Glasgow Math. J. **48** (2006) 93–109. © 2006 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089505002909. Printed in the United Kingdom

CONTACT-HOMOGENEOUS LOCALLY φ-SYMMETRIC MANIFOLDS

E. BOECKX
Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, 3001 Leuven, Belgium
e-mail: eric.boeckx@wis.kuleuven.ac.be

(Received 31 May, 2005; accepted 17 August, 2005)

Abstract. It is an open question whether every strongly locally φ-symmetric contact metric space is a (κ, μ)-space. We show that the answer is positive for locally homogeneous contact metric manifolds.

2000 *Mathematics Subject Classification.* 53D10, 53C25, 53C30.

1. Introduction. In K-contact and Sasakian geometry, local symmetry is a very strong condition: a locally K-contact manifold is necessarily a space of constant curvature equal to 1 ([**16**], [**19**]). So, there is a need for a new kind of symmetry, better adapted to the additional structure of Sasakian and K-contact spaces. T. Takahashi introduced the appropriate notion in [**18**]: a Sasakian space is *(locally) φ-symmetric* if its Riemann curvature tensor R satisfies

$$g((\nabla_X R)(Y, Z)V, W) = 0 \tag{1}$$

for all vector fields X, Y, Z, V and W orthogonal to the characteristic vector field ξ, where ∇ denotes the Levi Civita connection. Geometrically, this corresponds to the fact that the characteristic reflections (i.e., reflections with respect to the integral curves of ξ) are local automorphisms of the Sasakian structure. In fact, it is already sufficient that the reflections are local isometries ([**5**]). A K-contact manifold whose characteristic reflections are local isometries is necessarily a (locally) φ-symmetric Sasakian space ([**11**]).

At least two generalizations of the notion of local φ-symmetry to the class of contact metric spaces have appeared in the literature. The first one, in [**4**], defines a locally φ-symmetric contact metric space to be one for which the curvature property (1) holds. It is as yet unclear what this means geometrically in the contact metric setting. A second generalization was proposed by the author and L. Vanhecke in [**10**]: a contact metric space is called locally φ-symmetric if its characteristic reflections are local isometries. This gives rise to an infinite number of curvature restrictions (see further), including (1). Hence, this second generalization is a priori more restrictive than the first. To distinguish between the two, we speak about *weak local φ-symmetry* (for the first one) and *strong local φ-symmetry* (for the second). That the two classes do not agree was shown explicitly in [**9**]: there, left-invariant contact metric structures on three-dimensional non-unimodular Lie groups were constructed which are weakly, but not strongly locally φ-symmetric. In [**17**], D. Perrone has presented another three-dimensional contact metric space with this property, but which is moreover not locally homogeneous.

The first examples of strongly locally φ-symmetric contact metric spaces (which are not Sasakian) were found in [**10**]: these are the unit tangent sphere bundles of spaces of constant curvature c, $c \neq 1$, equipped with their natural contact metric structure. Later, this family of examples was extended further to include all (non-Sasakian) *contact metric* (κ, μ)*-spaces*. These are contact metric manifolds for which the Riemann curvature tensor R satisfies

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{2}$$

for some real numbers κ and μ and for all vector fields X and Y. Here h denotes, up to a scaling factor, the Lie derivative of the structure tensor φ in the direction of ξ. For convenience, we will call such contact metric spaces (κ, μ)*-spaces*. Note that Sasakian spaces also satisfy (2) ($\kappa = 1$ and $h = 0$). The class of (κ, μ)-spaces was introduced in [**3**], and there it was shown that the only unit tangent sphere bundles with this curvature property are precisely those of spaces of constant curvature c (with $\kappa = c(2 - c)$ and $\mu = -2c$). All (non-Sasakian) (κ, μ)-spaces are strongly locally φ-symmetric and locally contact-homogeneous, as shown by the present author in [**6**]. Finally, a full local classification of (κ, μ)-spaces was realized in [**7**].

Apart from the (non-Sasakian) contact metric (κ, μ)-spaces, not a single example is known of a non-Sasakian strongly locally φ-symmetric space. This raises the question whether any actually exist. In dimension three, the answer is known: G. Calvaruso, D. Perrone and L. Vanhecke have shown in [**12**] that the two classes agree. In previous work, the present author studied left-invariant contact metric structures on Lie groups and showed that, also here, strong local φ-symmetry forces the Lie group to be a (κ, μ)-space ([**8**]).

Now, we consider the broader class of (locally) contact-homogeneous contact metric spaces. By using the theory of homogeneous structures ([**20**], [**14**]), one can make pointwise calculations while still taking the local homogeneity into account. (This situation is quite similar to the study of Lie groups via computations only involving their Lie algebras.) If we require the homogeneous contact metric manifold to be moreover strongly φ-symmetric, this severely restricts the possible homogeneous structures and the form of the Riemann curvature tensor. Indeed, we prove the following.

Main Theorem. *Let $(M, \xi, \eta, \varphi, g)$ be a locally contact-homogeneous contact metric space. If it is strongly locally φ-symmetric, then it is a (κ, μ)-space.*

So, in order to prove that every strongly locally φ-symmetric contact metric space is actually a (κ, μ)-space, it suffices to show that strong local φ-symmetry implies local contact-homogeneity.

2. Strongly locally φ-symmetric contact metric spaces.

In this section we collect the formulas and results we need on contact metric manifolds. We refer to [**2**] for a more detailed treatment. All manifolds in this note are assumed to be connected and smooth.

An odd-dimensional differentiable manifold M^{2n+1} has an *almost contact structure* if it admits a vector field ξ, a one-form η and a $(1, 1)$-tensor field φ satisfying

$$\eta(\xi) = 1 \quad \text{and} \quad \varphi^2 = -\mathrm{id} + \eta \otimes \xi.$$

https://doi.org/10.1017/S0017089505002909 Published online by Cambridge University Press

In that case, one can always find a compatible Riemannian metric g, i.e., such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X and Y on M. $(M, \xi, \eta, \varphi, g)$ is an *almost contact metric manifold.* If the additional property $d\eta(X, Y) = g(X, \varphi Y)$ holds, then $(M, \xi, \eta, \varphi, g)$ is called a *contact metric manifold.* As a consequence, the characteristic curves (i.e., the integral curves of the characteristic vector field ξ) are geodesics.

On a contact metric manifold M, we define the (1, 1)-tensor h by

$$hX = \frac{1}{2}(\mathcal{L}_\xi \varphi)(X)$$

where $\mathcal{L}_\xi$ denotes Lie differentiation in the direction of ξ. The tensor h is self-adjoint, $h\xi = 0$, $\operatorname{tr} h = 0$ and $h\varphi = -\varphi h$. The covariant derivative of ξ is given explicitly by

$$\nabla_X \xi = -\varphi X - \varphi h X. \tag{3}$$

If the vector field ξ on a contact metric manifold $(M, \xi, \eta, \varphi, g)$ is a Killing vector field, then the manifold is called a *K-contact manifold.* This is the case if and only if $h = 0$. Finally, if the Riemann curvature tensor satisfies

$$R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi = \eta(Y)X - \eta(X)Y \tag{4}$$

for all vector fields X and Y on M, then the contact metric manifold is *Sasakian.* In that case, ξ is a Killing vector field, hence every Sasakian manifold is K-contact.

Recall that a contact metric space $(M, \xi, \eta, \varphi, g)$ is called a (strongly) locally φ-symmetric space if the local reflections with respect to the integral curves of ξ are local isometries. This geometric property is reflected in an infinite list of curvature conditions (see also [**13**]):

PROPOSITION. *Let $(M, \xi, \eta, \varphi, g)$ be a contact metric manifold. If it is a (strongly) locally φ-symmetric space, then the following infinite list of curvature conditions hold*:

$$g\big((\nabla^{2k}_{X\cdots X} R)(X, Y)X, \xi\big) = 0, \tag{5}$$

$$g\big((\nabla^{2k+1}_{X\cdots X} R)(X, Y)X, Z\big) = 0, \tag{6}$$

$$g\big((\nabla^{2k+1}_{X\cdots X} R)(X, \xi)X, \xi\big) = 0, \tag{7}$$

for all vectors X, Y and Z orthogonal to ξ and $k = 0, 1, 2, \ldots$. Moreover, if (M, g) is analytic, these conditions are also sufficient for the contact metric manifold to be a locally φ-symmetric space.

