CONGRUENCES INDUCED BY TRANSITIVE REPRESENTATIONS OF INVERSE SEMIGROUPS

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1. Introduction and summary. Transitive group representations have their analogue for inverse semigroups as discovered by Schein [**7**]. The right cosets in the group case find their counterpart in the right ω-cosets and the symmetric inverse semigroup plays the role of the symmetric group. The general theory developed by Schein admits a special case discovered independently by Ponizovskiĭ [**4**] and Reilly [**5**]. For a discussion of this topic, see [**1**, §7.3] and [**2**, Chapter IV].

One of the basic results of Schein [**7**] asserts that every effective representation of an inverse semigroup is a sum of transitive representations. Since the Wagner representation of an inverse semigroup is effective, we may reason as follows. Let ρ be a congruence on an inverse semigroup. Then the Wagner representation of S/ρ is effective and thus a sum of transitive representations. The equality congruence on S/ρ is therefore the intersection of the congruences induced by the transitive representations so obtained. When these congruences are lifted to S, we obtain ρ as the intersection of congruences on S induced by transitive representations of S (see [**3**] for another demonstration of this fact).

This provides the motivation for a closer look at the congruences on an inverse semigroup which are induced by transitive representations by one-to-one partial transformations. In addition, a deeper study of the congruences induced by transitive representations would provide a better understanding of the transitive representations themselves. For example, one might ask when two transitive representations induce the same congruence. One may want to single out those congruences, say in kernel-trace form, which are induced by transitive representations. Some related questions are treated in this paper.

A few preliminary and general results are proven in Section 2. The notation and terminology used throughout the paper is described in this section as well. The kernel and the trace of the congruence induced by a transitive representation are characterized in Section 3. A reasonably specific description of inverse semigroups with a faithful transitive representation and at least one primitive idempotent is established in Section 4. This result is generalized to provide some conditions on a congruence which ensure that it is induced by a transitive representation. In Section 5, completely semisimple inverse semigroups with a finite number of $\mathscr{D}$-classes all of whose congruences are induced by transitive representations are described by means of special ideal extensions of Brandt semigroups. These extensions are studied in some detail in Section 6 both for general inverse semigroups and for Brandt semigroups.

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2. Notation and preliminary results. For all undefined concepts and notation, the reader is referred to [**2**]. *Throughout this paper, S shall denote an inverse semigroup and E its semilattice of idempotents.* For any subsemigroup T of S, we define $E_T = E \cap T$.

The closure operator, denoted by ω, plays an essential role in the theory of transitive representations of an inverse semigroup. For convenience, we recall its definition here (see for example [**2**] for more detail). If H is a subset of S, then the closure $H\omega$ of H in S is given by

$$H\omega = \{x \in S \mid xE \cap H \neq \varnothing\} = \{x \in S \mid Ex \cap H \neq \varnothing\}.$$

A set H is said to be a *closed* subset of S if $H = H\omega$. If H is an inverse subsemigroup of S, then H is closed if and only if H is unitary. As a result, if H is a closed inverse subsemigroup of S, then for any $e \in E$ and $u \in S$, $eu \in H$ if and only if both $e \in H$ and $u \in H$. In particular, for any $u, v \in S$, if $uv \in H$ then $uu^{-1} \in H$ and $v^{-1}v \in H$.

The following facts concerning the closure operator will be useful.

LEMMA 2.1. *Let X_1 and X_2 be subsets of S. Then $(X_1X_2)\omega = ((X_1\omega)(X_2\omega))\omega$.*

Proof. Since $X \subseteq X\omega$ for any $X \subseteq S$, we obtain $X_1X_2 \subseteq (X_1\omega)(X_2\omega)$ and so $(X_1X_2)\omega \subseteq ((X_1\omega)(X_2\omega))\omega$. Conversely, let $u \in ((X_1\omega)(X_2\omega))\omega$. Then for some $e \in E$, we have $eu \in (X_1\omega)(X_2\omega)$, whence $eu = rs$ for some $r \in X_1\omega$, $s \in X_2\omega$. Consequently, there are idempotents f and g such that $fr \in X_1$ and $sg \in X_2$. Now $u \geq feug = frsg \in X_1X_2$ and so $u \in (X_1X_2)\omega$.

Another useful result which is quite interesting in its own right describes the relationship between an idempotent congruence class and its closure.

PROPOSITION 2.2. *Let ρ be a congruence on S. For each $a \in S$, the following are equivalent*:

(i) $x \in (a\rho)\omega$,
(ii) $aa^{-1}x \in a\rho$,
(iii) $xa^{-1}a \in a\rho$.

If $e \in E$, then $e\rho$ is an ideal of $(e\rho)\omega$.

Proof. In S/ρ, (i), (ii), and (iii) are equivalent ways of saying that $x\rho \geq a\rho$.

Now, for $e \in E$, let $x \in (e\rho)\omega$ and $y \in e\rho$. Then $xy\rho xe\rho e$ and $yx\rho ex\rho e$ and so $e\rho$ is an ideal of $(e\rho)\omega$.

LEMMA 2.3. *Let ρ be a congruence on S. Then for any $e \in E$ and $x \in S$, $x^{-1}x \in (e\rho)\omega$ implies that $(x(e\rho)\omega x^{-1})\omega = ((xex^{-1})\rho)\omega$.*

Proof. By Lemma 2.1, we obtain $(x(e\rho)\omega x^{-1})\omega = (x(e\rho)x^{-1})\omega$. But $x(e\rho)x^{-1} \subseteq (xex^{-1})\rho$ and so $(x(e\rho)\omega x^{-1})\omega \subseteq ((xex^{-1})\rho)\omega$. Conversely, let $u \in ((xex^{-1})\rho)\omega$. Then for some $f \in E$, $uf\rho xex^{-1}$ and so $x^{-1}ufx\rho x^{-1}xe$. Now $x^{-1}x \in (e\rho)\omega$ yields $x^{-1}xe\rho e$ by Proposition 2.2. Thus $x^{-1}ufx\rho e$, whence $u \geq xx^{-1}ufxx^{-1} \in x(e\rho)x^{-1}$ and so $u \in (x(e\rho)x^{-1})\omega$, as required.

We conclude this sequence of observations with a local version of the result which states that if ρ is a group congruence with kernel K, then for any $a \in S$, $a\rho = (Ka)\omega$.

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LEMMA 2.4. *Let ρ be a congruence on S and let $a \in S$. Then $(a\rho)\omega = ((aa^{-1})\rho a)\omega$.*

Proof. Observe that $((aa^{-1})\rho a)\omega \subseteq (a\rho)\omega$ since $(aa^{-1})\rho a \subseteq a\rho$. Let $x \in (a\rho)\omega$. Then by Proposition 2.2, $xa^{-1}a\rho a$ so that $xa^{-1}\rho aa^{-1}$. As a result, we have $xa^{-1}a \in (aa^{-1})\rho a$. Finally, since $x \geq xa^{-1}a$ we obtain $x \in ((aa^{-1})\rho a)\omega$.

Other fundamental concepts that we shall require are the *principal right congruences* and *principal congruences* due to Dubreil and Croisot, respectively (see [**1**, Sections 10.2 and 10.4] for a detailed discussion). For convenience, we recall their definitions here. Let X be any subset of S. Define $aP_X^r b$ if for all $u \in S$, $au \in X$ if and only if $bu \in X$. Define $aP_X b$ if for all $u,v \in S$, $uav \in X$ if and only if $ubv \in X$. Note that if X is a closed subset of S, then P_X^r is the greatest right congruence saturating X and P_X is the greatest congruence saturating X. Furthermore, the one and two-sided residue sets W_X^r and W_X are defined by

$$W^r = W_X^r = \{x \in S \mid xS \cap X = \emptyset\},$$
$$W = W_X = \{x \in S \mid SxS \cap X = \emptyset\}.$$

Observe that W^r is either empty or is a right ideal and a class of P_X^r while W is either empty or is an ideal and a class of P_X.

An important result in the theory of representations of an inverse semigroup by one-to-one partial transformations, due to Schein, is that every transitive representation of S is equivalent to a representation obtained as follows. Let H be a closed inverse subsemigroup of S. A *right ω-coset* of H in S is a set of the form $(Ha)\omega$ for $a \in S$ such that $aa^{-1} \in H$. Let $\mathscr{X}$ denote the set of all right ω-cosets of H in S and let $\mathscr{I}_{\mathscr{X}}$ denote the symmetric inverse semigroup on $\mathscr{X}$ (with functions written on the right). To each $s \in S$ assign the mapping $\phi_H^s \in \mathscr{I}_{\mathscr{X}}$ defined by

$$\phi_H^s : (Ha)\omega \rightarrow (Has)\omega \quad (aa^{-1},\, as(as)^{-1} \in H).$$

Then $\phi_H : S \rightarrow \phi_H^s$ is a transitive representation of S on $\mathscr{X}$. The domain of ϕ_H^s is given by

$$\mathbf{d}\phi_H^s = \{(Ha)\omega \mid aa^{-1},\, as(as)^{-1} \in H\}.$$

An important special case occurs when $H = G\omega$ for some subgroup G of S. We may obtain a transitive representation equivalent to ϕ_H in the following way (see [**2**, IV.5.2] for details). Let e denote the identity of G and let $\mathscr{Y} = \{Ga \mid aa^{-1} \geq e\}$. Then $\psi_G : S \rightarrow \mathscr{I}_{\mathscr{Y}}$ defined by

$$\psi_G^s : Ga \rightarrow Gas \quad (ass^{-1}a^{-1} \geq e)$$

is a transitive representation of S equivalent to ϕ_H. If $G = \{e\}$, then we simply write $\psi_e : S \rightarrow \mathscr{I}_{\mathscr{Y}}$.

