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A Characterization of the Compound-Exponential Type Distributions

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Abstract. In this paper, a fixed point equation of the compound-exponential type distributions is derived, and under some regular conditions, both the existence and uniqueness of this fixed point equation are investigated. A question posed by Pitman and Yor can be partially answered by using our approach.

Introduction

Let $X \geq 0$ be a random variable (r.v.) with finite mean $0 < E(X) = \mu_X < \infty$, and the distribution function of X will be denoted by F_X. Two important random variables are induced by X (or F_X), the length-biased r.v. Z and the integrated tail r.v. X_1. They are defined by

$$F_Z(x) = \frac{1}{\mu_X} \int_0^x t dF_X(t), \quad x \geq 0,$$

$$F_{X_1}(x) = \frac{1}{\mu_X} \int_0^x (1 - F_X(t)) dt, \quad x \geq 0,$$

respectively. Sometimes F_{X_1} is called the stationary excess distribution or the equilibrium distribution of F_X. The characterization problems in this vein can be found in [5–9] and the references therein.

In this paper, we consider the following distributional equation:

$$Z \stackrel{d}{=} X_1 + X_2 + T \cdot Z,$$

where the r.v. $T \geq 0$ is given and X, X_1, X_2 are independent and identically distributed (i.i.d.) random variables and X_1, X_2, T, and Z in the right-hand side are independent. This equation is closely related to the Pitman–Yor problem, but is different, since the distributional equation can be reduced to a fixed point equation, which is a type of compound-exponential distributions case, but not a type of compound-Poisson distributions [10].

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Note that the distributional equation is equivalent to the following fixed point equation

$$\widehat{F}_X(s) = \frac{1}{1 + \int_0^{+\infty} \frac{1 - \widehat{F}_X(st)}{t} dF_T(t)}, \quad s \geq 0,$$

where $\widehat{F}_X(s) = E(e^{-sX})$, $s \geq 0$ is the Laplace–Stieltjes transform of the r.v. X, the integrand is defined for $t = 0$ by continuity to be equal to $\mu_X \cdot s$ (see Theorem 1.1 below), and so we call this the characterization of the compound-exponential type distributions.

In Section 2, the problems of existence and uniqueness of the distributional equation are solved under some regular conditions,. The Pitman–Yor question [8, p.320] can be partially answered by the approach given here (see Theorem 1.3 below). Here and in similar assertions below, unique means, of course, unique in law.

Finally, we note that the regular conditions given here are slightly different from the above mentioned papers. The key point in this paper is the sharp bound on the second moment of the Laplace–Stieltjes transforms (see [1, 3, 4]).

1 The Main Results and Proofs

Throughout this section, we assume that all random variables are non-negative, that the distribution functions are right continuous, and that the interval of integration is closed (and may be replaced by $[0, \infty)$). We also use the same notations as given in the introduction. For example, the Laplace–Stieltjes transforms of Z and X_1 are well known and given by

$$\widehat{F}_Z(s) = \frac{-1}{\mu_X} \widehat{F}'_X(s), \qquad \widehat{F}_{X_1}(s) = \frac{1 - \widehat{F}_X(s)}{\mu_X \cdot s}, \; s \geq 0,$$

respectively, where $\widehat{F}_{X_1}(0) = 1$ is defined by the limit value as $s \to 0^+$. For convenience, we assume that the mean of the r.v. X is finite and is given, that is,

$$0 < E(X) = \mu_X < \infty \quad \text{and} \quad \mu_X = \mu \text{ is given.}$$

Theorem 1.1 *Let $T \geq 0$ be a given r.v. with $0 \leq E(T) < 1$ and assume that the r.v. $X \geq 0$ with $0 < E(X) = \mu_X < \infty$ and $0 < \mathrm{Var}(X) = \sigma_X^2 < \infty$. Then the distributional equation*

$$Z \stackrel{d}{=} X_1 + X_2 + T \cdot Z \tag{1.1}$$

has exactly one solution X with the mean $E(X)$, where X, X_1, and X_2 are independent and identically distributed, X_1, X_2, T, and Z are independent.

