \tag{10}$$

For, suppose $(r(t)y'(t))' > 0$ eventually. Then $(r(t)y'(t))''$ being monotonic must eventually be non-positive because if $(r(t)y'(t))'' > 0$, then $r(t)y'(t)$ being concave up and increasing will eventually be positive, a contradiction. Proceeding this way and remembering that all derivatives of $r(t)y'(t)$ are monotonic, we find $[r(t)y'(t)]^{(2n-1)} \geqq 0$, a contradiction to (9). Hence (10) holds.

Integrating (10) between t_4 and t we obtain $r(t)y'(t) \leqq r(t_4)y'(t_4) < 0$, or

$$y'(t) \leqq r(t_4)y'(t_4)\frac{1}{r(t)}. \tag{11}$$

Therefore from (11),

$$y(t) \leqq y(t_4) + r(t_4)y'(t_4) \int_{t_4}^t \frac{1}{r(s)}\, ds < 0. \tag{12}$$

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Now as $t \to \infty$, the right hand side of (12) tends to $-\infty$ which is a contradiction, since $y(t) > 0$ in $[t_4, \infty)$ and $r(t)$ is bounded. Hence either $y(t)$ is oscillatory or the following case holds.

Case 2. $y(t) > 0$, $y'(t) > 0$ for $t \in [t_4, \infty)$: Since from inequality (9), $(r(t)y'(t))^{(2n-1)} < 0$ and $r(t)y'(t) > 0$ in $[t_4, \infty)$, we must have

$$(r(t)y'(t))^{(2n-2)} > 0 \text{ eventually.} \tag{13}$$

For if $(ry')^{(2n-2)} < 0$, then $(ry')^{(2n-3)}$ is concave down decreasing and therefore ultimately negative. This will eventually make $y < 0$, a contradiction.

We now claim that

$$(-1)^i(r(t)y'(t))^{(i)} \geqq 0, \quad i = 0, 1, 2, \ldots, 2n-1, \tag{14}$$

where (i) denotes the order of differentiation. To see this suppose first that $y(t)$ is bounded. If $(r(t)y'(t)^{(2n-3)} > 0$ eventually then because of (13), $(r(t)y'(t))^{(2n-4)}$ will be eventually positive and tend to ∞. Proceeding this way we find that $r(t)y'(t) \to \infty$ as $t \to \infty$ and since $r(t)$ is bounded, this leads to the fact that $y(t) \to \infty$ as $t \to \infty$, a contradiction. Hence

$$(r(t)y'(t))^{2n-3} \leqq 0 \text{ eventually,}$$

and the claim holds by continuation of this process. Now suppose $y(t)$ is unbounded as $t \to \infty$. Integrating (8) between t_5 and t, t_5 being conveniently large we have

$$\begin{aligned}(r(t)y'(t)^{(2n-2)} &< (r(t_5)y'(t_5))^{(2n-2)} - \int_{t_5}^{t} P_j(s)F_j(y_{\tau_j}(s), y_{\sigma_j}'(s), \\ &\quad y_{\sigma_j}''(t), \ldots, y_{\sigma_j}^{(2n-1)}(s))ds = (r(t_5)y'(t_5))^{(2n-2)} \\ &\quad - \int_{t_5}^{t} s^{2n-1}P_j(s)\frac{F_j(y_{\tau_j}(s), y_{\sigma_j}'(s), \ldots, y_{\sigma_j}^{(2n-1)}(s))}{s^{2n-1}}ds.\end{aligned} \tag{15}$$

Since left hand side of (15) eventually becomes positive and by condition (b_2) of this theorem

$$\int_{t_5}^{\infty} t^{2n-1}p_j(t)dt = \infty,$$

we must have

$$\operatorname{Lim\,inf}_{t\to\infty}\left[\frac{F_j(y_{\tau_j}(t), \ldots, y_{\sigma_j}^{(2n-1)}(t))}{t^{2n-1}}\right] = 0. \tag{16}$$