In this paper we study the congruences on S which are induced by the transitive representations of S. Since equivalent representations induce the same congruence, it is sufficient to study just those representations obtained from the closed inverse subsemigroups of S in the manner described above. Moreover, it is known that two closed inverse subsemigroups H and K produce equivalent representations if and only if H and K are *conjugate*, that is to say, for some $x \in S$, $xHx^{-1} \subseteq K$ and $x^{-1}Kx \subseteq H$. The next lemma describes all conjugates of a closed inverse subsemigroup of S.

LEMMA 2.5. *Let H be a closed inverse subsemigroup of S. Then for any $x \in S$ such that $x^{-1}x \in H$, $K=(xHx^{-1})\omega$ is a conjugate of H; in fact $x^{-1}Kx \subseteq H$ and $xHx^{-1} \subseteq K$. Conversely, every conjugate of H is of this form.*

Proof. Since xHx^{-1} is an inverse subsemigroup of S, $(xHx^{-1})\omega$ is a closed inverse subsemigroup of S and $xHx^{-1} \subseteq (xHx^{-1})\omega$. We show that $x^{-1}((xHx^{-1})\omega)x \subseteq H$. Let $u \in (xHx^{-1})\omega$. Then for some $e \in E$, we have $ue \in xHx^{-1}$ so that $x^{-1}uex \in x^{-1}xHx^{-1}x \subseteq H$. Since $x^{-1}ux \geq x^{-1}uex$, we obtain $x^{-1}ux \in H\omega = H$. Thus $x^{-1}((xHx^{-1})\omega)x \subseteq H$, as required.

Conversely, suppose that K is a conjugate of H. Then there exists $x \in S$ such that $xHx^{-1} \subseteq K$ and $x^{-1}Kx \subseteq H$. We show that $K=(xHx^{-1})\omega$. Observe that $x^{-1}(xHx^{-1})x \subseteq H$. Since $x^{-1}ux \geq x^{-1}uex$, we obtain $x^{-1}ux \in H\omega = H$. Thus $x^{-1}((xHx^{-1})\omega)x \subseteq H$, as required.
obtain

$$K=(xx^{-1}Kxx^{-1})\omega \subseteq (xHx^{-1})\omega \subseteq K\omega = K.$$

For a closed inverse subsemigroup H of S we shall denote by $\bar{\phi}_H$ the congruence $\phi_H \circ \phi_H^{-1}$ induced on S by the transitive representation ϕ_H obtained from H as described above. We shall use ϕ and $\bar{\phi}$ to denote ϕ_H and $\bar{\phi}_H$ whenever there is no ambiguity in doing so.

It is shown in [**1**, Lemma 7.23] that $\bar{\phi}_H = P_H$. This fact can be presented conceptually as follows. Note that for all $a \in S$, $aP'_H=(Ha)\omega$ if $aa^{-1} \in H$ while if $aa^{-1} \notin H$ then $aP'_H = W'_H$. Thus if one passes from the 0-transitive representation of S that is obtained by right multiplication on S/P'_H to the 0-transitive representation of S by one-to-one partial transformations as described by Wagner (see for example [**2**, IV. 5.9]) and then drop the zero, the result is ϕ_H. Since the congruence induced on S by the representation on S/P'_H is P_H and since at each stage the resulting representations induce the same congruence, we see that $\bar{\phi}_H = P_H$.

In this context, we remark that in a group, any right coset Hg of a given subgroup H completely determines the congruence induced by the transitive representation on right cosets of H, namely $P_H = P_{Hg}$. The situation is similar for inverse semigroups.

PROPOSITION 2.6. *If H is a closed inverse subsemigroup of S and $aa^{-1} \in H$, then $P_H = P_{(Ha)\omega}$.*

Proof. If xP_Hy, then for any $u,v \in S$,

$$uxv \in (Ha)\omega \Leftrightarrow uxva^{-1} \in H \Leftrightarrow uyva^{-1} \in H \Leftrightarrow uyv \in (Ha)\omega$$

and so $xP_{(Ha)\omega}y$. Conversely, if $xP_{(Ha)\omega}y$, then for $u,v \in S$ we obtain

$$uxv \in H \Leftrightarrow uxva \in (Ha)\omega \Leftrightarrow uyva \in (Ha)\omega \Leftrightarrow uyv \in H,$$

whence xP_Hy.

COROLLARY 2.7. *Let ρ be a congruence on S and $a \in S$. Then $P_{(a\rho)\omega} = P_{((aa^{-1})\rho)\omega}$.*

Proof. By Lemma 2.4, $(a\rho)\omega = ((aa^{-1})\rho a)\omega$, which in turn is equal to $(((aa^{-1})\rho)\omega a)\omega$ by Lemma 2.1. Thus $P_{(a\rho)\omega} = P_{(((aa^{-1})\rho)\omega a)\omega} = P_{((aa^{-1})\rho)\omega}$ by Proposition 2.6.

LEMMA 2.8. *Let ρ be a congruence on S and let $X \subseteq S$ be saturated by ρ. Then the following hold.*

(i) *$X\omega$ is saturated by ρ and $(X\omega)/\rho = (X/\rho)\omega$.*

(ii) *$P_X/\rho = P_{X/\rho}$.*

Proof. (i). Let $x \in X$ and suppose that $x\rho y$. Then for some $e \in E$, we have $xe \in X$ and $xe\rho ye$. Since X is saturated by ρ we have $ye \in X$ whence $y \in X\omega$. Thus $X\omega$ is saturated by ρ.

Now, let $x\rho \in (X\omega)/\rho$, whence $x \in X\omega$. For some $e \in E$ we have $xe \in X$, whence $(x\rho)(e\rho) \in X/\rho$. Thus $x\rho \in (X/\rho)\omega$ and so $(X\omega)/\rho \subseteq (X/\rho)\omega$. Conversely, let $x\rho \in (X/\rho)\omega$. By Lallement's Lemma, there exists $e \in E$ with $(xe)\rho \in X/\rho$ and so $xe \in X$. Thus $x \in X\omega$ and so $x\rho \in (X\omega)/\rho$, whence $(X/\rho)\omega \subseteq (X\omega)/\rho$.

(ii). This follows immediately from the observation that for all $x,a,y \in S$, $xay \in X$ if and only if $(x\rho)(a\rho)(y\rho) \in X/\rho$.

COROLLARY 2.9. *Let ρ and τ be congruences on S with $\rho \subseteq \tau$. Then τ is induced by a transitive representation of S if and only if τ/ρ is induced by a transitive representation of S/ρ.*

Proof. By Lemma 2.8, $\tau = P_H$ for some closed inverse subsemigroup H of S if and only if $\tau/\rho = P_{H/\rho}$ and H/ρ is a closed inverse subsemigroup of S/ρ.

We conclude this section with the definition of the *kernel* and the *trace* of a congruence on an inverse semigroup (see [**2**, Chapter III] for a detailed discussion of these notions). For any congruence ρ on S, the kernel of ρ, denoted by $\ker \rho$, is defined to be the union of the ρ-classes which contain idempotents, and the trace of ρ, denoted by $\operatorname{tr} \rho$, is the restriction of ρ to E.

3. The kernel and trace of the congruence induced by a transitive representation. *In this section, H shall denote a closed inverse subsemigroup of S.*

PROPOSITION 3.1. *For any $a,b \in S$, $a\bar{\phi}b$ if and only if for every $x \in S$ with $x^{-1}x \in H$, the following conditions hold:*

1) $x^{-1}aa^{-1}x \in H \Leftrightarrow x^{-1}bb^{-1}x \in H$;

2) $x^{-1}aa^{-1}x \in H \Rightarrow x^{-1}ab^{-1}x \in H$.

Proof. Suppose that $a\bar{\phi}b$. Then for every $x \in S$ with $x^{-1}x \in H$, $(Hx^{-1})\omega \in \mathbf{d}\phi^a$ if and only if $(Hx^{-1})\omega \in \mathbf{d}\phi^b$, which is equivalent to $x^{-1}a(x^{-1}a)^{-1} \in H$ if and only if $x^{-1}b(x^{-1}b)^{-1} \in H$. Moreover, $x^{-1}aa^{-1}x \in H$ implies $(Hx^{-1})\omega \in \mathbf{d}\phi^a = \mathbf{d}\phi^b$ and $(Hx^{-1})\omega\phi^a = (Hx^{-1})\omega\phi^b$. Thus $x^{-1}aa^{-1}x \in H$ implies that $(Hx^{-1}a)\omega = (Hx^{-1}b)\omega$, which occurs exactly when $x^{-1}a(x^{-1}b)^{-1} \in H$.

Conversely, suppose that for each $x \in S$ with $x^{-1}x \in H$, $x^{-1}aa^{-1}x \in H$ if and only if

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$x^{-1}bb^{-1}x \in H$ and $x^{-1}aa^{-1}x \in H$ implies that $x^{-1}ab^{-1}x \in H$. The first condition asserts that $\mathbf{d}\phi^a = \mathbf{d}\phi^b$, while the second condition ensures that $(Hx^{-1}a)\omega = (Hx^{-1}b)\omega$ for all $x \in S$ for which $Hx^{-1} \in \mathbf{d}\phi^a = \mathbf{d}\phi^b$. Thus $\phi^a = \phi^b$ and so $a\bar{\phi}b$.

As a consequence of this proposition, the trace of $\bar{\phi}$ can readily be described.

Corollary 3.2. *For $e,f \in E$, $e\bar{\phi}f$ if and only if for all $x \in S$ with $x^{-1}x \in H$,*

$$e \in (xHx^{-1})\omega \Leftrightarrow f \in (xHx^{-1})\omega.$$

Proof. In view of Proposition 3.1, it is sufficient to observe that if $x^{-1}ex$ and $x^{-1}fx$ both belong to H, then $x^{-1}efx = x^{-1}exx^{-1}fx \in H$ as well.

We shall use the following symbolism.