Proof Under the given conditions, the distributional equation (1.1) is equivalent to $\widehat{F}_Z(s) = \widehat{F}_X^2(s) \cdot \widehat{F}_{TZ}(s)$, $s \geq 0$. Since $\widehat{F}_Z(s) = \frac{-1}{\mu_X} \widehat{F}'_X(s)$, the equation leads to

$$\widehat{F}'_X(s) + \mu_X \cdot \widehat{F}_X^2(s) \cdot \widehat{F}_{TZ}(s) = 0, \quad s \geq 0.$$

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By using $y = \frac{1}{\widehat{F}_X(s)} - 1$ and $\widehat{F}_X(0) = 1$, we get

$$\widehat{F}_X(s) = \frac{1}{1 + \mu_X \int_0^s \widehat{F}_{TZ}(x)\,dx}, \quad s \geq 0,$$

and the identities

$$\int_0^{+\infty} \frac{1 - \widehat{F}_X(st)}{t}\,dF_T(t) = \mu_X \cdot s \cdot \widehat{F}_{TX_1}(s)$$
$$= \mu_X \int_0^s \widehat{F}_{TZ}(x)\,dx, \quad s \geq 0,$$

imply the following fixed point equation

$$\widehat{F}_X(s) = \frac{1}{1 + \int_0^{+\infty} \frac{1 - \widehat{F}_X(st)}{t}\,dF_T(t)}, \quad s \geq 0.$$

Under the given conditions, we will prove that this equation has a unique fixed point. Note that if $P(T = 0) = 1$, then the equation is reduced to

$$\widehat{F}_X(s) = \frac{1}{1 + \mu_X \cdot s}, \quad s \geq 0$$

and clearly, the exponential distribution is the solution, so there is nothing to prove. Hence, we may assume that $0 < E(T) < 1$.

First, we prove the existence. For $n \geq 1$, define

$$\widehat{F}_n(s) = \frac{1}{1 + \int_0^{\infty} \frac{1 - \widehat{F}_{n-1}(st)}{t}\,dF_T(t)}, \quad s \geq 0,$$

and $\widehat{F}_0(s)$ is the Laplace–Stieltjes transform of an initial random variable Y_0. Note that $\widehat{F}_n(s)$ are well defined for all $n \geq 0$. In fact, for $n \geq 1$, $\widehat{F}_n(s)$ is a Laplace–Stieltjes transform of an infinitely divisible probability distribution [2, p. 441, Criterion 2, p. 450, Theorem 1] or [10, p. 99].

Let Y_n, $n \geq 0$, be an r.v. with the Laplace–Stieltjes transform $\widehat{F}_n$. Then, under the given conditions of the theorem, we have (here we assume that $E(Y_0^2) < \infty$)

$$E(Y_n) = E(Y_0), \quad n \geq 1,$$
$$E(Y_n^2) = E(Y_{n-1}^2) \cdot E(T) + 2[E(Y_0)]^2, \quad n \geq 1.$$

Since $0 < E(X) = \mu_X < \infty$ is a given real number, let us choose

$$\widehat{F}_{Y_0}(s) = 1 - \frac{\alpha_1^2}{\alpha_2} + \frac{\alpha_1^2}{\alpha_2} e^{-(\alpha_2/\alpha_1)s}, \quad s \geq 0,$$

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where $\alpha_1 = \mu_X$ and $\alpha_2 = \frac{2\mu_X^2}{1-E(T)}$, $\widehat{F}_0 = \widehat{F}_{Y_0}$.