Now

$$\begin{aligned}&\operatorname{Lim\,inf}_{t\to\infty}\left[\frac{F_j(y_{\tau_j}(t), \ldots, y_{\sigma_j}^{(2n-1)}(t))}{t^{2n-1}}\right] \\ &= \operatorname{Lim\,inf}_{t\to\infty}\frac{y^{2\beta+1}(t)F_j\left[\dfrac{y_{\tau_j}(t)}{y(t)}, \ldots, \dfrac{y_{\sigma_j}^{(2n-1)}(t)}{y(t)}\right]}{t^{2n-1}} \\ &\geqq \left[\operatorname{Lim\,inf}_{t\to\infty}\frac{y^{2\beta+1}}{t^{2n-1}}\right]\left[\operatorname{Lim\,inf}_{t\to\infty} F_j(y_{\tau_j}(t)/y(t), \ldots, y_{\sigma_j}^{(2n-1)}(t)/y(t))\right].\end{aligned} \tag{17}$$

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As will be shown later

$$\text{(18)}\quad \liminf_{t\to\infty} F_j(y_{\tau_j}(t)/y(t), \ldots, y_{\sigma_j}{}^{(2n-1)}(t)/y(t)) = \lim_{t\to\infty} F_j(1, 0, 0, \ldots, 0) > 0.$$

(16), (17) and (18) imply that

$$\text{(19)}\quad \liminf_{t\to\infty} \frac{y^{2\beta+1}}{t^{2n-1}} = 0.$$

Now

$$\liminf_{t\to\infty} \frac{y^{2\beta+1}}{t^{2n-1}} = \liminf_{t\to\infty} \frac{ry^{2\beta+1}(t)}{rt^{2n-1}}$$

$$\text{(20)}\quad \geqq \left[\liminf_{t\to\infty} \frac{(ry(t))}{t^{2n-3}}\right]\left[\liminf_{t\to\infty} \frac{y(t)}{t}\right]\left[\liminf_{t\to\infty} \frac{y^{2\beta-1}(t)}{rt}\right].$$

First suppose $\text{Lim}_{t\to\infty} y'(t) \neq 0$ and $\beta \geqq 1$.
Since $\beta \geqq 1$, $y'(t) > 0$, and $1/r(t)$ is bounded away from zero, it follows from (19) and (20) that

$$\text{(21)}\quad \liminf_{t\to\infty} \left(\frac{ry}{t^{2n-3}}\right) = 0.$$

But due to monotonicity of $(ry)^{(i)}$, $i = 0, 1, 2, \ldots, 2n - 2$, we have

$$\liminf_{t\to\infty} \frac{ry(t)}{t^{2n-3}} = \lim_{t\to\infty} \frac{r(t)y(t)}{t^{2n-3}} = \lim_{t\to\infty} (r(t)y(t))^{(2n-3)}/(2n-3)! = 0$$

and since $(r(t)y(t))^{(2n-2)} > 0$ eventually, we must have $(r(t)y(t))^{(2n-3)} < 0$ for some $t \geqq t_6 > t_5$, where t_6 is conveniently large. If, however, $y'(t) \to 0$ as $t \to \infty$, then $ry'(t) \to 0$ implies $(r(t)y'(t))^{(2n-3)} \to 0$ as $t \to \infty$ and the conclusion follows since $(ry)^{(2n-2)} > 0$ eventually. The rest of the lemma follows in an identical manner. Hence (14) holds.

Also since $r(t) > 0$, $r'(t) \geqq 0$, $r''(t) \leqq 0, \ldots, r^{(2n-1)}(t) \geqq 0$, we get from (14)

$$\text{(22)}\quad y(t) > 0,\quad y'(t) > 0,\quad y''(t) \leqq 0,\quad y'''(t) \geqq 0, \ldots, y^{(2n)}(t) \leqq 0$$

and

$$\text{(23)}\quad \lim_{t\to\infty} y^{(i)}(t) = 0, i = 2, 3, \ldots, 2n - 1.$$

By invoking homogeneity condition on F_j we obtain from (8)

$$\text{(24)}\quad [r(t)y'(t)]^{(2n-1)} + p_j(t)y^{2\beta+1}(t)F_j\left[\frac{y_{\tau_j}(t)}{y(t)}, \frac{y_{\sigma_j}'(t)}{y(t)}, \ldots, \frac{y_{\sigma_j}{}^{(2n-1)}(t)}{y(t)}\right] < 0.$$