Notation 3.3. Denote by $\mathscr{C} = \mathscr{C}_H$ the set of all conjugates of H, that is

$$\mathscr{C} = \{(xHx^{-1})\omega \mid x^{-1}x \in H\}.$$

The preceding corollary can then be stated as: $e\bar{\phi}f$ if and only if for all $C \in \mathscr{C}$, $e \in C \Leftrightarrow f \in C$.

Lemma 3.4. *Let $e \in E$. Then either $SeS \cap H = \varnothing$ or else $e \in C$ for some $C \in \mathscr{C}$.*

Proof. Suppose that $SeS \cap H \neq \varnothing$. Then for some $u,v \in S$, $uev \in H$ so that $v^{-1}v \in H$ and $(ev)^{-1}(ev) = v^{-1}ev \in H$. But then $vv^{-1}e = v(v^{-1}ev)v^{-1} \in vHv^{-1}$, whence $e \in (vHv^{-1})\omega$. Since $v^{-1}v \in H$, we have $(vHv^{-1})\omega \in \mathscr{C}$, as required.

Note that for any $C \in \mathscr{C}$, $P_C = P_H$ and so $W \cap C = \varnothing$. As a consequence, Lemma 3.4 allows us to conclude that

$$E \backslash W = \bigcup\{E_C \mid C \in \mathscr{C}\}.$$

Corollary 3.5. *The kernel of $\bar{\phi}$ is given by*

$$\ker \bar{\phi} = \{x \in S \mid \text{for all } C \in \mathscr{C},\ \{xx^{-1}, x^{-1}x\} \cap C \neq \varnothing \Rightarrow x \in \mathscr{C}\}.$$

Proof. Suppose that $x \in \ker \bar{\phi}$. Then $x\bar{\phi}x^{-1}x$, which by Proposition 3.1 implies that $u^{-1}xx^{-1}u \in H$ if and only if $u^{-1}x^{-1}xu \in H$ and that $u^{-1}xx^{-1}u \in H$ forces $u^{-1}xx^{-1}xu = u^{-1}xu \in H$ for any $u \in S$ for which $u^{-1}u \in H$. For each $C \in \mathscr{C}$ there exists $u \in S$ with $u^{-1}u \in H$ and $C = (uHu^{-1})\omega$. It follows from Lemma 2.5 that $x^{-1}x \in C$ if and only if $u^{-1}xx^{-1}u \in H$, and that $x^{-1}x \in C$ if and only if $u^{-1}x^{-1}xu \in H$. Thus we have $xx^{-1} \in C$ if and only if $x^{-1}x \in C$ and either one implies that $u^{-1}xu \in H$, whence $x \in (uHu^{-1})\omega = C$.

Conversely, suppose that $\{xx^{-1}, x^{-1}x\} \cap C \neq \varnothing$ implies that $x \in C$, for all $C \in \mathscr{C}$. We show that $x\bar{\phi}x^{-1}x$. By Proposition 3.1, this can be accomplished by demonstrating that $u^{-1}xx^{-1}u \in H$ if and only if $u^{-1}x^{-1}xu \in H$ and that if $u^{-1}xx^{-1}u \in H$, then $u^{-1}xu \in H$ for all $u \in S$ for which $u^{-1}u \in H$. By Lemma 2.5, this is equivalent to $xx^{-1} \in C$ if and only if $x^{-1}x \in C$, and if $xx^{-1} \in C$ then $x \in C$, for all $C \in \mathscr{C}$. But by hypothesis, either of $xx^{-1} \in C$ or $x^{-1}x \in C$ forces $x \in C$, whence both $xx^{-1} \in C$ and $x^{-1}x \in C$ are true if either is true. Thus $x\bar{\phi}x^{-1}x$.

We continue the analogy with groups with a description of the idempotent classes of $\bar{\phi}$ and finally with a description of $\bar{\phi}$ itself which is similar to the description of a congruence on a group.

NOTATION 3.6. For each $e \in E$, let

$$C_e = \bigcap\{C \in \mathscr{C} \mid e \in C\}.$$

We remark that by Corollary 3.2, idempotents e and f are related by $\bar{\phi}$ if and only if $C_e = C_f$.

PROPOSITION 3.7. *For each $e \in E$, we have $C_e = (e\bar{\phi})\omega$.*

Proof. For $e \in W$, $C_e = \bigcap\varnothing = S$ and $e\bar{\phi} = W$. Since W is an ideal of S, $W\omega = S$ and so $C_e = (e\bar{\phi})\omega$. On the other hand, if $e \notin W$, then for any $x \in e\bar{\phi}$, we have $x^{-1}x\bar{\phi}e$. Thus for all $C \in \mathscr{C}$, $x^{-1}x \in C$ if and only if $e \in C$. In view of this, $e \in C$ implies $x \in C$ by Corollary 3.5. But then $x \in C_e$. We have thus obtained $e\bar{\phi} \subseteq C_e$. Since C_e is closed, $(e\bar{\phi})\omega \subseteq C_e$. Conversely, let $x \in C_e$. In view of Proposition 2.2, it is sufficient to show that $ex\bar{\phi}e$. By Proposition 3.1 and Lemma 2.5, this can be accomplished by showing that $exx^{-1} \in C$ if and only if $e \in C$ and that $exx^{-1} \in C$ implies $exe \in C$ for all $C \in \mathscr{C}$. Since $x \in C_e$, we know that $e \in C$ implies that $x \in C$. Thus, if $e \in C$ then $x \in C$, whence $x^{-1} \in C$ and so $exx^{-1} \in C$. Conversely, $exx^{-1} \in C$ implies $e \in C$ since C is closed. Thus $exx^{-1} \in C$ if and only if $e \in C$. As well, if $exx^{-1} \in C$ then $e,x \in C$ and so $exe \in C$.

PROPOSITION 3.8. *For any $a,b \in S$, $a\bar{\phi}b$ if and only if $C_{aa^{-1}} = C_{bb^{-1}}$ and $(C_{aa^{-1}}a)\omega = (C_{bb^{-1}}b)\omega$.*

Proof. If $a\bar{\phi}b$, then $aa^{-1}\bar{\phi}bb^{-1}$, whence $C_{aa^{-1}} = C_{bb^{-1}}$. As well, by Lemmas 2.4 and 2.1 we have $(a\bar{\phi})\omega = ((aa^{-1})\bar{\phi}a)\omega = ((aa^{-1})\bar{\phi}\omega a)\omega$ and by Proposition 3.7, $(aa^{-1})\bar{\phi}\omega = C_{aa^{-1}}$. Thus $(a\bar{\phi})\omega = (C_{aa^{-1}}a)\omega$. Similarly, $(b\bar{\phi})\omega = (C_{bb^{-1}}b)\omega$ and so $(C_{aa^{-1}}a)\omega = (C_{bb^{-1}}b)\omega$.

Conversely, suppose that $C_{aa^{-1}} = C_{bb^{-1}}$ and that $(C_{aa^{-1}}a)\omega = (C_{bb^{-1}}b)\omega$. By Proposition 3.1, Lemma 2.5 and Corollary 3.2, $a\bar{\phi}b$ if $C_{aa^{-1}} = C_{bb^{-1}}$ and $ab^{-1} \in C_{aa^{-1}}$. We have $a \in C_{aa^{-1}}a \subseteq (C_{aa^{-1}}a)\omega = (C_{bb^{-1}}b)\omega$ and so by Proposition 3.7 and Lemma 2.1 we have

$$ab^{-1} \in ((bb^{-1})\bar{\phi}\omega b)\omega b^{-1} \subseteq (bb^{-1})\bar{\phi}\omega = C_{bb^{-1}} = C_{aa^{-1}}.$$

In view of Proposition 3.7, we may attempt to generalize the notion of the greatest normal subgroup contained within a given subgroup of a group in the following manner.

NOTATION 3.9. Let $K = K_H = \bigcup_{e \in E_H} C_e$.

PROPOSITION 3.10. *$K = (H \cap \ker \bar{\phi})\omega$ and so K is a closed inverse subsemigroup of S contained in H. Moreover, $\bar{\phi}_H = \bar{\phi}_K$.*

Proof. Since H is saturated by $\bar{\phi}$ and H is closed, Proposition 3.7 yields $K = \bigcup_{e \in E_H} C_e = \bigcup_{e \in E_H} (e\bar{\phi})\omega = \left(\bigcup_{e \in E_H} e\bar{\phi}\right)\omega = (H \cap \ker \bar{\phi})\omega.$

Now, by Proposition 2.2, it follows that for each $e \in E$, $(e\bar{\phi}_H)\omega$ is saturated by $\bar{\phi}_H$ and so K is saturated by $\bar{\phi}_H$. Thus $\bar{\phi}_H \subseteq P_K = \bar{\phi}_K$. Conversely, suppose that $a\bar{\phi}_K b$. Let $x \in S$ be such that $x^{-1}x \in H$. If $x^{-1}aa^{-1}x \in H$, then $x^{-1}x, x^{-1}aa^{-1}x \in E_H = E_K$ and so by Proposition 3.1, $x^{-1}bb^{-1}x, x^{-1}ab^{-1}x \in K \subseteq H$. Similarly, $x^{-1}bb^{-1}x \in H$ implies that $x^{-1}aa^{-1}x \in H$. Thus by Proposition 3.1 we have $a\bar{\phi}_H b$.

We remark that in general, one can expect to have different closed inverse subsemigroups of S contained in K which still induce the same congruence. Consider the bicyclic semigroup for example. If the ω-chain of idempotents is denoted as usual by $e_0 > e_1 > \dots$, then $H = \{e_0, e_1\}$ and $L = \{e_0\}$ are each closed inverse subsemigroups. Now $\bar{\phi}_H$ must saturate H and as a result, $\bar{\phi}_H = \varepsilon$. Similarly, $\bar{\phi}_L = \varepsilon$. But then $K = H$ and so $L \subsetneqq K$.