Note that (6) for $k = 0$ is precisely the condition (1), implying that any strongly locally φ-symmetric space is also weakly locally φ-symmetric.

3. Contact-homogeneous structures. Let $(M^{2n+1}, \xi, \eta, \varphi, g)$ be a contact metric space which is (locally) contact-homogeneous, i.e., the pseudo-group of local automorphisms of the contact metric structure (ξ, η, φ, g) acts transitively on M. The theory of *homogeneous structures* ([**1**], [**20**]) and its generalization by V. F. Kiričenko in [**14**] allow to describe local homogeneity in an infinitesimal way, which is very convenient for calculations. Namely, the contact metric space $(M, \xi, \eta, \varphi, g)$ is locally

https://doi.org/10.1017/S0017089505002909 Published online by Cambridge University Press

homogeneous if and only if there exists a $(1, 2)$-tensor field T, the homogeneous structure, such that

$$\tilde{\nabla} g = 0, \quad \tilde{\nabla} R = 0, \quad \tilde{\nabla} T = 0, \quad \tilde{\nabla} \xi = 0, \quad \tilde{\nabla} \eta = 0, \quad \tilde{\nabla} \varphi = 0 \tag{8}$$

where $\tilde{\nabla}$ is the connection determined by $\tilde{\nabla} = \nabla - T$.

The existence of the tensor field T makes it possible to calculate pointwise while still taking the local homogeneity into account. So, for all further calculations, we implicitly assume we are working at a fixed point $p \in M$. For the simplicity of the notation, however, we do not mention p in our formulas.

Consider the operator $h = \frac{1}{2} \mathcal{L}_\xi \varphi$. Since it is symmetric with respect to the metric g and since $h\varphi = -\varphi h$, we can find an orthonormal basis of $T_p M$ of the form $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$ satisfying

$$h(X_i) = \lambda_i X_i, \quad h(Y_i) = -\lambda_i Y_i, \quad \varphi(X_i) = Y_i, \quad \varphi(Y_i) = -X_i$$

for $i = 1, \ldots, n$ and such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. In this basis, we can express the tensor T explicitly in the form:

$$\begin{aligned}
T_{X_i} X_j &= \sum_{k=1}^{n} \left(A_{ij}^k X_k + a_{ij}^k Y_k\right) + x_{ij}\xi, \\
T_{X_i} Y_j &= \sum_{k=1}^{n} \left(b_{ij}^k X_k + B_{ij}^k Y_k\right) + z_{ij}\xi, \\
T_{X_i} \xi &= \sum_{k=1}^{n} \left(A_i^k X_k + a_i^k Y_k\right) + x_i\xi, \\
T_{Y_i} X_j &= \sum_{k=1}^{n} \left(C_{ij}^k X_k + c_{ij}^k Y_k\right) + w_{ij}\xi, \\
T_{Y_i} Y_j &= \sum_{k=1}^{n} \left(d_{ij}^k X_k + D_{ij}^k Y_k\right) + y_{ij}\xi, \\
T_{Y_i} \xi &= \sum_{k=1}^{n} \left(d_i^k X_k + D_i^k Y_k\right) + y_i\xi, \\
T_\xi X_j &= \sum_{k=1}^{n} \left(\alpha_j^k X_k + \beta_j^k Y_k\right) + \tilde{x}_j\xi, \\
T_\xi Y_j &= \sum_{k=1}^{n} \left(\gamma_j^k X_k + \epsilon_j^k Y_k\right) + \tilde{y}_j\xi, \\
T_\xi \xi &= \sum_{k=1}^{n} (\bar{x}^k X_k + \bar{y}^k Y_k) + \bar{z}\xi.
\end{aligned} \tag{9}$$

From the properties (8), we can immediately reduce the number of coefficients in (9). Condition $(8)_4$ implies $T_X \xi = \nabla_X \xi = -\varphi X - \varphi h X$. Therefore, $T_\xi \xi = 0$, $T_{X_i}\xi = -(1 + \lambda_i) Y_i$ and $T_{Y_i}\xi = (1 - \lambda_i) X_i$. So, we find

$$\begin{aligned}
&\bar{x}^k = \bar{y}^k = \bar{z} = A_i^k = x_i = D_i^k = y_i = 0, \\
&a_i^k = -(1 + \lambda_i)\delta_{ik}, \qquad d_i^k = (1 - \lambda_i)\delta_{ik}.
\end{aligned}$$

Next, we derive from $(8)_1$:

$$(T_Z \cdot g)(X, Y) = (\nabla_Z g)(X, Y) - (\tilde{\nabla}_Z g)(X, Y) = 0.$$

On the other hand, it holds by definition:

$$(T_Z \cdot g)(X, Y) = -g(T_Z X, Y) - g(X, T_Z Y).$$

Hence, T_Z is a skew-symmetric operator for each $Z \in T_p M$. Putting $Z = X_i, Y_i, \xi$, it follows

$$\begin{aligned}
&A^k_{ij} + A^j_{ik} = 0, \quad B^k_{ij} + B^j_{ik} = 0, \quad a^k_{ij} + b^j_{ik} = 0,\\
&x_{ij} = 0, \quad z_{ij} = (1+\lambda_i)\delta_{ij},\\
&C^k_{ij} + C^j_{ik} = 0, \quad D^k_{ij} + D^j_{ik} = 0, \quad c^k_{ij} + d^j_{ik} = 0,\\
&y_{ij} = 0, \quad w_{ij} = -(1-\lambda_i)\delta_{ij},\\
&\alpha^k_j + \alpha^j_k = 0, \quad \epsilon^k_j + \epsilon^j_k = 0, \quad \beta^k_j + \gamma^j_k = 0,\\
&x_j = 0, \quad y_j = 0.
\end{aligned} \tag{10}$$

Finally, we use the equality $\nabla_\xi \varphi = 0$, which holds for every contact metric space ([**2**]), together with $(8)_6$ to obtain

$$0 = (\nabla_\xi \varphi)Z = (T_\xi \cdot \varphi)Z = T_\xi(\varphi Z) - \varphi(T_\xi Z).$$

Substituting $Z = X_i, Y_i$, we find

$$\beta^k_j + \gamma^k_j = 0, \quad \epsilon^k_j - \alpha^k_j = 0. \tag{11}$$

In particular, combining (10) and (11), we have $\beta^k_j = -\gamma^k_j = \beta^j_k$.

Combining all these conditions on the coefficients, the expressions (9) for T simplify to

$$\begin{aligned}
T_{X_i} X_j &= \sum_{k=1}^{n} \left(A^k_{ij} X_k + e^k_{ij} Y_k\right),\\
T_{X_i} Y_j &= \sum_{k=1}^{n} \left(-e^j_{ik} X_k + B^k_{ij} Y_k\right) + \delta_{ij}(1+\lambda_i)\xi,\\
T_{X_i} \xi &= -(1+\lambda_i) Y_i,\\
T_{Y_i} X_j &= \sum_{k=1}^{n} \left(C^k_{ij} X_k - f^j_{ik} Y_k\right) - \delta_{ij}(1-\lambda_i)\xi,\\
T_{Y_i} Y_j &= \sum_{k=1}^{n} \left(f^k_{ij} X_k + D^k_{ij} Y_k\right),\\
T_{Y_i} \xi &= (1-\lambda_i) X_i,\\
T_\xi X_j &= \sum_{k=1}^{n} \left(\alpha^k_j X_k + \beta^k_j Y_k\right),\\
T_\xi Y_j &= \sum_{k=1}^{n} \left(-\beta^k_j X_k + \alpha^k_j Y_k\right),\\
T_\xi \xi &= 0
\end{aligned} \tag{12}$$

https://doi.org/10.1017/S0017089505002909 Published online by Cambridge University Press

where A^k_{ij}, B^k_{ij}, C^k_{ij}, D^k_{ij} and α^k_j are skew-symmetric in j and k and β^k_j is symmetric in j and k.