The condition $0 < E(T) < 1$ implies that $\widehat{F}_{Y_0}$ is a well-defined Laplace–Stieltjes transform and we have $E(Y_0) = \alpha_1$ and $E(Y_0^2) = \alpha_2$ and the previous results give (setting $n = 1$)

$$E(Y_1) = \alpha_1 \quad \text{and} \quad E(Y_1^2) = \alpha_2 E(T) + 2\alpha_1^2 = \alpha_2.$$

Now applying the inequality for Laplace–Stieltjes transforms (see [1, 3, 4]), we get

$$\widehat{F}_1(s) \le 1 - \frac{\alpha_1^2}{\alpha_2} + \frac{\alpha_1^2}{\alpha_2} e^{-(\alpha_2/\alpha_1)s} = \widehat{F}_0(s), \quad s \ge 0.$$

The definition of $\widehat{F}_n$ implies, for $n \ge 2$,

$$\widehat{F}_n(s) - \widehat{F}_{n-1}(s) = \frac{\displaystyle\int_0^\infty \frac{\widehat{F}_{n-1}(st) - \widehat{F}_{n-2}(st)}{t}\, dF_T(t)}{1 + \displaystyle\int_0^\infty \frac{1 - \widehat{F}_{n-1}(st)}{t}\, dF_T(t) \quad 1 + \displaystyle\int_0^\infty \frac{1 - \widehat{F}_{n-2}(st)}{t}\, dF_T(t)}, \quad s \ge 0.$$

Hence, we get $\widehat{F}_n(s) \le \widehat{F}_{n-1}(s)$, $s \ge 0, n \ge 1$. Since $0 < E(Y_n) = \mu_X < \infty$, the Jensen's inequality gives $\widehat{F}_n(s) \ge e^{-\mu_X \cdot s}$, $s \ge 0, n \ge 0$. Combining these inequalities, we get $e^{-\mu_X \cdot s} \le \widehat{F}_n(s) \le \widehat{F}_0(s)$, $s \ge 0, n \ge 0$. Thus, the monotone and bounded sequence $\widehat{F}_n$, $n \ge 0$, has a unique limit, say $\widehat{F}_\infty$, since $\lim_{s\to 0^+} \widehat{F}_\infty(s) = 1$.

By the continuity theorem [2, p. 431], this $\widehat{F}_\infty$ is the Laplace–Stieltjes transform of an r.v. X_∞. In the following, we will show that this $\widehat{F}_\infty$ is a fixed point with $E(X_\infty) = \mu_X$ and $E(X_\infty^2) < \infty$. The following two identities [2, p. 435]

$$\frac{1 - \widehat{F}_X(s)}{s} = \int_0^\infty e^{-sx}(1 - F_X(x))\, dx, \quad s > 0,$$

$$\frac{\widehat{F}_X(s) - 1 + \mu_X \cdot s}{s^2} = \mu_X \int_0^\infty e^{-sx}(1 - F_{X_1}(x))\, dx, \quad s > 0.$$

imply that the two functions are decreasing in s; on the other hand, the previous inequalities imply that

$$\frac{1 - e^{-\mu_X s}}{s} \ge \frac{1 - \widehat{F}_n(s)}{s} \ge \frac{1 - \widehat{F}_0(s)}{s}, \quad s > 0, n \ge 0,$$

$$\frac{e^{-\mu_X s} - 1 + \mu_X \cdot s}{s^2} \le \frac{\widehat{F}_n(s) - 1 + \mu_X \cdot s}{s^2} \le \frac{\widehat{F}_0(s) - 1 + \mu_X \cdot s}{s^2}, \quad s > 0, n \ge 0.$$

Now by using the monotonic property, letting $n \to \infty$, and then $s \to 0^+$, we get

$$E(X_\infty) = \mu_X \quad \text{and} \quad E(X_\infty^2) \le \alpha_2 < \infty.$$

Finally, by the dominated convergence theorem, this $\widehat{F}_\infty(s)$ satisfies the fixed point equation, and the proof of existence is complete.