The next proposition establishes that K is maximal in a sense which is analogous to the notion of the greatest normal subgroup contained in a given subgroup of a group.

Proposition 3.11. *If L is a closed inverse subsemigroup of S such that*
(i) $E_L = E_H$.
(ii) $L \subseteq (H \cap \ker P_L)\omega$,
then $L \subseteq K$.

Proof. We show that $P_L \subseteq P_H$. Then since $P_H = P_K$, we obtain $L \subseteq (H \cap \ker P_L)\omega \subseteq (H \cap \ker P_H)\omega = K$. Suppose now that $aP_L b$. Let $x, y \in S$ be such that $xay \in H$. Then $xay(xay)^{-1} \in E_H = E_L$ whence $xay(xay)^{-1} \in L$. By hypothesis, we have $xby(xay)^{-1} \in L$. As a result $xby(xay)^{-1}xay \in H$ since H is a subsemigroup, and finally $xby \in H$ since H is closed. By symmetry, we have $aP_H b$.

Note that if S is a group, then condition (i) is automatically satisfied while condition (ii) simply asserts that L is a normal subgroup of S contained in H.

In this context, we offer an alternative description of $\ker \bar{\phi}$ (cf. Corollary 3.5).

Lemma 3.12. *The kernel of P_H is given by*:

$$\ker P_H = \{k \in S \mid xky \in H \Rightarrow xy \in H\}.$$

Proof. Let $k \in \ker P_H$. Then $kP_H kk^{-1}$ and thus for $x, y \in S$, $xkyP_H xkk^{-1}y$. Since H is saturated by P_H, $xky \in H$ implies $xkk^{-1}y \in H$ and thus $xy \in H$.

Conversely, let $k \in S$ be such that $xky \in H$ implies that $xy \in H$. We show that $kP_K kk^{-1}$. For any $x, y \in S$, $xkk^{-1}y \in H \Rightarrow y^{-1}kk^{-1}x^{-1} \in H \Rightarrow y^{-1}k^{-1}x^{-1} \in H \Rightarrow xky \in H \Rightarrow xkk^{-1}ky \in H \Rightarrow xkk^{-1}y \in H$, as required.

Corollary 3.13. *If $L \subseteq H$ is a closed inverse subsemigroup of S for which $L \subseteq \ker P_L$ and $E_L = E_H$, then $xLx^{-1} \cap H \subseteq L$ for all $x \in S$.*

Proof. Let $l \in L$ and $x \in S$ be such that $xlx^{-1} \in H$. Then $xl(xl)^{-1} \in E_H = E_L$. By Lemma 3.12, $x(xl)^{-1} \in L$ and thus $xlx^{-1} \in L$.

We shall require the following results.

LEMMA 3.14. *Let G be a subgroup of S and let e be the identity of G. Further, let $H = G\omega$. Then $K = (e\bar{\phi})\omega$.*

Proof. By [**2**, IV.5.1], E_H has e as a zero element. We show that for any $f \in E_H$, $f\bar{\phi} \subseteq (e\bar{\phi})\omega$. Let $x \in f\bar{\phi}$. Then $xe\bar{\phi}fe = e$ and so by Proposition 2.2 we obtain that $x \in (e\bar{\phi})\omega$, as required. But then $(f\bar{\phi})\omega \subseteq (e\bar{\phi})\omega$ and so $K = \bigcup_{f \in E_H} C_f = \bigcup_{f \in E_H} (f\bar{\phi})\omega = (e\bar{\phi})\omega$.

COROLLARY 3.15. *Let G be a subgroup of S with identity e and let $\rho = \bar{\phi}_{G\omega}$. Then $\rho = \bar{\phi}_{(e\rho)\omega}$.*

Proof. This follows immediately from Proposition 3.10 and Lemma 3.14.

We remark that two different conjugates of a closed inverse subsemigroup need not intersect. Any non-trivial Brandt semigroup will illustrate this. At the other extreme, in a group, the set of all conjugates of a subgroup has non-empty intersection, this intersection being of course the greatest normal subgroup contained in the given subgroup. The next lemma shows that in a Clifford semigroup (i.e. a semilattice of groups), all conjugates of a given closed inverse subsemigroup have exactly the same idempotents.

LEMMA 3.16. *Let S be a Clifford semigroup and let H be a closed inverse subsemigroup of S. Then $E_H = E_C$ for all $C \in \mathscr{C}$.*

Proof. By symmetry, it is sufficient to show that $E_H \subseteq (xHx^{-1})\omega$ if $x^{-1}x \in H$. Let $e \in E_H$. Then $e \geq exx^{-1} = xex^{-1} \in xHx^{-1}$ and so $e \in (xHx^{-1})\omega$ as required.

COROLLARY 3.17. *Let H be a closed inverse subsemigroup of a Clifford semigroup. Then $K = \bigcap_{x^{-1}x \in H} (xHx^{-1})\omega$.*

Proof. By definition, $K = \bigcup_{e \in E_H} \left\{ \bigcap_{x^{-1}ex \in H} (xHx^{-1})\omega \right\}$. From Lemma 3.16 we have for $e \in E_H$ that $x^{-1}ex \in H$ if and only if $x^{-1}x \in H$. Thus $K = \bigcup_{e \in E_H} \left\{ \bigcap_{x^{-1}x \in H} (xHx^{-1})\omega \right\} = \bigcap_{x^{-1}x \in H} (xHx^{-1})\omega$, as required.

4. Inverse semigroups with a faithful transitive representation. We characterize here inverse semigroups with primitive idempotents and a faithful transitive representation. We also describe congruences ρ for which $\rho = \bar{\phi}_{(e\rho)\omega}$ for an idempotent e such that $e\rho$ is primitive in S/ρ. We start with the case of Clifford semigroups (semilattices of groups).

PROPOSITION 4.1. *Let S be a Clifford semigroup. Then S has a faithful transitive representation if and only if S is either a group or a group with zero.*

Proof. Let H be a closed inverse subsemigroup of S such that $\bar{\phi}_H = \varepsilon$. Then $W_H = \varnothing$ or else W_H is an ideal of S and a class of $\bar{\phi}_H$. If the latter case occurs then $|W_H| = 1$ and so

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S has a zero. For any $e \in E \setminus W_H$, by Lemma 3.16 we have

$$C_e = \bigcap\{C \mid e \in C\} = \bigcap\{C \mid C \in \mathscr{C}\}$$

and so $C_e = C_f$ for $e, f \in E$. Thus $|E \setminus W_H| = 1$ and so S is either a group (if $W_H = \varnothing$) or else a group with zero.

The converse is clear.

Consequently one may observe that a congruence ρ on a Clifford semigroup is induced by a transitive representation if and only if S/ρ is a group or a group with zero. Furthermore, for any congruence ρ on an inverse semigroup, $\rho = \bigcap_{e \in E} P_{(e\rho)\omega}$ (see [3]) and so every Clifford semigroup is a subdirect product of groups with a zero possibly adjoined (see [2, II.2.6]). Observe that $E_H = E_C$ for all $C \in \mathscr{C}_H$ if and only if $S\bar{\phi}_H$ is a group or a group with zero. That this can occur for semigroups other than Clifford semigroups is seen from the following example. Let S be any inverse semigroup with a completely prime ideal I. Let $H = S \setminus I$. Then $x^{-1}x \in H$ implies $(xHx^{-1})\omega \subseteq H$.

THEOREM 4.2. *An inverse semigroup S has a zero, a primitive idempotent e, and a faithful transitive representation if and only S is a dense ideal extension of a Brandt semigroup B. Moreover, under these conditions, $B = J(e)$.*

Proof. Necessity. Let e be a primitive idempotent of S and H be a closed inverse subsemigroup of S for which $\phi_H : S \to \mathscr{I}(X)$ is faithful. Assume first that $0 \in H$. Then X has only one element and thus S has at most two elements. Hence $S = \{0, e\}$ is a two element semilattice, a special case of a Brandt semigroup.

Now suppose that $0 \notin H$. Then $0 \in W_H$ and since ϕ_H is faithful, we must have $W_H = \{0\}$. Since $e \neq 0$, we get that e is contained in some conjugate C of H. Since C and H induce equivalent representations, we may assume that $e \in H$. Now e must be the zero of E_H by primitivity and $0 \notin H$, which gives

$$K = \bigcup_{f \in E_H} f\omega = e\omega$$

and $\bar{\phi}_K = \bar{\phi}_H = \varepsilon$.

Let $B = J(e)$. Then B is a Brandt semigroup since e is primitive, and S is an ideal extension of B. Recall from Section 2 that $\psi_e : S \to \mathscr{I}(R_e)$ and that ψ_e is faithful. This means that S acts faithfully by right multiplication on the $\mathscr{R}$-class R_e and thus *a fortiori* on B. In view of [2, I.9.18], we conclude that S is a dense extension of B.

Sufficiency. Let S be a dense extension of a Brandt semigroup B. Then S has a zero and a primitive idempotent e. In order to prove that ψ_e is faithful, by the density of extension, it suffices to prove that $\bar{\psi}_e|_B = \varepsilon$. Let $B = B(G, I)$, $e = (1, 1, 1)$ and assume that $\psi_e^{(i,g,j)} = \psi_e^{(k,h,l)}$. Then

$$0 \neq (1, 1, i)(i, g, j) = (1, 1, i)\psi_e^{(i,g,j)} = (1, 1, i)\psi_e^{(k,h,l)} = (1, 1, i)(k, h, l)$$

so that $(i, g, j) = (k, h, l)$, as required.

We consider next the case without zero.