4. The first curvature condition. Recall that we are looking for locally homogeneous contact metric spaces which are strongly locally φ-symmetric. The first necessary condition on the curvature tensor is given by (5) for $k = 0$: $R(X, Y)\xi = 0$ for all vector fields X and Y orthogonal to ξ. On the other hand, since $\tilde{\nabla}\xi = 0$, also $\tilde{R}(X, Y)\xi = 0$. Because the two connections ∇ and $\tilde{\nabla}$ are related via T, we will use this to obtain additional restrictions on the coefficients of T.

First, we need the precise relation between R and $\tilde{R}$, which is given in [**20**]. For arbitrary vector fields X, Y and Z, it holds

$$\tilde{R}(X, Y)Z = R(X, Y)Z + T_{T_X Y}Z - T_{T_Y X}Z - T_X T_Y Z + T_Y T_X Z. \tag{13}$$

In particular

$$R(X, Y)\xi = [T_X, T_Y]\xi - (T_{T_X Y} - T_{T_Y X})\xi \tag{14}$$

for all vector fields X and Y on M. If we express the condition $R(X, Y)\xi = 0$ for strong local φ-symmetry for vector fields X and Y orthogonal to ξ, we obtain a system of *linear* equations in the coefficients A^k_{ij}, B^k_{ij}, C^k_{ij}, D^k_{ij}, e^k_{ij} and f^k_{ij}, which can be solved explicitly. We give some more details now, giving the explicit solutions.

4.1. One index. Using (14) and (12), we compute

$$\begin{aligned} 0 &= g(R(X_i, Y_i)\xi, X_i) = 2\lambda_i f^i_{ii}, \\ 0 &= g(R(X_i, Y_i)\xi, Y_i) = -2\lambda_i e^i_{ii}. \end{aligned}$$

Hence,

- if $\lambda_i > 0$: $e^i_{ii} = f^i_{ii} = 0$;
- if $\lambda_i = 0$: e^i_{ii} and f^i_{ii} are arbitrary.

4.2. Two different indices. This time we look at the conditions of the form $g(R(X, Y)\xi, Z) = 0$ where $X, Y, Z \in \{X_i, Y_i, X_j, Y_j\}$, $i < j$, with both indices occurring. This gives rise to a list of linear equations in the coefficients of T with two different indices. However, this system subdivides into systems of five equations for only five coefficients. Since all the subsystems are similar, we treat only the one involving C^i_{ij}, D^i_{ij}, e^j_{ii}, $e^i_{ij}$and e^i_{ji}. This subsystem corresponds to the conditions

$$\begin{aligned} 0 &= g(R(X_i, X_j)\xi, X_i), \\ 0 &= g(R(X_i, Y_i)\xi, Y_j), \\ 0 &= g(R(X_i, Y_j)\xi, Y_i), \\ 0 &= g(R(X_j, Y_i)\xi, Y_i), \\ 0 &= g(R(Y_i, Y_j)\xi, X_i). \end{aligned}$$

After some calculations, using (14) and (12), this system can be written as

$$M\begin{pmatrix} C^i_{ij} \\ D^i_{ij} \\ e^j_{ii} \\ e^i_{ij} \\ e^i_{ji} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{15}$$

where

$$M = \begin{pmatrix} 0 & 0 & 1+\lambda_j & -(1-\lambda_i) & -2\lambda_i \\ 1+\lambda_j & -(1+\lambda_i) & 1-\lambda_i & -(1+\lambda_j) & 0 \\ 0 & 0 & 1+\lambda_i & -(1-\lambda_j) & 0 \\ 1+\lambda_i & -(1+\lambda_j) & 0 & 0 & 2\lambda_i \\ 1-\lambda_j & -(1-\lambda_i) & 0 & 0 & 0 \end{pmatrix}.$$

The determinant of the matrix M is given by $8\lambda_i{}^2(\lambda_i+\lambda_j)(\lambda_j-\lambda_i)$. Keeping in mind that $\lambda_i \geq \lambda_j \geq 0$, we have the following solutions for the system (15):

- if $\lambda_i > \lambda_j \geq 0$: $C^i_{ij} = D^i_{ij} = e^j_{ii} = e^i_{ij} = e^i_{ji} = 0$;
- if $\lambda_i = \lambda_j > 0$: $C^i_{ij} = D^i_{ij}$, $e^j_{ii} = e^i_{ij} = e^i_{ji} = 0$, D^i_{ij} is arbitrary;
- if $\lambda_i = \lambda_j = 0$: $C^i_{ij} = D^i_{ij}$, $e^j_{ii} = e^i_{ij}$, D^i_{ij}, e^j_{ii} and e^i_{ji} are arbitrary.

Working similarly for the other subsystems $\{C^j_{ji}, D^j_{ji}, e^i_{jj}, e^j_{ji}, e^j_{ij}\}$, $\{A^i_{ij}, B^i_{ij}, f^j_{ii}, f^i_{ij}, f^i_{ji}\}$ and $\{A^j_{ji}, B^j_{ji}, f^i_{jj}, f^j_{ji}, f^j_{ij}\}$, we find

- if $\lambda_i > \lambda_j > 0$:

$$\begin{aligned} A^i_{ij} &= B^i_{ij} = f^j_{ii} = f^i_{ij} = f^i_{ji} = 0, \\ A^j_{ji} &= B^j_{ji} = f^i_{jj} = f^j_{ji} = f^j_{ij} = 0, \\ C^i_{ij} &= D^i_{ij} = e^j_{ii} = e^i_{ij} = e^i_{ji} = 0, \\ C^j_{ji} &= D^j_{ji} = e^i_{jj} = e^j_{ji} = e^j_{ij} = 0; \end{aligned}$$

- if $\lambda_i > \lambda_j = 0$:

$$\begin{aligned} & A^i_{ij} = B^i_{ij} = f^j_{ii} = f^i_{ij} = f^i_{ji} = 0, \\ & A^j_{ji} = B^j_{ji} = f^i_{jj} = f^j_{ji} = 0, \quad f^j_{ij} \text{ is arbitrary}, \\ & C^i_{ij} = D^i_{ij} = e^j_{ii} = e^i_{ij} = e^i_{ji} = 0, \\ & C^j_{ji} = D^j_{ji} = e^i_{jj} = e^j_{ji} = 0, \quad e^j_{ij} \text{ is arbitrary}; \end{aligned}$$

- if $\lambda_i = \lambda_j > 0$:

$$\begin{aligned} & A^i_{ij} = B^i_{ij}, \quad f^j_{ii} = f^i_{ij} = f^i_{ji} = 0, \quad A^i_{ij} \text{ is arbitrary}, \\ & A^j_{ji} = B^j_{ji}, \quad f^i_{jj} = f^j_{ji} = f^j_{ij} = 0, \quad A^j_{ji} \text{ is arbitrary}, \\ & C^i_{ij} = D^i_{ij}, \quad e^j_{ii} = e^i_{ij} = e^i_{ji} = 0, \quad D^i_{ij} \text{ is arbitrary}, \\ & C^j_{ji} = D^j_{ji}, \quad e^i_{jj} = e^j_{ji} = e^j_{ij} = 0, \quad D^j_{ji} \text{ is arbitrary}; \end{aligned}$$

https://doi.org/10.1017/S0017089505002909 Published online by Cambridge University Press