To prove the uniqueness, let us assume that there are two fixed points $\widehat{F}_X$ and $\widehat{F}_Y$ with $\mu \equiv \mu_X = \mu_Y$ and $E(X^2) < \infty$, $E(Y^2) < \infty$, where $\mu_X = E(X)$ and $\mu_Y = E(y)$, and hence the random variables X_1 and Y_1 are well defined with

$$\mu_{X_1} \equiv E(X_1) = \frac{E(X^2)}{2\mu_X} < \infty \quad \text{and} \quad \mu_{Y_1} \equiv E(Y_1) = \frac{E(Y^2)}{2\mu_Y} < \infty.$$

The fixed point equation implies

$$\begin{aligned} |\widehat{F}_{X_1}(s) - \widehat{F}_{Y_1}(s)| &= \Big| \frac{\widehat{F}_X(s)}{\mu s} - \frac{\widehat{F}_Y(s)}{\mu s} \Big| = \Big| \frac{\widehat{F}_X(s) - \widehat{F}_Y(s)}{\mu s} \Big| \\ &\leq |\widehat{F}_{TX_1}(s) - \widehat{F}_{TY_1}(s)|, \quad s \geq 0. \end{aligned}$$

Iterating, we get

$$\begin{aligned} |\widehat{F}_{X_1}(s) - \widehat{F}_{Y_1}(s)| &\leq \int_0^\infty |\widehat{F}_{X_1}(st_1) - \widehat{F}_{Y_1}(st_1)|\, dF_T(t_1) \\ &\leq \int_0^\infty \int_0^\infty |\widehat{F}_{X_1}(st_1t_2) - \widehat{F}_{Y_1}(st_1t_2)|\, dF_T(t_1)dF_T(t_2) \\ &\leq \Lambda \\ &\leq \int_0^\infty \Lambda \int_0^\infty |\widehat{F}_{X_1}(st_1\Lambda t_n) - \widehat{F}_{Y_1}(st_1\Lambda t_n)|\, dF_T(t_1)\Lambda dF_T(t_2) \\ &\leq s \cdot (\mu_{X_1} + \mu_{Y_1}) \cdot [E(T)]^n, \quad n \geq 1, s > 0. \end{aligned}$$

Using the condition $0 < E(T) < 1$ and letting $n \to \infty$, we get

$$\widehat{F}_{X_1}(s) = \widehat{F}_{Y_1}(s), \quad s > 0,$$

that is,

$$\frac{1 - \widehat{F}_{X_1}(s)}{\mu s} = \frac{1 - \widehat{F}_{Y_1}(s)}{\mu s}, \quad s > 0.$$

Hence, $\widehat{F}_X(s) = \widehat{F}_Y(s)$, $s > 0$, or equivalently $X \stackrel{d}{=} Y$. Finally, combining the previous results, the proof is complete. ■

Corollary 1.2 *Under the conditions of Theorem* 1.1 *and assuming that* $P(T = 0) = 1$, *the distributional equation* (1.1) *has the exponential distribution solution.*

See [7] for a closely related result. Note that in this case the conditions of Theorem 1.1 can be weakened. The proof of Corollary 1.2 is obvious.

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Theorem 1.3 *Let $T \geq 0$ be a given random variable with $0 < E(T) < 1$ and assume that the r.v. $X \geq 0$ with $0 < E(X) = \mu_X < \infty$ and $0 < \operatorname{Var}(X) = \sigma_x^2 < \infty$. Then the distributional equation*

$$z \stackrel{d}{=} X + T \cdot Z \tag{1.2}$$

has exactly one solution X with the mean $E(X)$, where X, T, and Z are independent.

Proof The proof of Theorem 1.3 is similar to the proof of Theorem 1.1. A brief proof is given below. The distributional equation (1.2) is equivalent to

$$\widehat{F}_X'(s) + \mu_X \cdot \widehat{F}_X(s) \cdot \widehat{F}_{TZ}(s) = 0, \quad s \geq 0,$$

which in turn leads to the following fixed point equation:

$$\widehat{F}_X(s) = e^{-\int_0^{+\infty} \frac{1-\widehat{F}_X(st)}{t} dF_T(t)}, \quad s \geq 0.$$