Now suppose (14) and (22) hold for $t \geqq t_5 \geqq t_4$. Then multiplying (24) by

t^{2n-1}, dividing by $y^{2\beta+1}(t)$ and integrating between t_5 and t we obtain

$$\text{(24a)}\quad \int_{t_5}^{t} \frac{s^{2n-1}[r(s)y'(s)]^{(2n-1)}ds}{(y(s))^{2\beta+1}} + \int_{t_5}^{t} s^{2n-1}p_j(s)F_j\left(\frac{y_{\tau j}(s)}{y(s)}, \frac{y_{\sigma j}'(s)}{y(s)}, \ldots, \frac{y_{\sigma j}^{(2n-1)}(s)}{y(s)}\right) ds < 0.$$

Now

$$0 \leqq \frac{y_{\sigma j}'(t)}{y_{\tau j}(t)} \leqq \frac{y'(t-M)}{y(t-M)}.$$

Therefore, by Lemma 1

$$\text{(25)}\qquad \lim_{t\to\infty} \frac{y'(t-M)}{y(t-M)} = 0,$$

and by Lemma 2

$$\text{(26)}\qquad \lim_{t\to\infty} \frac{y_{\tau j}(t)}{y(t)} = 1.$$

From (23), (25) and (26) and continuity in all the variables of F_j, it follows that

$$\lim_{t\to\infty} F_j\left[\frac{y_{\tau j}(t)}{y(t)}, \frac{y_{\sigma j}'(t)}{y(t)}, \ldots, \frac{y_{\sigma j}^{(2n-1)}(t)}{y(t)}\right] = F_j(1, 0, 0, 0 \ldots, 0) > 0$$

and hence the second integral in (24a) tends to ∞ as $t \to \infty$. Now the first integral in (24a) gives an integration by parts,

$$\text{(27)}\quad \int_{t_5}^{t} \frac{[r(s)y'(s)]^{(2n-1)}s^{2n-1}ds}{(y(s))^{2\beta+1}} = \frac{t^{2n-1}[r(t)y'(t)]^{(2n-2)}}{(y(t))^{2\beta+1}} - \frac{t_5^{2n-1}[r(t_5)y'(t_5)]^{(2n-2)}}{(y(t_5))^{2\beta+1}} - \int_{t_5}^{t} \frac{(ry')^{(2n-2)}(2n-1)s^{2n-2}ds}{(y(s))^{2\beta+1}} + (2\beta+1)\int_{t_5}^{t} \frac{(ry')^{(2n-2)}s^{2n-1}y'(s)ds}{(y(s))^{2\beta+2}}$$

$$\geqq k_1 - (2n-1)\int_{t_5}^{t} \frac{(ry')^{(2n-2)}s^{2n-2}ds}{(y(s))^{2\beta+1}},$$

since on the right hand side of (27), the first and the last term are positive in view of (14) and k_1 is a constant equal to second term in (27). Integrating again and again by parts we get

$$\text{(28)}\qquad \int_{t_5}^{t} \frac{[ry']^{(2n-1)}s^{2n-1}ds}{(y(s))^{2\beta+1}} \geqq R_1' - R_2'\int_{t_5}^{t} \frac{ry'(s)ds}{(y(s))^{2\beta+1}}$$

where R_1' and R_2' are constants and $R_2' > 0$. Since $r(t)$ is non-decreasing we have from (28),

$$\int_{t_5}^{t} \frac{[ry']^{(2n-1)} s^{2n-1} ds}{(y(s))^{2\beta+1}} \geqq R_1' - R_2' r(t) \int_{t_5}^{t} \frac{y'(s)ds}{(y(s))^{2\beta+1}}$$

$$(28a) \qquad = R_1' + R_2' r(t) \left(\frac{1}{2\beta}\right) \left[\frac{1}{(y(t))^{2p}} - \frac{1}{(y(t_5))^{2p}}\right]$$

$$< \infty \quad \text{as } t \to \infty$$

since $r(t)$ is bounded and $y(t)$ is increasing. Hence the left-hand side of (24a) tends to ∞ as $t \to \infty$ which is a contradiction. The proof is complete if $\beta \geqq 1$ in (b_2). If $\beta = 0$ in (b_2) then right hand side of (28a) is

$$R_1' + R_2' r(t)[\ln|y(t)| - \ln|y(t_5)|]$$

and the result follows by boundedness and increasingness of $y(t)$.

For necessity criteria we will consider the equation

$$(29) \quad [r(t)y'(t)]^{(2n-1)} + p_j(t) F_j(y_{\tau_j}(t), y_{\sigma_j}'(t), \ldots, y_{\sigma_j}^{(2n-1)}(t)] = 0$$

where p_j, F_j and r satisfy the same conditions as in Theorem 1 and in addition we will assume that $p_j(t)$ is bounded. For convenience we will drop the subscript j.