- if $\lambda_i = \lambda_j = 0$:

$$A^i_{ij} = B^i_{ij}, \quad f^j_{ii} = f^i_{ij}, \quad A^i_{ij}, f^i_{ij} \text{ and } f^i_{ji} \text{ are arbitrary,}$$
$$A^j_{ji} = B^j_{ji}, \quad f^i_{jj} = f^j_{ji}, \quad A^j_{ji}, f^j_{ji} \text{ and } f^j_{ij} \text{ are arbitrary,}$$
$$C^i_{ij} = D^i_{ij}, \quad e^j_{ii} = e^i_{ij}, \quad D^i_{ij}, e^i_{ij} \text{ and } e^i_{ji} \text{ are arbitrary,}$$
$$C^j_{ji} = D^j_{ji}, \quad e^i_{jj} = e^j_{ji}, \quad D^j_{ji}, e^j_{ji} \text{ and } e^j_{ij} \text{ are arbitrary.}$$

4.3. Three different indices. As in the previous case, the conditions $g(R(X, Y)\xi, Z) = 0$ where $X, Y, Z \in \{X_i, Y_i, X_j, Y_j, X_k, Y_k\}$, $i < j < k$, and with three different indices occurring, lead to subsystems of twelve equations in twelve coefficients: $\{A^k_{ij}, A^i_{jk}, A^j_{ki}, B^k_{ij}, B^i_{jk}, B^j_{ki}, f^k_{ij}, f^j_{ik}, f^k_{ji}, f^i_{jk}, f^j_{ki}, f^i_{kj}\}$ and $\{C^k_{ij}, C^i_{jk}, C^j_{ki}, D^k_{ij}, D^i_{jk}, D^j_{ki}, e^k_{ij}, e^j_{ik}, e^k_{ji}, e^i_{jk}, e^j_{ki}, e^i_{kj}\}$. Again, both cases are quite similar. We do not give the explicit equations here. It suffices to say that the rank of the system is equal to

- 6 if $\lambda_i = \lambda_j = \lambda_k = 0$;
- 9 if $\lambda_i = \lambda_j = \lambda_k > 0$;
- 10 if $\lambda_i > \lambda_j = \lambda_k = 0$;
- 11 in all other cases.

The corresponding solutions are given by

- if $\lambda_i = \lambda_j = \lambda_k = 0$:

$$A^k_{ij} = B^k_{ij}, \quad A^i_{jk} = B^i_{jk}, \quad A^j_{ki} = B^j_{ki},$$
$$C^k_{ij} = D^k_{ij}, \quad C^i_{jk} = D^i_{jk}, \quad C^j_{ki} = D^j_{ki},$$
$$e^k_{ij} = e^j_{ik}, \quad e^i_{jk} = e^k_{ji}, \quad e^j_{ki} = e^i_{kj},$$
$$f^k_{ij} = f^j_{ik}, \quad f^i_{jk} = f^k_{ji}, \quad f^j_{ki} = f^i_{kj},$$
$$A^k_{ij}, A^i_{jk}, A^j_{ki}, D^k_{ij}, D^i_{jk}, D^j_{ki}, e^k_{ij}, e^i_{jk}, e^j_{ki}, f^k_{ij}, f^i_{jk} \text{ and } f^j_{ki} \text{ are arbitrary;}$$

- if $\lambda_i = \lambda_j = \lambda_k > 0$:

$$A^k_{ij} = B^k_{ij}, \quad A^i_{jk} = B^i_{jk}, \quad A^j_{ki} = B^j_{ki},$$
$$C^k_{ij} = D^k_{ij}, \quad C^i_{jk} = D^i_{jk}, \quad C^j_{ki} = D^j_{ki},$$
$$e^k_{ij} = e^j_{ik} = e^i_{jk} = e^k_{ji} = e^j_{ki} = e^i_{kj} = 0,$$
$$f^k_{ij} = f^j_{ik} = f^i_{jk} = f^k_{ji} = f^j_{ki} = f^i_{kj} = 0,$$
$$A^k_{ij}, A^i_{jk}, A^j_{ki}, D^k_{ij}, D^i_{jk} \text{ and } D^j_{ki} \text{ are arbitrary;}$$

- if $\lambda_i > \lambda_j = \lambda_k = 0$:

$$A^k_{ij} = B^k_{ij}, \quad A^i_{jk} = B^i_{jk} = A^j_{ki} = B^j_{ki} = 0,$$
$$C^k_{ij} = D^k_{ij}, \quad C^i_{jk} = D^i_{jk} = C^j_{ki} = D^j_{ki} = 0,$$
$$e^k_{ij} = e^j_{ik}, \quad e^i_{jk} = e^k_{ji} = e^j_{ki} = e^i_{kj} = 0,$$
$$f^k_{ij} = f^j_{ik}, \quad f^i_{jk} = f^k_{ji} = f^j_{ki} = f^i_{kj} = 0,$$
$$A^k_{ij}, D^k_{ij}, e^k_{ij} \text{ and } f^k_{ij} \text{ are arbitrary;}$$

https://doi.org/10.1017/S0017089505002909 Published online by Cambridge University Press

- otherwise:

$$
\begin{aligned}
&A^k_{ij} = B^k_{ij} = (\lambda_i+\lambda_j)(\lambda_j+\lambda_k)(\lambda_k+\lambda_i)(\lambda_i-\lambda_j)(\lambda_k-\lambda_i)\,a_{ijk},\\
&A^i_{jk} = B^i_{jk} = (\lambda_i+\lambda_j)(\lambda_j+\lambda_k)(\lambda_k+\lambda_i)(\lambda_i-\lambda_j)(\lambda_j-\lambda_k)a_{ijk},\\
&A^j_{ki} = B^j_{ki} = (\lambda_i+\lambda_j)(\lambda_j+\lambda_k)(\lambda_k+\lambda_i)(\lambda_j-\lambda_k)(\lambda_k-\lambda_i)\,a_{ijk},\\
&C^k_{ij} = D^k_{ij} = (\lambda_i+\lambda_j)(\lambda_j+\lambda_k)(\lambda_k+\lambda_i)(\lambda_i-\lambda_j)(\lambda_k-\lambda_i)\,d_{ijk},\\
&C^i_{jk} = D^i_{jk} = (\lambda_i+\lambda_j)(\lambda_j+\lambda_k)(\lambda_k+\lambda_i)(\lambda_i-\lambda_j)(\lambda_j-\lambda_k)d_{ijk},\\
&C^j_{ki} = D^j_{ki} = (\lambda_i+\lambda_j)(\lambda_j+\lambda_k)(\lambda_k+\lambda_i)(\lambda_j-\lambda_k)(\lambda_k-\lambda_i)\,d_{ijk},\\
&e^k_{ij} = e^j_{ik} = (\lambda_i+\lambda_j)(\lambda_k+\lambda_i)(\lambda_i-\lambda_j)(\lambda_j-\lambda_k)(\lambda_k-\lambda_i)\,d_{ijk},\\
&e^i_{jk} = e^k_{ji} = (\lambda_i+\lambda_j)(\lambda_j+\lambda_k)(\lambda_i-\lambda_j)(\lambda_j-\lambda_k)(\lambda_k-\lambda_i)\,d_{ijk},\\
&e^j_{ki} = e^i_{kj} = (\lambda_j+\lambda_k)(\lambda_k+\lambda_i)(\lambda_i-\lambda_j)(\lambda_j-\lambda_k)(\lambda_k-\lambda_i)\,d_{ijk},\\
&f^k_{ij} = f^j_{ik} = (\lambda_i+\lambda_j)(\lambda_k+\lambda_i)(\lambda_i-\lambda_j)(\lambda_j-\lambda_k)(\lambda_k-\lambda_i)\,a_{ijk},\\
&f^i_{jk} = f^k_{ji} = (\lambda_i+\lambda_j)(\lambda_j+\lambda_k)(\lambda_i-\lambda_j)(\lambda_j-\lambda_k)(\lambda_k-\lambda_i)\,a_{ijk},\\
&f^j_{ki} = f^i_{kj} = (\lambda_j+\lambda_k)(\lambda_k+\lambda_i)(\lambda_i-\lambda_j)(\lambda_j-\lambda_k)(\lambda_k-\lambda_i)\,a_{ijk},\\
&a_{ijk} \text{ and } d_{ijk} \text{ are arbitrary.}
\end{aligned}
$$

5. Curvature components. The results in the previous section have already greatly reduced the number of independent coefficients for the homogeneous structure T of a strongly locally φ-symmetric contact metric space. In this section, we obtain even more information as a consequence of the conditions (8). The operator h plays a crucial role here. Recall that the vanishing of h is equivalent to the contact metric structure being K-contact. Since a locally φ-symmetric K-contact manifold is always Sasakian (hence also a (κ, μ)-space), we may suppose that $h \neq 0$ or, equivalently, $\lambda_1 > 0$. First, we show that also h is parallel for the connection $\tilde{\nabla}$.