PROPOSITION 4.3. *If S has a primitive idempotent, a faithful transitive representation and no zero, then it is a group.*

Proof. Let e be a primitive idempotent of S. Since S has no zero, e must be the unique minimal idempotent of S and hence the zero of E. By [**2**, IV.5.5], H_e is a group ideal of S. By hypothesis, S has a faithful transitive representation, say ϕ_H for some closed inverse subsemigroup H of S. Since S has no zero and ϕ_H is faithful, $W_H = \varnothing$. According to Lemma 3.4, e is contained in some conjugate C of H. We may thus assume that $e \in H$. Since e is the zero of E, [**2**, IV.5.5] implies that $H = G\omega$ for some subgroup G of H_e. In view of Corollary 3.2, any two idempotents of S are $\bar{\phi}_H$-related since e is its only conjugate (recall that H_e is an ideal of S) and so $e \in (xHx^{-1})\omega$ for any $x \in S$, which implies that $E \subseteq (xHx^{-1})\omega$. But ϕ_H is faithful, so S has only one idempotent and thus must be a group.

The next theorem gives necessary and sufficient conditions on a congruence ρ on S to be induced by $\phi_{(e\rho)\omega}$ for an idempotent e for which $e\rho$ is primitive in S/ρ. This theorem will be useful in the next section.

We will need the following simple result.

LEMMA 4.4. *Let S be a dense extension of a Brandt semigroup B. Then every non-zero ideal of S contains B.*

Proof. Let I be an ideal of S and suppose that $B \not\subseteq I$. Then since B is 0-simple, we have $I \cap B = \{0\}$. But then the Rees congruence ρ_I when restricted to B is equality, whence $\rho_I = \varepsilon$. Thus $I = \{0\}$.

For any congruence ρ on S, let $\rho^{\#}$ be the natural homomorphism of S onto S/ρ.

THEOREM 4.5. *Let ρ be a congruence on S and $e \in E$. Then the following are equivalent.*

(i) *$e\rho^{\#}$ is a primitive idempotent of S/ρ and $\rho = \bar{\phi}_{(e\rho)\omega}$.*

(ii) *The ideals $J = \bigcup\{x\rho \mid x \in SeS\}$ and $I = \begin{cases} \varnothing & \text{if } S/\rho \text{ has no zero,} \\ f\rho & \text{if } f\rho^{\#} \text{ is the zero of } S/\rho \end{cases}$* satisfy the following conditions:

(a) *if $I \neq \varnothing$, then I is a prime ideal,*

(b) *$e \in J \setminus I$,*

(c) *for all $g, h \in (J \setminus I) \cap E$, $g\rho gh$ implies that $g\rho h$,*

(d) *for any congruence ξ on S, $\xi|_J = \rho|_J$ implies that $\xi \subseteq \rho$.*

Proof. If S/ρ has no zero, we may adjoin a zero to S and extend ρ in the obvious way. The result would then follow from the case with zero upon removal of the zero. Thus we may assume that S/ρ has a zero $f\rho^{\#}$.

(i) *implies* (ii). By Corollary 2.9, the equality relation on S/ρ is induced by a transitive representation of S/ρ, whence S/ρ has a faithful transitive representation.

Since S/ρ has a zero and a primitive idempotent $e\rho^{\#}$, we may apply Theorem 4.2 to obtain that S/ρ is a dense extension of the Brandt semigroup $B = J(e\rho^{\#})$.

We now consider condition (a). Let $a,b \in S$ and suppose that $aSb \subseteq I$. Then $J(a\rho^{\#})J(b\rho^{\#}) = \{0\}$ in S/ρ. Since $B^2 \neq 0$, we must have either $B \nsubseteq J(a\rho^{\#})$ or $B \nsubseteq J(b\rho^{\#})$. Thus by Lemma 4.4, either $a\rho^{\#} = 0$ or $b\rho^{\#} = 0$, whence $a \in I$ or $b \in I$ and so I is prime.

Condition (b) follows immediately from $e\rho^{\#} \neq 0$. In order to establish condition (c), let $g,h \in E_{J\setminus I}$ and suppose that $g\rho gh$. Then $g\rho^{\#} = (g\rho^{\#})(h\rho^{\#})$ with $g\rho^{\#}$, $h\rho^{\#} \in E_B$. Thus $g\rho^{\#} = h\rho^{\#}$ whence $g\rho h$.

Finally, let ξ be a congruence on S such that $\xi|_J = \rho|_J$. We prove that $(\xi \vee \rho)|_J = \rho|_J$. Indeed, let $a\xi \vee \rho b$ for $a,b \in J$. There exist $x_1, x_2, \ldots, x_n \in S$ such that

$$a\xi x_1 \rho x_2 \xi x_3 \ldots x_n \rho b.$$

Multiplying on the right by $a^{-1}a$, we obtain

$$a\xi x_1 a^{-1}a\rho x_2 a^{-1}a \ldots x_n a^{-1}a\rho ba^{-1}a,$$

where $x_1a^{-1}a, x_2a^{-1}a, \ldots, x_na^{-1}a \in J$. The hypothesis then implies that $a\rho ba^{-1}a$. Similarly, we get $bb^{-1}a\rho b$ and thus $bb^{-1}a\rho ba^{-1}a$. Consequently $a\rho ba^{-1}a\rho b$, as required. Now $(\xi \vee \rho)/\rho$ is a congruence on S/ρ whose restriction to B is the equality relation. But S/ρ is a dense extension of B and thus $(\xi \vee \rho)/\rho = \varepsilon$. This implies that $\xi \vee \rho = \rho$, that is to say, $\xi \subseteq \rho$. Thus condition (d) holds.

(ii) *implies* (i). By definition, $B = J\rho^{\#}$ is the ideal of S/ρ generated by $e\rho^{\#}$. Moreover, by (b) we have $J\rho^{\#} \neq \{0\}$. Then using Lallement's Lemma and (c), we obtain that all non-zero idempotents of B are primitive. Since I is a prime ideal of S saturated by ρ, it follows that $I\rho^{\#} = \{0\}$ is a prime ideal of S/ρ, hence of B. It is well-known that this is equivalent to the assertion that B is a Brandt semigroup. Moreover, we have by (b) that $e\rho^{\#} \neq 0$, whence $e\rho^{\#}$ is primitive. Observe as well that (d) implies that S/ρ is a dense ideal extension of B. We shall use this fact to demonstrate that the residue of $(e\rho)\omega$ is I.

Let $a \in I$. Then $J(a) \cap e\rho \subseteq I \cap e\rho = \varnothing$ since $e \in J\setminus I$ and I is a ρ-class. Suppose now that $a \in S\setminus I$. Then $J(a\rho^{\#})$ is a non-zero ideal of S/ρ and so by Lemma 4.4 we have $B \subseteq J(a\rho^{\#})$. Thus $e\rho^{\#} \in J(a\rho^{\#})$ and so $J(a) \cap e\rho \neq \varnothing$. We have shown that $I = W_{e\rho}$. It is easy to see that $W_{(e\rho)\omega} = W_{e\rho}$ whence $I = W_{(e\rho)\omega}$.

Now let $a,b \in J$ be such that $a\bar{\phi}_{(e\rho)\omega}b$. Since $I = W_{(e\rho)\omega}$, we have that $a \in I$ if and only if $b \in I$, in which case $a\rho b$. Hence assume that $a \notin I$. Then $u^{-1}av \in (e\rho)\omega$ for some $u,v \in S$ by the above. It follows that $u^{-1}aa^{-1}u = (u^{-1}a)(u^{-1}a)^{-1} \in (e\rho)\omega$. Since $a\bar{\phi}_{(e\rho)\omega}b$, we have also

$$u^{-1}bb^{-1}u,\ u^{-1}ab^{-1}u \in (e\rho)\omega. \tag{1}$$

The inclusion $u^{-1}aa^{-1}u \in (e\rho)\omega$ also implies that $u^{-1}u \in (e\rho)\omega$. Consequently $aa^{-1} \in (u(e\rho)\omega u^{-1})\omega$ and thus $aa^{-1} \in ((ueu^{-1})\rho)\omega$. In view of Proposition 2.2 we now obtain $aa^{-1}ueu^{-1}\rho ueu^{-1}$. If $ueu^{-1} \in I$, then $u^{-1}(ueu^{-1})u \in I$ so that $(u^{-1}u)e(u^{-1}u) \in I \cap (e\rho)\omega$, a contradiction. We thus have aa^{-1}, $ueu^{-1} \in J\setminus I$, which together with $aa^{-1}ueu^{-1}\rho ueu^{-1}$ by condition (c) yields $aa^{-1}\rho ueu^{-1}$. Using (1), we similarly get $bb^{-1}\rho ueu^{-1}$, which then gives

$aa^{-1}\rho bb^{-1}$. Moreover, (1) yields $ab^{-1}ueu^{-1}\rho ueu^{-1}$, which then implies $ab^{-1}\rho bb^{-1}$. Now substituting a^{-1} for a and b^{-1} for b, we obtain $a^{-1}a\rho b^{-1}b$. Hence

$$a = aa^{-1}a\rho a(b^{-1}b) = (ab^{-1})b\rho bb^{-1}b = b.$$

This proves that $\bar{\phi}_{(e\rho)\omega}|_J \subseteq \rho|_J$. It follows easily from the expression for $\bar{\phi}_{(e\rho)\omega}$ that $\rho \subseteq \bar{\phi}_{(e\rho)\omega}$. In particular, we have $\bar{\phi}_{(e\rho)\omega}|_J = \rho|_J$. Now condition (d) yields $\bar{\phi}_{(e\rho)\omega} \subseteq \rho$, which finally gives $\bar{\phi}_{(e\rho)\omega} = \rho$.

5. Inverse semigroups all of whose congruences are induced by transitive representations. The main result here is a description of inverse semigroups in the title which are also completely semisimple and have only a finite number of $\mathcal{J}$-classes.

We shall require the following well-known results.

LEMMA 5.1. *Let S be completely semisimple with $S/\mathcal{J}$ finite. Then any subchain of E is finite.*

Proof. In a completely semisimple inverse semigroup we have $\mathcal{J} = \mathcal{D}$ and no two distinct $\mathcal{D}$-related idempotents are comparable.