THEOREM 2. *If all the nontrivial continuous solutions of* (29) *are oscillatory, then*

$$\int^{\infty} t^{2n-1} p(t) dt = \infty .$$

Proof. We will prove this theorem by constructing a solution with a prescribed limit at ∞, should the hypothesis

$$\int^{\infty} t^{2n-1} p(t) dt < \infty$$

hold. From equation (29)

$$(30) \qquad r y'(t) = \int_{t}^{\infty} \frac{(s-t)^{2n-2}}{(2n-2)} p(s) F(y_\tau(s), y_\sigma'(s), \ldots, y_\sigma^{(2n-1)}(s)) ds.$$

We consider the integral equation

$$(31) \quad y'(t) = 1 - \int_{t}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \frac{(x-s)^{2n-2}}{(2n-2)} \times p(x) F(y_\tau(x), y_\sigma'(x), \ldots, y_\sigma^{(2n-1)}(x)) dx\, ds.$$

Here we shall employ a process similar to the one used by Onose [**10**]. We first observe that for $t \geqq t_5$

$$(32)\quad \int_t^\infty \frac{1}{r(s)} \int_s^\infty (x-s)^{2n-1} p(x)dxds \leqq \frac{1}{r(t_5)} \int_t^\infty \int_s^\infty (x-s)^{2n-2} p(x)dxds$$
$$= \frac{1}{r(t_5)} \int_t^\infty \frac{(s-t)^{2n-1}}{(2n-1)} p(s)ds < \infty .$$

Because of conditions on $r(t)$, there exist constants $P_i > 0$ such that

$$|r(t))^{(i)}| \leqq P_i, \quad i = 0, 1, 2, \ldots 2n-1.$$

Let

$$(33)\qquad P = \max_{0 \leqq i \leqq 2n-1} P_i .$$

Define sets

$$D = \{(x_0, x_1, \ldots, x_{2n-1}) : 1/2 \leqq x_0 \leqq 1, |x_i| \leqq 1/2, i = 1, 2, \ldots, 2n-1\},$$
$$N = \{0, 1, 2, 3, \ldots, 2n-1\}.$$

We choose T large enough so that for $t \geqq T \geqq t_5$

$$(34)\qquad \left| \left[\operatorname{Sup}_D F \right] \operatorname*{Max}_{i \in N} \left[\int_t^\infty \frac{1}{r(s)} \int_s^\infty \frac{(x-s)^{2n-2}}{(2n-2)!} p(x)dx\, ds \right]^{(i)} \right| \leqq \tfrac{1}{2}.$$

This is possible due to (32), (33) and continuity of F. We now define a sequence of functions which will converge to a solution of (31). Let $t \geqq T + M$. Let

$$(35)\qquad y_0(t) \equiv 1, \quad y_0^{(i)}(t) \equiv 0, \quad i = 1, 2, \ldots, 2n,$$

and for $n = 1, 2, 3, \ldots$ let

$$(36)\quad y_n(t) = 1 - \int_t^\infty \frac{1}{r(s)} \int_s^\infty \frac{(x-s)^{2n-2}}{(2n-2)!}$$
$$\times p(x) F(y_{n-1}(x-\sigma(x)), \ldots, y_{n-1}^{(2n-1)}(x-\sigma(x))dxds.$$

From (35) and (36),

$$y_1(t) = 1 - F(1, 0, 0, 0 \ldots, 0) \int_t^\infty \frac{1}{r(s)} \int_s^\infty \frac{(x-s)^{2n-2}}{(2n-2)!} p(x)dx\, ds.$$

Therefore in view of (34)

$$1/2 \leqq y_1(t) \leqq 1 \quad \text{and} \quad |y_1^{(i)}(t)| \leqq 1/2, \quad i = 1, 2, \ldots, 2n-1.$$

Similarly,

$$(37)\qquad 1/2 \leqq y_k(t) \leqq 1, \quad k = 1, 2, \ldots,$$

and

$$(38)\qquad |y_k^{(i)}(t)| \leqq 1/2, \quad i = 0, 1, 2, \ldots, 2n-1; k = 1, 2, \ldots .$$

Also from equation (29) and boundedness of $p(t)$, $r(t)$ and $(r(t))^{(i)}$, $i = 1, 2, \ldots, 2n - 1$, it follows that

$$|y_k^{(2n)}(t)| \leqq m_1. \tag{39}$$

Since the family $\{y_k^{(i)}\}$ is uniformly bounded and equicontinuous by (38) and (39), there exists a uniformly convergent subsequence $y_{k_r}^{(i)}$ such that

$$\lim_{k_r \to \infty} y_{k_r}^{(i)} = y^{(i)}, \quad i = 0, 1, 2, \ldots, 2n - 1.$$

The proof is complete.