LEMMA. $\tilde{\nabla} h = 0$.

Proof. In the following calculation, we use the definition of h, the properties (8) and in particular the equalities:

$$
\begin{aligned}
[\xi, Z] &= \nabla_\xi Z - \nabla_Z \xi = \tilde{\nabla}_\xi Z + T_\xi Z - \tilde{\nabla}_Z \xi - T_Z \xi = \tilde{\nabla}_\xi Z + T_\xi Z - T_Z \xi,\\
0 &= \nabla_\xi \varphi = \tilde{\nabla}_\xi \varphi + T_\xi \cdot \varphi = T_\xi \circ \varphi - \varphi \circ T_\xi .
\end{aligned}
$$

We compute

$$
\begin{aligned}
2(\tilde{\nabla}_X h)Y &= 2\tilde{\nabla}_X(hY) - 2h(\tilde{\nabla}_X Y) = \tilde{\nabla}((\mathcal{L}_\xi \varphi)Y) - (\mathcal{L}_\xi \varphi)(\tilde{\nabla}_X Y)\\
&= \tilde{\nabla}_X([\xi, \varphi Y] - \varphi[\xi, Y]) - [\xi, \varphi(\tilde{\nabla}_X Y)] + \varphi[\xi, \tilde{\nabla}_X Y]\\
&= \tilde{\nabla}_X(\tilde{\nabla}_\xi(\varphi Y) + T_\xi(\varphi Y) - T_{\varphi Y}\xi) - \tilde{\nabla}_X(\varphi(\tilde{\nabla}_\xi Y + T_\xi Y - T_Y \xi))\\
&\quad - \tilde{\nabla}_\xi(\varphi \tilde{\nabla}_X Y) - T_\xi(\varphi \tilde{\nabla}_X Y) + T_{\varphi \tilde{\nabla}_X Y}\xi\\
&\quad + \varphi(\tilde{\nabla}_\xi \tilde{\nabla}_X Y) + \varphi(T_\xi \tilde{\nabla}_X Y) - \varphi(T_{\tilde{\nabla}_X Y}\xi)
\end{aligned}
$$

https://doi.org/10.1017/S0017089505002909 Published online by Cambridge University Press

$$
\begin{aligned}
&= \varphi(\tilde\nabla_X\tilde\nabla_\xi Y) + T_\xi(\varphi\tilde\nabla_X Y) - T_{\varphi\tilde\nabla_X Y}\xi - \varphi(\tilde\nabla_X\tilde\nabla_\xi Y)\\
&\quad -\varphi(T_\xi\tilde\nabla_X Y) + \varphi(T_{\tilde\nabla_X Y}\xi) - \varphi(\tilde\nabla_\xi\tilde\nabla_X Y)\\
&\quad -T_\xi(\varphi\tilde\nabla_X Y) + T_{\varphi\tilde\nabla_X Y}\xi, + \varphi(\tilde\nabla_\xi\tilde\nabla_X Y)\\
&\quad +\varphi(T_\xi\tilde\nabla_X Y) - \varphi(T_{\tilde\nabla_X Y}\xi)\\
&= 0. \qquad \square
\end{aligned}
$$

As a consequence, we have $\tilde R(X, Y)\cdot h = \tilde\nabla_X\tilde\nabla_Y h - \tilde\nabla_Y\tilde\nabla_X h - \tilde\nabla_{[X,Y]}h = 0$ for all vector fields X and Y by the Ricci identity. From (13), we then get

$$R(X, Y)\cdot h = ([T_X, T_Y] - T_{T_X Y} + T_{T_Y X})\cdot h$$

or

$$
\begin{aligned}
&R(X, Y)\circ h - h\circ R(X, Y)\\
&\quad = ([T_X, T_Y] - (T_{T_X Y} - T_{T_Y X}))\circ h - h\circ([T_X, T_Y] - (T_{T_X Y} - T_{T_Y X})). \qquad (16)
\end{aligned}
$$

We can use this formula to compute some curvature components $g(R(X, Y)Z, W)$. Indeed, if $hZ = \lambda Z$ and $hW = \mu W$ with $\lambda \neq \mu$, then we have

$$g(R(X, Y)hZ - hR(X, Y)Z, W) = (\lambda - \mu)g(R(X, Y)Z, W),$$

while

$$
\begin{aligned}
&g([T_X, T_Y]hZ - (T_{T_X Y} - T_{T_Y X})hZ - h([T_X, T_Y]Z - (T_{T_X Y} - T_{T_Y X})Z))\\
&\quad = (\lambda - \mu)g([T_X, T_Y]Z - (T_{T_X Y} - T_{T_Y X})Z, W).
\end{aligned}
$$

From (16), we then conclude

$$g(R(X, Y)Z, W) = g([T_X, T_Y]Z - (T_{T_X Y} - T_{T_Y X})Z, W). \qquad (17)$$

Note that this equality only holds for eigenvectors Z and W of h with different eigenvalues! In the rest of this section, we use (17) to compute certain curvature components and then use the symmetries of the curvature tensor to derive information on the eigenvalues λ_i of h and their multiplicities. We perform one such calculation in detail, and afterwards only provide the resulting expressions for the curvature components.

In order to take the multiplicities of the eigenvalues of h into account, we now change the notation slightly and work with an orthonormal basis $\{X_{11}, \ldots, X_{1k_1}, X_{21}, \ldots, X_{2k_2}, \ldots, X_{t1}, \ldots, X_{tk_t}, Y_{11}, \ldots, Y_{1k_1}, \ldots, Y_{t1}, \ldots, Y_{tk_t}, \xi\}$ such that

$$h(X_{si}) = \lambda_s X_{si}, \quad h(Y_{si}) = -\lambda_s Y_{si}, \quad \varphi(X_{si}) = Y_{si}, \quad \varphi(Y_{si}) = -X_{si}$$

for $s = 1, \ldots, t$, $i = 1, \ldots, k_s$ and such that $\lambda_1 > \lambda_2 > \cdots > \lambda_t = 0$. The indexing of the coefficients in (12) will be changed accordingly.

5.1. The zero eigenvalue. Suppose first that h has a zero eigenvalue on $\xi^\perp$. With the notation above, this means $k_t > 0$. From (17), we have

$$g(R(X_{11}, X_{t1})Y_{11}, Y_{t1}) = g\big([T_{X_{11}}, T_{X_{t1}}]Y_{11} - \big(T_{T_{X_{11}}X_{t1}} - T_{T_{X_{t1}}X_{11}}\big)Y_{11}, Y_{t1}\big).$$

https://doi.org/10.1017/S0017089505002909 Published online by Cambridge University Press