LEMMA 5.2. *Let I be an ideal of S and ρ a congruence on I. Then $\rho \cup \varepsilon_S$ is a congruence on S.*

Proof. Straightforward.

The notion of special ideal extension plays an important role in the work that follows.

DEFINITION 5.3. Let I be an ideal of S. We shall say that S is a *special ideal extension* of I if every idempotent separating congruence on I extends uniquely to a congruence on S.

Observe that in view of Lemma 5.2 when S is a special ideal extension of I, then for each idempotent separating congruence ρ on I, $\rho \cup \varepsilon_S$ is the unique extension of ρ to S. Furthermore, a special ideal extension is of course dense.

LEMMA 5.4. *Let $I \subsetneq J$ be ideals of S for which S/I is a dense extension of J/I. Then for each $a \in S \setminus J$ there exists $b \in J \setminus I$ with $J_a > J_b$.*

Proof. Suppose for some $a \in S \setminus J$ there is no $b \in J \setminus I$ with $J_a > J_b$. Then $SaS \cap (J \setminus I) = \varnothing$. Let $K = SaS \cup I$. Since $a \in K \setminus J \subseteq K \setminus I$ we have $I \subsetneq K$. Then the Rees congruence ρ_I refines the Rees congruence ρ_K and so $\rho_K/\rho_I = \rho_{K/I}$ is a congruence on S/I whose restriction to J/I is equality. Since S/I is a dense extension of J/I, and $\rho_{K/I}$ restricted to J/I is equality, we must have $\rho_{K/I} = \varepsilon$. But this is not possible since $K \neq I$.

Our main theorem characterizes a class of inverse semigroups on which every congruence is induced by a transitive representation. As we shall see, this class includes the symmetric inverse semigroup on a finite set.

THEOREM 5.5. *Let S be an inverse semigroup with zero. Then the following are equivalent.*

(i) (a) *S is completely semisimple and $S/\mathcal{J}$ is finite;*

(b) *every congruence on S is induced by a transitive representation.*

(ii) *S satisfies* (a) *and*

(c) *for every congruence ρ on S, there exists $e \in E$ such that $\rho = \bar{\phi}_{(e\rho)\omega}$.*

(iii) (d) *S has a principal series*

$$0 = S_0 \subset S_1 \subset \ldots \subset S_n = S$$

such that for each $i = 1, 2, \ldots, n$, the quotient S_i/S_{i-1} is a Brandt semigroup;

(e) *for each $i = 1, 2, \ldots, n-1$, S_{i+1}/S_{i-1} is a special extension of S_i/S_{i-1}.*

(iv) *S satisfies* (d) *and*

(f) *for any $i = 1, 2, \ldots, n$ and any idempotent separating congruence β on S_i/S_{i-1}, the relation $\omega_{S_{i-1}} \cup \beta|_{S_i \setminus S_{i-1}} \cup \varepsilon_S$ is a congruence on S and conversely, every non-universal congruence on S is of this form.*

Proof. (i) *implies* (ii). Let ρ be a congruence on S. Then by (b) there is a closed inverse subsemigroup H of S such that $\rho = \bar{\phi}_H$. By Lemma 5.1 we obtain that E_H has a zero e, whence by [**2**, IV.5.5], there exists a subgroup G with identity e for which $H = G\omega$. An application of Corollary 3.15 then yields $\rho = \bar{\phi}_{(e\rho)\omega}$, as required.

(ii) *implies* (iii). We observe that (b) implies that each congruence on S is induced by a transitive representation. By Corollary 2.9, each congruence on any homomorphic image of S is also induced by a transitive representation. Moreover, every homomorphic image of an inverse semigroup with property (a) also has property (a). Thus every homomorphic image of S has properties (a) and (b).

Now, apply Lemma 5.1 to conclude that S has a primitive idempotent. Moreover, since ε is induced by a transitive representation, we see that S has a faithful transitive representation. By Theorem 4.2, S is a dense extension of a Brandt semigroup B_1. If $S \neq B_1$, then we may replace S by S/B_1 and repeat the argument to obtain that S/B_1 is a dense extension of a Brandt semigroup B_2. Because of the finiteness of $S/\mathcal{J}$, this process must terminate after n steps, where $n = |S/\mathcal{J}|$. We obtain the principal series required for (d) by setting $S_0 = \{0\}$ and $S_i \setminus S_{i-1} = B_i^*$ for $i = 1, 2, \ldots, n$.

To verify that (e) holds, let $i \in \{1, 2, \ldots, n-1\}$ and let β be any congruence on S_{i+1}/S_{i-1} whose restriction to S_i/S_{i-1} is idempotent separating. Let ρ denote the congruence on S obtained by applying Lemma 5.2 to the congruence on S_i that results when the restriction of β to S_i/S_{i-1} is pulled back to S_i. Then $\rho = \omega_{S_{i-1}} \cup \beta|_{S_i \setminus S_{i-1}} \cup \varepsilon_S$ since S_i/S_{i-1} is a Brandt semigroup and an idempotent separating congruence on a Brandt semigroup has the zero as a singleton congruence class. Let λ denote the congruence on S obtained by applying Lemma 5.2 to the congruence on S_{i+1} that results when β is pulled back to S_{i+1}. Now by (c), there exists an idempotent $e \in S$ with $\rho = \bar{\phi}_{(e\rho)\omega}$. We wish to show that $e\rho$ is primitive in S/ρ. Since $S_{i-1} = 0\rho$, we see that $e \notin S_{i-1}$. Suppose that $e \notin S_i$. Then $e\rho \cap S_i = \varnothing$ since S_i is saturated by ρ and so $(e\rho)\omega \cap S_i = \varnothing$. But then $S_i \subseteq W_{(e\rho)\omega} = S_{i-1}$, a contradiction. Thus $e \in S_i$. Since S/S_{i-1} is an ideal extension of the Brandt semigroup S_i/S_{i-1} and $e \in S_i \backslash S_{i-1}$, we see that e is primitive in S/S_{i-1}. The fact that the Rees congruence modulo S_{i-1} refines ρ allows us to conclude that $e\rho$ is primitive in S/ρ. We may therefore apply Theorem 4.5. Let J be the union of all ρ-classes which meet SeS. Since $e \in S_i$ and S_i is saturated by ρ, we obtain that $J \subseteq S_i$ whence $\lambda|_J = \rho|_J$. By Theorem 4.5(d) we have $\lambda \subseteq \rho$ and since it is evident that $\rho \subseteq \lambda$, we conclude that $\lambda = \rho$. Now since β is the quotient of $\lambda|_{S_{i+1}} = \rho|_{S_{i+1}}$ by the Rees congruence modulo S_{i-1}, it follows that $\beta = \beta|_{S_i/S_{i-1}} \cup \varepsilon_{S_{i+1}/S_{i-1}}$ whence S_{i+1}/S_{i-1} is a special ideal extension of S_i/S_{i-1}.

(iii) *implies* (iv). Let $i \in \{1, 2, \ldots, n\}$ and let β be any idempotent separating congruence on S_i/S_{i-1}. Then the zero of the Brandt semigroup S_i/S_{i-1} is a congruence class of β, and so $\omega_{S_{i-1}} \cup \beta|_{S_i \backslash S_{i-1}}$ is the pullback of β to S_i via the canonical homomorphism of S_i onto S_i/S_{i-1}. We then apply Lemma 5.2 to obtain the desired result.

Conversely, let ρ be an idempotent separating congruence on S and let i be maximal subject to the requirement that $\rho|_{S_{i-1}} = \omega_{S_{i-1}}$. Let $\beta = \rho|_{S_i \backslash S_{i-1}}$. We prove by induction on k that

$$\rho|_{S_k} = \omega_{S_{i-1}} \cup \beta \cup \varepsilon_{S_k}$$

for $i \leq k \leq n$. Consider first the case when $k = i$. Let $x \in S_i \backslash S_{i-1}$ and $y \in S_{i-1}$ be such that $x\rho y$. Then $x\rho 0$ since $S_{i-1} \subseteq y\rho$. As a result we have $S_i = SxS \cup S_{i-1} \subseteq 0\rho$ whence $\rho|_{S_i} = \omega_{S_i}$, contradicting the maximality of i. Thus $\rho|_{S_i}$ saturates S_{i-1}, whence $\rho|_{S_i} = \omega_{S_{i-1}} \cup \beta = \omega_{S_{i-1}} \cup \beta \cup \varepsilon_{S_i}$.

Next, suppose for some k, $i \leq k \leq n$, that $\rho|_{S_k}$ has the required form. Let $\alpha = \rho|_{S_{k+1}}$. We show that S_{k-1} is saturated by α. For, suppose that it is not. Then for some idempotent $e \in S_{k+1} \backslash S_k$ there is an idempotent $f \in S_{k-1}$ with $e\rho f$. By (e) and Lemma 5.4 applied to $S_{k-1} \subset S_k \subset S_{k+1}$, there exists $b \in S_k \backslash S_{k-1}$ with $J_e > J_b$. But then there exists an idempotent $g \in J_b$ with $e \geq g$. Since $J_b = S_k \backslash S_{k-1}$, we have $g \in S_k \backslash S_{k-1}$ and $g = eg\rho fg \in S_{k-1}$, contradicting the fact that $\rho|_{S_k}$ saturates S_{k-1}. Thus S_{k-1} is saturated by α and so the Rees congruence modulo S_{k-1} refines α. But then α induces a congruence on S_{k+1}/S_{k-1} whose restriction to S_k/S_{k-1} is induced by $\rho|_{S_k}$ and thus is an idempotent separating congruence on S_k/S_{k-1}. From (e) we infer that α saturates S_k and $\alpha|_{S_{k+1} \backslash S_k}$ is equality, whence

$$\alpha = \alpha|_{S_k} \cup \alpha|_{S_{k+1} \backslash S_k} = \rho|_{S_k} \cup \varepsilon_{S_{k+1} \backslash S_k} = \omega_{S_{i-1}} \cup \beta \cup \varepsilon_{S_k} \cup \varepsilon_{S_{k+1} \backslash S_k}$$

as required.