COROLLARY 1. *Under the hypotheses of Theorem* 1 *regarding* F_j, $r(t)$ *and* p_j *and the additional condition that* $p_j(t)$ *is bounded, and* $\beta \geqq 1$ *in assumption* (b_1) *of Theorem* 1, *a necessary and sufficient condition that equation* (29) *oscillates is that*

$$\int^{\infty} t^{2n-1} p_j(t) dt = \infty.$$

If, however, $\beta \geqq 0$ *in* (b_1) *of Theorem* 1, *then the above is a necessary and sufficient condition for all bounded continuous solutions of* (29) *to be oscillatory.*

3. More on sufficiency. The following theorem generalizes Theorem 3 of [**13**, p. 700] by a relatively simpler technique.

THEOREM 3. *Let equation* (1) *satisfy conditions* (a) *and* (b) *of Theorem* (1), *as well as the following:*

(c) $F_i(\lambda x_0, \lambda x_1, \lambda x_2, \ldots, \lambda x_{2n-1}) = \lambda F_i(x_0, x_1, \ldots, x_{2n-1})$ *for every* $(x_0, x_1, \ldots, x_{2n-1}) \in R^{2n}$ *and* $\lambda \in R$;

(d) $I \neq \phi$, *where* I *denotes the set of all indices for which the function* $F_i(x_0, x_1, \ldots, x_{2n-1})$ *is non-decreasing with respect to each variable* $x_0, x_1, x_3, x_5, \ldots, x_{2n-1}$ *separately and decreasing with respect to* $x_2, x_4, \ldots, x_{2n-2}$ *as well as the function* $[F_i(x, 0, 0, 0, \ldots, 0)]/x$ *is non-increasing on* $(0, \infty)$;

(e) *there exists a positive and differentiable function* $\phi(t)$, $t \geqq t_0$ *for some* t_0, *such that* $\phi' \leqq 0$ and

$$\int^{\infty} \left[\phi(t) \sum_{i \in I} p_i(t) F_i(1, 0, 0, \ldots, 0) - \frac{\phi'^2(t) |r^{(2n-2)}(t)|}{4\phi(t)} \right] dt = \infty.$$

Then equation (1) *is oscillatory.*

Proof. From equation (1), as in the proof of Theorem 1,

$$(r(t)y'(t))^{(2n-1)} + \sum_{i \in I} p_i(t) F_i(y_{\tau_i}(t), y_{\sigma_i}'(t), \ldots, y_{\sigma_i}^{(2n-1)}(t)) < 0 \tag{40}$$

for $t \geqq t_5$ for some convenient t_5. Also for any non-oscillatory solution $y(t)$ conclusions (14) and (22) of the proof of Theorem 1 hold for $t \geqq t_5$. Multiplying and dividing (40) by $\phi(t)$ and $y(t)$ respectively and invoking conditions (c) of this theorem we get

$$(41)\quad [r(t)y'(t)]^{(2n-1)}\phi(t)/y(t) + \phi(t)\sum_{i\in I} p_i(t)F_i\left[\frac{y_{\tau_i}(t)}{y(t)}, \frac{y_{\sigma_i}'(t)}{y(t)}, \ldots, \frac{y_{\sigma_i}^{(2n-1)}(t)}{y(t)}\right] < 0.$$