We calculate the right-hand side term by term, using the results from the previous section:

$$
\begin{aligned}
T_{X_{t1}}Y_{11} &= -\sum_{s=1}^{t}\sum_{i=1}^{k_s} e_{t1\,si}^{11}X_{si} + \sum_{s=1}^{t}\sum_{i=1}^{k_s} B_{t1\,11}^{si}\,Y_{si}\\
&= -e_{t1\,11}^{11}X_{11} - \sum_{i=2}^{k_1} e_{t1\,1i}^{11}X_{1i} - \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} e_{t1\,si}^{11}X_{si}\\
&\quad - e_{t1\,t1}^{11}X_{t1} - \sum_{i=2}^{k_t} e_{t1\,ti}^{11}X_{ti}\\
&\quad + B_{t1\,11}^{11}\,Y_{11} + \sum_{i=2}^{k_1} B_{t1\,11}^{1i}\,Y_{1i} + \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} B_{t1\,11}^{si}\,Y_{si}\\
&\quad + B_{t1\,11}^{t1}\,Y_{t1} + \sum_{i=2}^{k_t} B_{t1\,11}^{ti}\,Y_{ti}\\
&= \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} \lambda_1^2\lambda_s^2(\lambda_1-\lambda_s)\,d_{11\,si\,t1}X_{si}\\
&\quad + \sum_{i=2}^{k_1} A_{t1\,11}^{1i}\,Y_{1i} - \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} \lambda_1^2\lambda_s^2(\lambda_1+\lambda_s)\,a_{11\,si\,t1}\,Y_{si},
\end{aligned}
$$

$$
\begin{aligned}
g(T_{X_{11}}T_{X_{t1}}Y_{11},\,Y_{t1}) &= \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} \lambda_1^2\lambda_s^2(\lambda_1-\lambda_s)\,d_{11\,si\,t1}e_{11\,si}^{t1}\\
&\quad + \sum_{i=2}^{k_1} A_{t1\,11}^{1i}B_{11\,1i}^{t1} - \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} \lambda_1^2\lambda_s^2(\lambda_1+\lambda_s)\,a_{11\,si\,t1}B_{11\,si}^{t1}\\
&= -\sum_{s=2}^{t-1}\sum_{i=1}^{k_s} \lambda_1^4\lambda_s^3(\lambda_1-\lambda_s)^2(\lambda_1+\lambda_s)d_{11\,si\,t1}^2\\
&\quad + \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} \lambda_1^4\lambda_s^3(\lambda_1-\lambda_s)(\lambda_1+\lambda_s)^2a_{11\,si\,t1}^2\\
&= \sum_{s=2}^{t-1}\sum_{i=1}^{k_s} \lambda_1^4\lambda_s^3(\lambda_1^2-\lambda_s^2)\big((\lambda_1+\lambda_s)a_{11\,si\,t1}^2-(\lambda_1-\lambda_s)d_{11\,si\,t1}^2\big),
\end{aligned}
$$

$$
\begin{aligned}
T_{X_{11}}Y_{11} &= -\sum_{s=1}^{t}\sum_{i=1}^{k_s} e_{11\,si}^{11}X_{si} + \sum_{s=1}^{t}\sum_{i=1}^{k_s} B_{11\,11}^{si}\,Y_{si} + (1+\lambda_1)\xi\\
&= \sum_{i=2}^{k_1} A_{11\,11}^{1i}\,Y_{1i} + (1+\lambda_1)\xi,
\end{aligned}
$$

$$
g(T_{X_{t1}}T_{X_{11}}Y_{11},\,Y_{t1}) = \sum_{i=2}^{k_1} A_{11\,11}^{1i}B_{t1\,1i}^{t1} - (1+\lambda_1) = -(1+\lambda_1),
$$

https://doi.org/10.1017/S0017089505002909 Published online by Cambridge University Press

$$
\begin{aligned}
T_{X_{11}}X_{t1} &= \sum_{s=1}^{t}\sum_{i=1}^{k_s} A_{11\,t1}^{si}X_{si} + \sum_{s=1}^{t}\sum_{i=1}^{k_s} e_{11\,t1}^{si}Y_{si}\\
&= \sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^2\lambda_s\big(\lambda_1^2-\lambda_s^2\big)a_{11\,si\,t1}X_{si} + \sum_{i=2}^{k_t}A_{11\,t1}^{ti}X_{ti}\\
&\quad - \sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^2\lambda_s\big(\lambda_1^2-\lambda_s^2\big)d_{11\,si\,t1}Y_{si} + \sum_{i=1}^{k_t}e_{11\,t1}^{ti}Y_{ti},
\end{aligned}
$$

$$
\begin{aligned}
g\big(T_{T_{X_{11}}X_{t1}}Y_{11},\,Y_{t1}\big) &= \sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^2\lambda_s\big(\lambda_1^2-\lambda_s^2\big)a_{11\,si\,t1}B_{si\,11}^{t1} + \sum_{i=2}^{k_t}A_{11\,t1}^{ti}B_{ti\,11}^{t1}\\
&\quad - \sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^2\lambda_s\big(\lambda_1^2-\lambda_s^2\big)d_{11\,si\,t1}D_{si\,11}^{t1} + \sum_{i=1}^{k_t}e_{11\,t1}^{ti}D_{ti\,11}^{t1}\\
&= -\sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^3\lambda_s^3\big(\lambda_1^2-\lambda_s^2\big)^2a_{11\,si\,t1}^2\\
&\quad + \sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^3\lambda_s^3\big(\lambda_1^2-\lambda_s^2\big)^2d_{11\,si\,t1}^2\\
&= \sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^3\lambda_s^3\big(\lambda_1^2-\lambda_s^2\big)^2\big(d_{11\,si\,t1}^2-a_{11\,si\,t1}^2\big),
\end{aligned}
$$

$$
\begin{aligned}
T_{X_{t1}}X_{11} &= \sum_{s=1}^{t}\sum_{i=1}^{k_s} A_{t1\,11}^{si}X_{si} + \sum_{s=1}^{t}\sum_{i=1}^{k_s} e_{t1\,11}^{si}Y_{si}\\
&= \sum_{i=2}^{k_1}A_{t1\,11}^{1i}X_{1i} - \sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^2\lambda_s^2(\lambda_1+\lambda_s)a_{11\,si\,t1}X_{si}\\
&\quad - \sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^2\lambda_s^2(\lambda_1-\lambda_s)d_{11\,si\,t1}Y_{si},
\end{aligned}
$$

$$
\begin{aligned}
g\big(T_{T_{X_{t1}}X_{11}}Y_{11},\,Y_{t1}\big) &= \sum_{i=2}^{k_1}A_{t1\,11}^{1i}B_{1i\,11}^{t1} - \sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^2\lambda_s^2(\lambda_1+\lambda_s)a_{11\,si\,t1}B_{si\,11}^{t1}\\
&\quad - \sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^2\lambda_s^2(\lambda_1-\lambda_s)d_{11\,si\,t1}D_{si\,11}^{t1}\\
&= \sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^3\lambda_s^4(\lambda_1+\lambda_s)^2(\lambda_1-\lambda_s)a_{11\,si\,t1}^2\\
&\quad + \sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^3\lambda_s^4(\lambda_1-\lambda_s)^2(\lambda_1+\lambda_s)d_{11\,si\,t1}^2\\
&= \sum_{s=2}^{t-1}\sum_{i=1}^{k_s}\lambda_1^3\lambda_s^4\big(\lambda_1^2-\lambda_s^2\big)\big((\lambda_1+\lambda_s)a_{11\,si\,t1}^2+(\lambda_1-\lambda_s)d_{11\,si\,t1}^2\big).
\end{aligned}
$$

https://doi.org/10.1017/S0017089505002909 Published online by Cambridge University Press

Bringing all this together, we find

$$\begin{aligned} &g(R(X_{11}, X_{t1})Y_{11}, Y_{t1}) \\ &\quad = (1+\lambda_1) + 2\sum_{s=2}^{t-1}\sum_{i=1}^{k_s} \lambda_1^4\lambda_s^3\big(\lambda_1^2-\lambda_s^2\big)\big((\lambda_1+\lambda_s)a_{11\,si\,t1}^2 - (\lambda_1-\lambda_s)d_{11\,si\,t1}^2\big). \end{aligned}$$

Similarly, we compute

$$\begin{aligned} &g(R(Y_{11}, Y_{t1})X_{11}, X_{t1}) \\ &\quad = (1-\lambda_1) + 2\sum_{s=2}^{t-1}\sum_{i=1}^{k_s} \lambda_1^4\lambda_s^3\big(\lambda_1^2-\lambda_s^2\big)\big((\lambda_1+\lambda_s)d_{11\,si\,t1}^2 - (\lambda_1-\lambda_s)a_{11\,si\,t1}^2\big). \end{aligned}$$