By induction, we obtain that $\rho = \rho|_{S_n} = \omega_{S_{i-1}} \cup \beta \cup \varepsilon_S$.

(iv) *implies* (ii). It is immediate that (d) implies (a). To verify (c), let ρ be a congruence on S. If $\rho = \omega$, then $\rho = \bar{\phi}_{(0\rho)\omega}$. Otherwise, ρ is a non-universal congruence on S. By (f), there exists i with

$$\rho = \omega_{S_{i-1}} \cup \rho|_{S_i \backslash S_{i-1}} \cup \varepsilon_S.$$

Let $A = \{a_0, a_1, \ldots, a_n\}$ where $a_0 = 0$ and $a_i \in S_i \backslash S_{i-1} = D_{a_i}$ for $i = 1, 2, \ldots, n$. Then by [3, Proposition 3.4], we have $\rho = \bigcap_{a \in A} P_{(a\rho)\omega}$. Since $a_j\rho = 0\rho = S_{i-1}$ for $0 \leq j \leq i-1$ and $P_{(0\rho)\omega} = \omega$, we obtain $\rho = \bigcap_{j \geq i} P_{(a_j\rho)\omega}$. Moreover, since $W_{a_i\rho} = W_{(a_i\rho)\omega} = S_{i-1}$, we have by

(f) that $P_{(a_i\rho)\omega} = \omega_{S_{i-1}} \cup \alpha \cup \varepsilon_S$ where α is the restriction of $P_{(a_i\rho)\omega}$ to $S_i \backslash S_{i-1}$. Thus $P_{(a_i\rho)\omega}$ saturates $\{a_j\} = a_j\rho$ for all $j > i$ and so $P_{(a_i\rho)\omega} \subseteq P_{a_j\rho} \subseteq P_{(a_j\rho)\omega}$ for all $j > i$. As a result, we have $\rho = P_{(a_i\rho)\omega}$. Finally, by [3, Lemma 3.3], for any idempotent $e \in S_i \backslash S_{i-1}$, we have $\rho = P_{(e\rho)\omega}$. Thus $\rho = \bar{\phi}_{(e\rho)\omega}$, as required.

(ii) *implies* (i). This is immediate.

There is a large class of inverse semigroups all of whose congruences are induced by transitive representations.

PROPOSITION 5.6. *Every congruence on a* (0-)*bisimple inverse semigroup is induced by a transitive representation.*

Proof. Let ρ be a congruence on a (0-)bisimple inverse semigroup S. By [3, Proposition 3.4] we have $\rho = P_{(0\rho)\omega} \cap P_{(e\rho)\omega} = P_{(e\rho)\omega}$ for any idempotent $e \neq 0$ if S has zero, while $\rho = P_{(e\rho)\omega}$ if S is without zero. In either case, we have $\rho = P_{(e\rho)\omega} = \bar{\phi}_{(e\rho)\omega}$ and so ρ is induced by a transitive representation.

6. Special ideal extensions. In view of Theorem 5.5, it is of interest to have a closer look into the nature of special extensions for general inverse semigroups and, in particular, for Brandt semigroups.

LEMMA 6.1. *Let S be a dense ideal extension of I. Then I is saturated by any congruence whose restriction to I is idempotent separating.*

Proof. Let ρ be a congruence on S for which $\rho|_I$ is idempotent separating. Suppose that I is not saturated by ρ. Then there exist $x \in S \backslash I$ and $y \in I$ with $x\rho y$. Since S is a dense extension of I (and I is inverse), no two elements of S can act the same on I via left multiplication. Since $e = xx^{-1} \in S \backslash I$ and $f = yy^{-1} \in I$ we have $e\rho f$ and $e \neq f$. Thus there exists $t \in I$ with $et \neq ft$, whence $ett^{-1} \neq ftt^{-1}$. But $ett^{-1}\rho ftt^{-1}$ and ett^{-1}, $ftt^{-1} \in I$, contradicting the hypothesis that $\rho|_I$ is idempotent separating. Thus I is saturated by ρ.

PROPOSITION 6.2. *Let S be an ideal extension of I. Then S is a special extension of I if and only if*

(i) *S is a dense extension of I,*

(ii) *$S/(\mu_I \cup \varepsilon)$ is a dense extension of I/μ_I,*

that is, if and only if the least and the greatest idempotent separating congruences on I extend uniquely to S.

Proof. Obviously, if S is a special extension of I, then (i) and (ii) hold. Conversely, suppose that (i) and (ii) hold. Let ρ be a congruence on S for which $\rho|_I$ is idempotent separating. Then by (i) and Lemma 6.1 we see that I is saturated by ρ. Let $\tau = \rho \vee (\mu_I \cup \varepsilon)$. Since I is saturated by ρ, it follows that $\tau|_I = \mu_I$ and $\tau|_{S\backslash I} = \rho|_{S\backslash I}$. Now $\tau' = \tau/(\mu_I \cup \varepsilon)$ is a congruence on $S' = S/(\mu_I \cup \varepsilon)$ whose restriction to $I' = I/\mu_I$ is equality, whence by (ii) we have $\tau'|_{S'\backslash I'} = \varepsilon$. But $S\backslash I = S'\backslash I'$ and $\rho|_{S\backslash I} = \tau|_{S\backslash I} = \tau'|_{S'\backslash I'} = \varepsilon$. Thus I is saturated by ρ and $\rho|_{S\backslash I} = \varepsilon$, whence $\rho = \rho|_I \cup \varepsilon$.

We now turn to extensions of Brandt semigroups by considering first dense extensions.

LEMMA 6.3. *Let I be a Brandt semigroup and S be an ideal extension of I for which S/I is a Brandt semigroup. Then S is a dense extension of I if and only if for all $e \in E_{S\backslash I}$,*

(i) $|[e] \cap I^*| \geq 2$,

(ii) *for each $x \in H_e$, $[x] \cap I = [e] \cap I$ implies that $x = e$.*

Proof. Suppose that S is a dense extension of I. Then S is not a retract ideal extension of I, whence (i) holds for all $e \in E_{S\backslash I}$. Now let $e \in E_{S\backslash I}$ and $x \in H_e$ be such that $[x] \cap I = [e] \cap I$. Then for any $y \in I$, we have $xyy^{-1} \in [x] \cap I$ and $eyy^{-1} \in [e] \cap I$. Then $xyy^{-1} \leq e$ and $eyy^{-1} \leq x$, whence $xyy^{-1} \leq eyy^{-1} \leq xyy^{-1}$. It follows that $xy = ey$. Since S is a dense extension of I, this implies that $x = e$, and so (ii) holds for each $e \in E_{S\backslash I}$.

Conversely, suppose that (i) and (ii) hold. Let ρ be a congruence on S for which $\rho|_I$ is idempotent separating. Suppose that I is not saturated by ρ. Then for some $e \in E_{S\backslash I}$ and $f \in E_I$ we have $e\rho f$, whence $ef\rho f$ and so $ef = f$. Thus $f \leq e$. By (i), there exists $g \in [e] \cap I^*$ with $g \neq f$. We obtain then that $g = eg\rho fg \neq g$, contradicting the fact that $\rho|_I$ is idempotent separating. Thus I is saturated by ρ.

Now suppose that for some $x,y \in S/I$, we have $x\rho y$, whence $e = xx^{-1}\rho yy^{-1} = f$ and $e,f \in E_{S\backslash I}$. Suppose that $e \neq f$ so that $ef \in I$ since $S\backslash I$ is a Brandt semigroup. If $[e] \cap I = [f] \cap I$, then $[e] \cap I = [ef]$, whence $|[e] \cap I^*| \leq 1$, a contradiction. Thus $[e] \cap I \neq [f] \cap I$. By symmetry, we may suppose that there exists $g \in [e] \cap I$ with $g \notin [f]$. Then $eg = g \neq fg$ whence $P_{\{g\}}$ separates e and f. Since ρ saturates $\{g\}$, we have $\rho \subseteq P_{\{g\}}$, whence $e\rho \neq f\rho$. Thus ρ is idempotent separating on S.

Finally, observe that since I is saturated by ρ, ρ induces a congruence on the Brandt semigroup S/I. This congruence is therefore completely determined by the congruence class of any idempotent in $S\backslash I$. Let $e \in E_{S\backslash I}$ and $x \in H_e$. Suppose that $x \neq e$. By (ii), we have $[x] \cap I \neq [e] \cap I$, and so there exists $y \in I$ for which either $y \leq x$ and $y \not\leq e$ or else $y \not\leq x$ and $y \leq e$. In the first instance, we obtain $y = yy^{-1}x \neq yy^{-1}e$ while in the latter case we have $yy^{-1}x \neq y = yy^{-1}e$. In either case we see that $P_{\{y\}}$ separates x and e. Since ρ saturates $\{y\}$ we have $\rho \subseteq P_{\{y\}}$ and so $x \notin e\rho$. Consequently, $e\rho = \{e\}$ and so $\rho|_{S\backslash I} = \varepsilon$. Since I is saturated by ρ and $\rho|_{S\backslash I} = \varepsilon$, we have $\rho = \varepsilon$, as required.

We are now ready for the principal result of this section.