Now

$$F_i\left[\frac{y_{\tau_i}(t)}{y(t)}, \frac{y_{\sigma_i}'(t)}{y(t)}, \ldots, \frac{y_{\sigma_i}^{(2n-1)}(t)}{y(t)}\right] \geqq F_i\left[\frac{y(t-M)}{y(t)}, 0, 0, \ldots, 0\right],$$

in view of (14) and (22) and condition (d) of this theorem. Therefore

$$F_i\left(\frac{y_{\tau_i}(t)}{y(t)}, \frac{y_{\sigma_i}'(t)}{y(t)}, \ldots, \frac{y_{\sigma_i}^{(2n-1)}(t)}{y(t)}\right)$$

$$\geqq \left\{F_i\left[y(t-M)/y(t), 0, 0, \ldots, 0\right] \bigg/ \frac{y(t-M)}{y(t)}\right\}\frac{y(t)}{y(t-M)} \geqq F_i[1, 0, 0, \ldots, 0],$$

since $y(t-M)/y(t)$ increases to 1 as $t \to \infty$ by Lemma 2. Hence from (41) and this fact,

$$(42)\quad [r(t)y'(t)]^{(2n-1)}\phi(t)/y(t) + \phi(t)\sum_{i\in I} p_i(t)F_i(1, 0, 0, 0 \ldots, 0) < 0.$$

Adding and subtracting $\phi'^2(t)|r^{(2n-2)}(t)|/4\phi(t)$ to (42) and integrating between t_5 and t we get

$$(43)\quad \int_{t_5}^{t}\left[\frac{(ry')^{(2n-1)}\phi(s)}{y(s)} + \frac{\phi'^2(s)|r^{(2n-2)}(s)|}{4\phi(s)}\right]ds + \int_{t_5}^{t}\left[\phi(s)\sum_{i\in I} p_i(s)F_i(1, 0, 0, \ldots, 0) - \frac{\phi'^2(s)|r^{(2n-2)}(s)|}{4\phi(s)}\right]ds < 0.$$

Since the second integral in (43) tends to ∞ as $t \to \infty$ we only need to consider the first integral.

Let

$$P = \int_{t_5}^{t}\left[\frac{(ry')^{(2n-1)}\phi(s)}{y(s)} + \frac{\phi'^2(s)|r^{(2n-2)}(s)|}{4\phi(s)}\right]ds$$

$$= \frac{(ry')^{(2n-2)}\phi(t)}{y(t)} - \frac{(r(t_5)y'(t_5))^{(2n-2)}\phi(t_5)}{y(t_5)} - \int_{t_5}^{t}\left[\frac{(ry')^{(2n-2)}\phi'(s)}{y(s)} - \frac{(ry')^{(2n-2)}\phi(s)y'(s)}{y^2(s)} - \frac{\phi'^2(s)|r^{(2n-2)}(s)|}{4\phi(s)}\right]ds$$

$$\geqq L_0 + \int_{t_5}^{t}\left[\frac{(ry')^{(2n-2)}\phi(s)y'(s)}{y^2(s)} - \frac{(ry')^{(2n-2)}\phi'(s)}{y(s)} + \frac{\phi'^2(s)|r^{(2n-2)}(s)|}{4\phi(s)}\right]ds,$$

where

$$L_0 = \frac{(r(t_5)y'(t_5))^{(2n-2)}\phi(t_5)}{y(t_5)}.$$

$$P \geqq L_0 + \int_{t_5}^{t} \frac{(ry')^{(2n-2)}y'(s)}{\phi(s)} \left(\frac{\phi^2(s)}{y^2(s)} - \frac{\phi(s)\phi'(s)}{y(s)y'(s)} + \frac{\phi'^2(s)|r^{(2n-2)}(s)|}{4(ry')^{(2n-2)}y'(s)}\right) ds.$$

Now

$$\frac{|r^{(2n-2)}(s)|y'(s)}{(ry')^{(2n-2)}} \geqq \frac{|r^{(2n-2)}(s)|y'(s)}{|r^{(2n-2)}|y'(s) + (2n-2)|r^{(2n-3)}|y''| + \ldots + r|y^{2n-1}|} = l^2$$

in view of (14) and (22) and $0 < l < 1$.
Hence

$$(44) \qquad P \geqq P_0 + \int_{t_5}^{t} \frac{(ry')^{(2n-2)}y'(s)}{\phi(s)} \left[\frac{\phi^2(s)}{y^2(s)} - \frac{\phi'(s)\phi(s)l}{yy'} + \frac{\phi'^2 l^2}{4y'^2}\right] ds$$

$$= P_0 + \int_{t_5}^{t} \frac{(ry')y'^{(2n-2)}(s)}{\phi(s)} \left[\frac{\phi(s)}{y(s)} - \frac{\phi' l}{2y'}\right]^2 ds.$$

which indicates that left hand side of 43 tends to ∞ as $t \to \infty$. This is a contradiction and the proof is complete.

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