Since both expressions must agree because of the symmetries of the curvature tensor, we find the following equality:

$$\lambda_1 = 2\sum_{s=2}^{t-1}\sum_{i=1}^{k_s} \lambda_1^5\lambda_s^3\big(\lambda_1^2-\lambda_s^2\big)\big(d_{11\,si\,t1}^2 - a_{11\,si\,t1}^2\big). \tag{18}$$

On the other hand, working in a similar way, we find

$$\begin{aligned} &g(R(X_{t1}, Y_{11})X_{11}, Y_{t1}) \\ &\quad = (1-\lambda_1) + 2\sum_{s=2}^{t-1}\sum_{i=1}^{k_s} \lambda_1^4\lambda_s^3\big(\lambda_1^2-\lambda_s^2\big)\big((\lambda_1+\lambda_s)a_{11\,si\,t1}^2 - (\lambda_1-\lambda_s)d_{11\,si\,t1}^2\big), \\ &g(R(X_{11}, Y_{t1})X_{t1}, Y_{11}) \\ &\quad = (1+\lambda_1) + 2\sum_{s=2}^{t-1}\sum_{i=1}^{k_s} \lambda_1^4\lambda_s^3\big(\lambda_1^2-\lambda_s^2\big)\big((\lambda_1+\lambda_s)d_{11\,si\,t1}^2 - (\lambda_1-\lambda_s)a_{11\,si\,t1}^2\big). \end{aligned}$$

Again, both expressions must agree and we get

$$\lambda_1 = 2\sum_{s=2}^{t-1}\sum_{i=1}^{k_s} \lambda_1^5\lambda_s^3\big(\lambda_1^2-\lambda_s^2\big)\big(a_{11\,si\,t1}^2 - d_{11\,si\,t1}^2\big). \tag{19}$$

Comparing (18) and (19), we see that $\lambda_1 = 0$, which is contrary to our assumption. Hence, $k_t = 0$ and zero is not an eigenvalue of h on $\xi^\perp$.

5.2. The number of eigenvalues. Suppose next that h has at least two different positive eigenvalues on $\xi^\perp$, i.e., $k_2 > 0$. Calculating as before, we get first

$$\begin{aligned} g(R(X_{11}, X_{21})Y_{11}, Y_{21}) &= (1+\lambda_1)(1+\lambda_2) + 2\sum_{s\geq 3}\sum_{i=1}^{k_s}\big(\lambda_1^2-\lambda_2^2\big)^2\big(\lambda_2^2-\lambda_s^2\big)\big(\lambda_s^2-\lambda_1^2\big) \\ &\quad \cdot\big((\lambda_2+\lambda_s)(\lambda_s+\lambda_1)a_{11\,21\,si}^2 - (\lambda_2-\lambda_s)(\lambda_s-\lambda_1)d_{11\,21\,si}^2\big), \\ g(R(Y_{11}, Y_{21})X_{11}, X_{21}) &= (1-\lambda_1)(1-\lambda_2) + 2\sum_{s\geq 3}\sum_{i=1}^{k_s}\big(\lambda_1^2-\lambda_2^2\big)^2\big(\lambda_2^2-\lambda_s^2\big)\big(\lambda_s^2-\lambda_1^2\big) \\ &\quad \cdot\big((\lambda_2+\lambda_s)(\lambda_s+\lambda_1)d_{11\,21\,si}^2 - (\lambda_2-\lambda_s)(\lambda_s-\lambda_1)a_{11\,21\,si}^2\big). \end{aligned}$$

https://doi.org/10.1017/S0017089505002909 Published online by Cambridge University Press

These two expressions must agree because of the symmetries of the Riemann curvature tensor. This gives

$$\lambda_1+\lambda_2=2\sum_{s\geq 3}\sum_{i=1}^{k_s}\big(\lambda_1^2-\lambda_2^2\big)^2\big(\lambda_2^2-\lambda_s^2\big)\big(\lambda_s^2-\lambda_1^2\big)\lambda_s(\lambda_1+\lambda_2)\big(d_{11\,21\,si}^2-a_{11\,21\,si}^2\big)$$

or

$$1=2\sum_{s\geq 3}\sum_{i=1}^{k_s}\big(\lambda_1^2-\lambda_2^2\big)^2\big(\lambda_2^2-\lambda_s^2\big)\big(\lambda_s^2-\lambda_1^2\big)\lambda_s\big(d_{11\,21\,si}^2-a_{11\,21\,si}^2\big). \tag{20}$$

On the other hand, we also have

$$\begin{aligned} g(R(X_{11},Y_{21})Y_{11},X_{21}) &= -(1+\lambda_1)(1-\lambda_2)+2\sum_{s\geq 3}\sum_{i=1}^{k_s}\big(\lambda_1^2-\lambda_2^2\big)^2\big(\lambda_2^2-\lambda_s^2\big)\big(\lambda_s^2-\lambda_1^2\big)\\ &\quad\cdot\big((\lambda_2-\lambda_s)(\lambda_s+\lambda_1)d_{11\,21\,si}^2-(\lambda_2+\lambda_s)(\lambda_s-\lambda_1)a_{11\,21\,si}^2\big),\\ g(R(Y_{11},X_{21})X_{11},Y_{21}) &= -(1-\lambda_1)(1+\lambda_2)+2\sum_{s\geq 3}\sum_{i=1}^{k_s}\big(\lambda_1^2-\lambda_2^2\big)^2\big(\lambda_2^2-\lambda_s^2\big)\big(\lambda_s^2-\lambda_1^2\big)\\ &\quad\cdot\big((\lambda_2-\lambda_s)(\lambda_s+\lambda_1)a_{11\,21\,si}^2-(\lambda_2+\lambda_s)(\lambda_s-\lambda_1)d_{11\,21\,si}^2\big)\end{aligned}$$

and hence

$$\lambda_1-\lambda_2=2\sum_{s\geq 3}\sum_{i=1}^{k_s}\big(\lambda_1^2-\lambda_2^2\big)^2\big(\lambda_2^2-\lambda_s^2\big)\big(\lambda_s^2-\lambda_1^2\big)\lambda_s(\lambda_1-\lambda_2)\big(a_{11\,21\,si}^2-d_{11\,21\,si}^2\big)$$

or

$$1=2\sum_{s\geq 3}\sum_{i=1}^{k_s}\big(\lambda_1^2-\lambda_2^2\big)^2\big(\lambda_2^2-\lambda_s^2\big)\big(\lambda_s^2-\lambda_1^2\big)\lambda_s\big(a_{11\,21\,si}^2-d_{11\,21\,si}^2\big). \tag{21}$$

If we compare the right-hand sides of (20) and (21), we see that they are each other's negative. Hence, we obtain a contradiction once more. Consequently, k_2 must be zero and h has only one positive eigenvalue on $\xi^\perp$.

5.3. (κ,μ)-spaces. Since there is now only one positive eigenvalue $\lambda=\lambda_1>0$ left for h, we can revert to our original notation. So, we have an orthonormal basis $\{X_1,\dots,X_n,Y_1,\dots,Y_n,\xi\}$ such that

$$h(X_i)=\lambda X_i,\quad h(Y_i)=-\lambda Y_i,\quad \varphi(X_i)=Y_i,\quad \varphi(Y_i)=-X_i$$

for $i=1,\dots,n$. Furthermore, based on Section 4, the homogeneous structure T is given in this basis as

$$\begin{aligned} T_{X_i}X_j &= \sum_{k=1}^{n}A_{ij}^kX_k,\\ T_{X_i}Y_j &= \sum_{k=1}^{n}A_{ij}^kY_k+\delta_{ij}(1+\lambda)\xi,\end{aligned}$$

https://doi.org/10.1017/S0017089505002909 Published online by Cambridge University Press

$$\begin{aligned} T_{X_i}\xi &= -(1+\lambda)Y_i, \\ T_{Y_i}X_j &= \sum_{k=1}^{n} D_{ij}^k X_k - \delta_{ij}(1-\lambda)\xi, \\ T_{Y_i}Y_j &= \sum_{k=1}^{n} D_{ij}^k Y_k, \\ T_{Y_i}\xi &= (1-\lambda)X_i, \\ T_\xi X_j &= \sum_{k=1}^{n}\big(\alpha_j^k X_k + \beta_j^k Y_k\big), \\ T_\xi Y_j &= \sum_{k=1}^{n}\big(-\beta_j^k X_k + \alpha_j^k Y_k\big), \\ T_\xi\xi &= 0 \end{aligned} \tag{22}$$

where A_{ij}^k, D_{ij}^k and α_j^k are skew-symmetric in j and k and β_j^k is symmetric in j and k.