Note that the integrand is defined for $t = 0$ by continuity to be equal to $\mu_X \cdot s$. To prove the existence, let us define

$$\widehat{F}_n(s) = e^{-\int_0^{+\infty} \frac{1-\widehat{F}_{n-1}(st)}{t} dF_T(t)}, \quad s \geq 0, n \geq 1,$$

and $\widehat{F}_0(s)$ is the Laplace–Stieltjes transform of the initial r.v. Y_0. Note that $\widehat{F}_n(s)$ are all well defined for $n \geq 0$. For $n \geq 1$, $\widehat{F}_n(s)$ is a Laplace–Stieltjes transform of an infinitely divisible probability distribution [2, Theorem 1, p. 450] Under the conditions of the theorem, we have (here we assume that $E(Y_0^2) < \infty$) $E(Y_n) = E(Y_0)$, $n \geq 1$, and $E(Y_n^2) = E(Y_{n-2}^2) \cdot E(T) + [E(Y_0)]^2$, $n \geq 1$, where Y_n is an r.v. with the Laplace–Stieltjes transform $\widehat{F}_n$. Now by using the same argument as the proof of Theorem 1.1, choose

$$\widehat{F}_0(s) = 1 - \frac{\alpha_1^2}{\alpha_2} + \frac{\alpha_1^2}{\alpha_2} e^{-(\alpha_2/\alpha_1)s}, \quad s \geq 0,$$

where $\alpha_1 = \mu_X$ and $\alpha_2 = \frac{\mu_X^2}{1-E(T)}$. The condition $0 < E(T) < 1$ implies that $\widehat{F}_0$ is well defined. The same argument as the proof of Theorem 1.1 completes the proof of the existence. The proof of the uniqueness is obvious. ■

Corollary 1.4 *Let $T \geq 0$ be a non-degenerate r.v. concentrated on $(0, b]$, $0 < b \leq 1$, and assume that the r.v. $X \geq 0$ with $0 < E(X) = \mu_X < \infty$ and $0 < \operatorname{Var}(X) = \sigma_X^2 < \infty$. Then the distributional equation (1.2) has exactly one solution C with the mean $E(X) = \mu_X$.*

Proof Corollary 1.4 can be found in Iksanov and Kim [6], where they prove Corollary 1.4 without the condition $0 < \operatorname{Var}(X) = \sigma_X^2 < \infty$. The condition of T implies that $0 < E(T) < 1$, and Theorem 1.3 is in force. ■

Theorem 1.5 *Let $T \geq 0$ be a given r.v. with $0 \leq E(T) < 1$ and assume that the r.v. $X \geq 0$ with $0 < E(X) = \mu_X < \infty$ and $0 < \operatorname{Var}(X) = \sigma_X^2 < \infty$. Then the distributional equation*

$$X_1 \stackrel{d}{=} X - 1 + T \cdot X_1 \tag{1.3}$$

has exactly one solution X with the mean $E(X)$, where $X_1 \stackrel{d}{=} X$ and X_1, T and X_1 are independent.

Proof Let U have a uniform distribution over $(0, 1)$. It is easy to obtain the distributional identity $X_1 \stackrel{d}{=} U \cdot Z$, where U and Z are independent (see Sen and Khattree [9]). Now applying this identity to the right-hand side of (1.3), it follows that the equation (1.3) is equivalent to $\widehat{F}_X'(s) + \mu_X \cdot \widehat{F}_X^2(s) \cdot \widehat{F}_{TZ}(s) = 0$, $s \geq 0$. This equation leads to the same fixed point equation as given in the proof of Theorem 1.1. Under the given conditions, Theorem 1.1 is in force. ∎

Remarks (i) The problem here is closely connected to some general results in Steutel and van Harn [10, p. 443–445]. Let X be a nonnegative r.v. with finite mean $0 < \mu < \infty$. Actually, under this condition, it is easily shown that the r.v. X as in formula (1.1) is necessarily compound-exponential (see Steutel and van Harn [10, p. 445]). Theorem 1.5 is a contribution of the referee.

(ii) All problems above can be proved by using the Banach contraction principle. Here we give a different and simplified proof by using the sharp bounds of the Laplace–Stieltjes transforms.

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