THEOREM 6.4. *Let I be a Brandt semigroup and S be an ideal extension of I for which S/I is a Brandt semigroup. Then S is a special extension of I if and only if for every $e \in E_{S\backslash I}$,*

(i) $|[e] \cap I^*| \geq 2$,

(ii) *for each $x \in H_e$, $H_f x \subseteq H_f$ for all $f \in [e] \cap I$ implies that $x = e$.*

Proof. Sufficiency. Let $e \in E_{S\backslash I}$ and $x \in H_e$ with $x \neq e$. By (ii) there exists $f \in [e] \cap I$ for which $H_f x \not\subseteq H_f$. Since $H_f x \subseteq H_{fx}$, we obtain $fx \neq f$. If $fx \leq e$, then $fx = fx(fx)^{-1} = fxx^{-1} = fe = f$, a contradiction. Thus $fx \notin [e] \cap I$ but $fx \in [x] \cap I$, whence $[x] \cap I \neq [e] \cap I$. We have shown that condition (ii) of Lemma 6.3 is satisfied, while condition (i) holds by hypothesis. By Lemma 6.3 therefore, we have that S is a dense extension of I.

Now let $\tau = \mathscr{H}_I \cup \varepsilon$, $S' = S/\tau$ and $I' = I/\tau$. We show that (ii) is equivalent to condition (ii) of Lemma 6.3 formulated for the extension of the Brandt semigroup I' by the Brandt semigroup $S'/I' = S/I$. In fact, for $e \in E_{S\setminus I}$ and $x \in H_e$, we show that $[x] \cap I' = [e] \cap I'$ if and only if $H_f x \subseteq H_f$ for all $f \in [e] \cap I$. Suppose that $[x] \cap I' = [e] \cap I'$. Let $f \in [e] \cap I$. Then $f\tau \in [e] \cap I'$, whence $f\tau \le x$. But then $fx \in f\tau = H_f$. This implies that $(f\tau)x = H_f x \subseteq H_f$. Conversely, suppose that $H_f x \subseteq H_f$ for each $f \in [e] \cap I'$. Let $y\tau \in [e] \cap I'$. Then $(y\tau)e = y\tau$, whence for $f = y^{-1}y$ we have $H_y = H_f$ and $fe \in H_f$ so $fe = f$. Thus $f \in [e] \cap I$ and so $H_f x \subseteq H_f$. But then $(y\tau)x = (f\tau)x = f\tau = y\tau$ and $y\tau \in E_{I'}$, whence $y\tau \in [x] \cap I'$. Thus $[e] \cap I' \subseteq [x] \cap I'$. For $y\tau \in [x] \cap I'$, we have $(y^{-1}y)\tau x = y\tau$, whence for $f = yy^{-1}$ we have $H_f x \subseteq H_y$. Since $fe = yy^{-1}xx^{-1} \ne 0$, we have $f \in [e] \cap I$, whence $H_f x \subseteq H_f$. Thus $H_f = H_y$ and so $y\tau = f\tau \in [e] \cap I'$, whence $[x] \cap I' \subseteq [e] \cap I'$.

We may now observe that S' is a dense extension of I'. Condition (i) is inherited by the extension S' of I' and thus condition (i) of Lemma 6.3 holds for this extension of I'. By the above argument, (ii) implies that Lemma 6.3(ii) holds for this extension as well. By Lemma 6.3 we obtain that S' is a dense extension of I'.

Finally, we apply Proposition 6.2 to conclude that S is a special extension of I.

Necessity. Suppose that S is a special extension of I. Then S is a dense extension of I, whence Lemma 6.3(i) asserts that (i) holds. Next, by Proposition 6.2(ii), we know that, with the notation above, S' is a dense extension of I'. By the above argument, in the presence of (i) this is equivalent to (ii). Thus both (i) and (ii) hold.

REMARK 1. We observe that condition (ii) of Theorem 6.4 is equivalent to the following: for $e \in E_{S\setminus I}$,

(ii)′ H_e acts faithfully by conjugation on $[e] \cap I$.

REMARK 2. In view of condition (i) of Lemma 6.3, we may observe that the proof of Lemma 6.3 demonstrates that condition (ii) of Lemma 6.3 is equivalent to the requirement that no two elements of $S \setminus I$ cover exactly the same elements of I, that is, for all $x,y \in S \setminus I$ with $x \ne y$, we have $[x] \cap I \ne [y] \cap I$. In order to appreciate the difference between conditions (ii) of Lemma 6.3 and Theorem 6.4, respectively, we observe that condition (ii) of Theorem 6.4 is equivalent to the following: for any $x,y \in S \setminus I$ with $x \ne y$, there exists an $\mathscr{H}$-class X in I such that $[x] \cap X \ne \varnothing$ and $[y] \cap X = \varnothing$ or vice-versa.

It is shown in [**2**, V.4.8] that every ideal extension of a Brandt semigroup $I = B(G, X)$ by a Brandt semigroup $Q = B(H, Y)$ can be constructed from data of the following form. Let ν be a cardinal number with $\nu \le |X|$, and $\mathscr{P}_\nu$ the family of all subsets of X of cardinality ν. Select $P_0 \in \mathscr{P}_\nu$. Let

(i) $\pi : Y \to P_\nu$ be any function such that $|x\pi \cap y\pi| \le 1$ if $x \ne y$;

(ii) $\theta : H \to G \text{ wr } \mathscr{S}(P_0)$ be any homomorphism and let $h\theta = (\sigma_h, \tau_h)$;

(iii) for each $y \in Y$, let $\xi_y : P_0 \to y\pi$ be a bijection and $\eta_y : P_0 \to G$ any function.

We shall let Π_2 denote the projection $G \text{ wr } \mathscr{S}(P_0) \to \mathscr{S}(P_0)$.

LEMMA 6.5. *Let S be an ideal extension of $I = B(G, X)$ by $Q = B(H, Y)$ determined by parameters ν, P_0, π, θ, ξ, η as described above. Let $e \in E_{S\setminus I}$, so $e = (y, 1_H, y)$ for some $y \in Y$. Then the following hold.*

(i) *$f \in [e] \cap I$ if and only if $f = (x, 1_G, x)$ for some $x \in y\pi$.*

(ii) *For $q \in H_e$, $H_f q \subseteq H_f$ for all $f \in [e] \cap I$ if and only if the projection of q into H is contained in* $\ker(\theta\Pi_2)$.

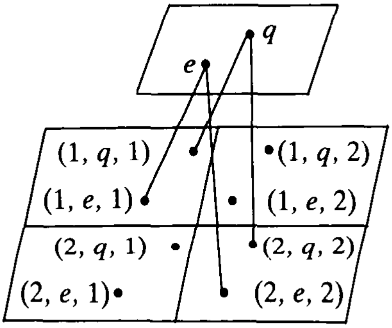
Proof. (i) This follows by routine computation.

(ii) Let $q \in H_e$, say $q = (y, h, y)$. We must show that $\tau_h = \varepsilon_{P_0}$ if and only if for all $x \in y\pi$, $x\xi_y^{-1}\tau_h\xi_y = x$, which in turn is equivalent to $(x\xi_y^{-1})\tau_h = x\xi_y^{-1}$ for all $x \in y\pi$. But ξ_y is a bijection from P_0 to $y\pi$, whence the result follows.

Corollary 6.6. *An ideal extension S of $I = B(G, X)$ by $B(H, Y)$, determined by parameters v, P_0, π, θ, ξ, η, is special if and only if $v \geq 2$ and $\theta\Pi_2$ is injective.*

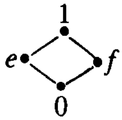
Proof. In view of Lemma 6.5(i), $v \geq 2$ is equivalent to condition (i) of Theorem 6.4 while by Lemma 6.5(ii), the injectivity of $\theta\Pi_2$ is equivalent to condition (ii) of Theorem 6.4. The result follows from Theorem 6.4.

To illustrate the situation, we offer an example of a dense extension S of a Brandt semigroup by a Brandt semigroup which is not a special extension. Let $I = B(C_2, \{1, 2\})$ and $Q = B(C_2, \{1\})$ where C_2 is a cyclic group of order 2 generated by q, so that $C_2 = \{e, q\}$ with $q^2 = e$. Let $v = 2$, $P_0 = \{1, 2\}$, $\pi : \{e\} \to \mathscr{P}_2$ be the function $e\pi = P_0$, $\theta : C_2 \to C_2 \text{ wr } \mathscr{S}(\{1, 2\})$ be the homomorphism determined by $q\theta = (\kappa_q, \varepsilon_{P_0})$, where κ_q is the constant function $1\kappa_q = 2\kappa_q = q$, $\xi_e = \varepsilon_{P_0}$ and $1\eta_e = 2\eta_e = e$. For convenience, we draw the poset graph for S^* under the natural partial order.



By Remark 2, it is evident that S is a dense extension of I which is not special.

On the other hand, by Remark 2 we may see that the semidirect product of the semilattice $\{0, e, f, 1\}$ with poset graph



and its automorphism group, upon forming the quotient by the minimum ideal, yields a special extension of $B(C_2, \{1, 2\})$ by $B(C_2, \{1\})$.

As an interesting consequence of our results, we have

PROPOSITION 6.7 (cf. [**6**]). *Let X be a finite set. Then the symmetric inverse semigroup $\mathcal{I}(X)$ has a principal series which satisfies Theorem* 5.5(*iii*). *Therefore $\mathcal{I}(X)$ satisfies the equivalent conditions of Theorem* 5.5, *and in particular, the congruences on $\mathcal{I}(X)$ are as described in Theorem* 5.5(*iv*).

Proof. It is well-known that $\mathcal{I}(X)$ has a unique principal series given by $S_i = \{\alpha \in \mathcal{I}(X) \mid \operatorname{rank}(\alpha) \leq i\}$ for $i = 0, 1, \ldots, |X|$ for which each quotient semigroup S_i/S_{i-1} is a Brandt semigroup. Conditions (i) and (ii) of Theorem 6.4 are readily verified for each extension S_{i+1}/S_{i-1} of S_i/S_{i-1}, $i = 0, 1, \ldots, |X| - 1$. Thus Theorem 6.4 asserts that S_{i+1}/S_{i-1} is a special extension of S_i/S_{i-1} for $i = 0, 1, \ldots, |X| - 1$, whence Theorem 5.5(iii) holds.

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