Let us first investigate under which conditions on the coefficients the curvature tensor R satisfies

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY).$$

For X and Y both orthogonal to ξ, this is already the case (see Section 4). This leaves only the case $R(X, \xi)\xi$ for X orthogonal to ξ. Using (14) and the expressions (22), we compute

$$\begin{aligned} R(X_i, \xi)\xi &= \big(1 - \lambda^2 - 2\lambda\beta_i^i\big)X_i - 2\lambda\sum_{k\neq i}\beta_i^k X_k, \\ R(Y_i, \xi)\xi &= \big(1 - \lambda^2 + 2\lambda\beta_i^i\big)Y_i + 2\lambda\sum_{k\neq i}\beta_i^k Y_k. \end{aligned}$$

On the other hand, for a (κ, μ)-contact metric structure, we have

$$R(X_i, \xi)\xi = (\kappa + \lambda\mu)X_i, \quad R(Y_i, \xi)\xi = (\kappa - \lambda\mu)Y_i.$$

So, we need

$$\beta_1^1 = \cdots = \beta_n^n, \qquad \beta_i^k = 0, \quad i \neq k.$$

Then $\kappa = 1 - \lambda^2$ and $\mu = -2\beta_1^1$. We show now that (β_i^k) is indeed a multiple of the identity.

We do this in a similar way as before, by calculating curvature components in the basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$ starting from (17). For $i \neq j$, we easily compute

$$g(R(X_i, Y_i)X_j, Y_i) = -2\beta_j^i, \quad g(R(X_j, Y_i)X_i, Y_i) = 0$$

and hence $\beta_i^j = 0$ for $i \neq j$. Further, still for $i \neq j$, we get

$$g(R(X_i, Y_i)X_j, Y_j) = -2\beta_j^j.$$

https://doi.org/10.1017/S0017089505002909 Published online by Cambridge University Press

So,

$$\beta_j^j = -\frac{1}{2}\, g(R(X_i, Y_i)X_j, Y_j) = -\frac{1}{2}\, g(R(X_j, Y_j)X_i, Y_i) = \beta_i^i.$$

Summarizing, we have shown that the Riemann curvature tensor of a locally contact-homogeneous strongly locally φ-symmetric space satisfies

$$R(X, Y)\xi = \kappa\,(\eta(Y)X - \eta(X)Y) + \mu\,(\eta(Y)hX - \eta(X)hY). \tag{23}$$

This does not quite prove the main theorem yet! Indeed, since all calculations were made *at a fixed point* $p \in M$, we only know that the κ and μ in (23) are functions on M, and not necessarily constants, i.e., we have a *generalized* (κ, μ)*-space*. These were introduced in [**15**]. It was shown in that paper that non-Sasakian spaces in this class with *non-constant* κ and μ are necessarily three-dimensional. However, as we mentioned in the introduction, it was proved by G. Calvaruso, D. Perrone and L. Vanhecke in [**12**] that three-dimensional strongly locally φ-symmetric spaces are always (κ, μ)-spaces, i.e., with constant κ and μ. These remarks complete the proof of the main theorem.

Note that we have only used the very first curvature condition to prove the main result. We could therefore strengthen it as follows:

Proposition. *Let* $(M, \xi, \eta, \varphi, g)$ *be a locally contact-homogeneous contact metric space satisfying* $R(X, Y)\xi = 0$ *for all vector fields* X *and* Y *orthogonal to* ξ. *Then the manifold is a* (κ, μ)*-space.*

Proof. Since we used $\lambda_1 > 0$ to prove the main theorem, we still have to consider the case when the contact structure is actually K-contact. Then $h = 0$ or, equivalently, $\lambda_i = 0$ for $i = 1, \ldots, n$. By (14) and (12), we easily calculate

$$R(X_i, \xi)\xi = X_i, \quad R(Y_i, \xi)\xi = Y_i.$$

Together with the condition $R(X, Y)\xi = 0$ for all vector fields X and Y orthogonal to ξ, it follows

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

for arbitrary vector fields X and Y and the structure is Sasakian, hence (κ, μ). □

REFERENCES

1. W. Ambrose and I. M. Singer, On homogeneous Riemannian manifolds, *Duke Math. J.* **25** (1958), 647–669.

2. D. E. Blair, *Riemannian geometry of contact and symplectic manifolds,* Progress in Mathematics 203 (Birkhäuser, 2001).

3. D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, Contact metric manifolds satisfying a nullity condition, *Israel J. Math.* **91** (1995), 189–214.

4. D. E. Blair, T. Koufogiorgos and R. Sharma, A classification of 3-dimensional contact metric manifolds with $Q\varphi = \varphi Q$, *Kodai Math. J.* **13** (1990), 391–401.

5. D. E. Blair and L. Vanhecke, Symmetries and φ-symmetric spaces, *Tôhoku Math. J.* **39** (1987), 373–383.

6. E. Boeckx, A class of locally φ-symmetric contact metric spaces, *Arch. Math.* **72** (1999), 466–472.

7. E. Boeckx, A full classification of contact metric (k, μ)-spaces, *Illinois J. Math.* **44** (2000), 212–219.

8. E. Boeckx, Invariant locally φ-symmetric contact structures on Lie groups, *Bull. Belg. Math. Soc.—Simon Stevin* **10** (2003), 391–407.

9. E. Boeckx, P. Bueken and L. Vanhecke, φ-symmetric contact metric spaces, *Glasgow Math. J.* **41** (1999), 409–416.

10. E. Boeckx and L. Vanhecke, Characteristic reflections on unit tangent sphere bundles, *Houston J. Math.* **23** (1997), 427–448.

11. P. Bueken and L. Vanhecke, Reflections in K-contact geometry, *Math. Rep. Toyama Univ.* **12** (1989), 41–49.

12. G. Calvaruso, D. Perrone and L. Vanhecke, Homogeneity on three-dimensional contact Riemannian manifolds, *Israel J. Math.* **114** (1999), 301–321.

13. B.-Y. Chen and L. Vanhecke, Isometric, holomorphic and symplectic reflections *Geom. Dedicata* **29** (1989), 259–277.

14. V. F. Kiričenko, On homogeneous Riemannian spaces with invariant tensor structure, *Soviet Math. Dokl.* **21** (1980), 734–737.

15. Th. Koufogiorgos and Ch. Tsichlias, On the existence of a new class of contact metric manifolds, *Canad. Math. Bull.* **43** (2000), 440–447.

16. M. Okumura, Some remarks on spaces with a certain contact structure, *Tôhoku Math. J.* **14** (1962), 135–145.

17. D. Perrone, Weakly ϕ-symmetric contact metric spaces, *Balkan J. Geom. Appl.* **7** (2002), 67–77.

18. T. Takahashi, Sasakian ϕ-symmetric spaces, *Tôhoku Math. J.* **29** (1977), 91–113.

19. S. Tanno, Locally symmetric K-contact Riemannian manifolds, *Proc. Japan Acad.* **43** (1968), 581–583.

20. F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lecture Notes Series No. 83 (Cambridge University Press, 1983.)