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TENSOR PRODUCT REPRESENTATION OF THE (PRE)DUAL OF THE L^p-SPACE OF A VECTOR MEASURE

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Abstract

The duality properties of the integration map associated with a vector measure m are used to obtain a representation of the (pre)dual space of the space $L^p(m)$ of p-integrable functions (where $1 < p < \infty$) with respect to the measure m. For this, we provide suitable topologies for the tensor product of the space of q-integrable functions with respect to m (where p and q are conjugate real numbers) and the dual of the Banach space where m takes its values. Our main result asserts that under the assumption of compactness of the unit ball with respect to a particular topology, the space $L^p(m)$ can be written as the dual of a suitable normed space.

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1. Introduction

Let (Ω, Σ) be a measurable space and X be a (real) Banach space. Let $m : \Sigma \to X$ be a countably additive vector measure and let $p, q > 1$ be conjugate. The aim of this paper is to provide a general representation of the space $L^p(m)$ of real p-integrable functions with respect to m as the dual space of a certain topological tensor product. Such a representation has already been obtained in [12]. However, the results obtained there give only a partial answer to a general representation problem since they are only valid under certain restrictions on the measure m, namely positivity, and on the space $L^p(m)$ that can be difficult to describe (see [11, Section 3]). In this paper, we provide a general tensor product representation of the predual of these spaces.

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The motivation of this paper is that the space of p-integrable functions with respect to a vector measure provides a general representation technique for p-convex Banach lattices: each order continuous p-convex Banach lattice with a weak order unit is (order and topologically) isomorphic to a space $L^p(m)$ for some vector measure m (see [5, Proposition 2.4]). Consequently, the results of this paper lead to representations of the (pre)dual spaces of general Banach lattices; we show some concrete examples at the end of the paper.

It seems natural to represent the dual space of $L^p(m)$ in terms of the space $L^q(m)$, as in the case of classical L^p-spaces. However, two facts suggest that this representation cannot be direct. The first is that it is well known that the dual of $L^p(m)$ coincides with $L^q(m)$ only in the trivial cases (that is, when $L^p(m)$ is isomorphic to $L^p(\mu)$ of a scalar measure μ). Furthermore, $L^p(m)$ could be a weighted c_0-space, hence reflexivity cannot be expected in general for these spaces. From the technical point of view, the natural weak topology associated with the integration map, the so-called m-weak topology, is the keystone of our arguments. For the case $p = 1$, a representation of the elements of the dual space of $L^1(m)$ has been given in [10].

2. Preliminaries

We use standard Banach space and vector measure notation. If X is a (real) Banach space, its unit ball is denoted by $B(X)$. We write X' for its dual space. The space of continuous linear operators from the Banach space Y into the Banach space X is denoted by $L(Y, X)$. We also need some basic facts regarding weak topologies on locally convex spaces. If X is a linear space, recall that a linear functional $\varphi : X \to \mathbb{R}$ is continuous with respect to a topology generated by a fundamental system of seminorms $\{p_\iota \mid \iota \in I\}$ in X if and only if there is a finite family $\{\iota_1, \ldots, \iota_n\} \subset I$ such that $|\varphi(x)| \le \sum_{j=1}^n p_{\iota_j}(x)$ for all x in X. This remark holds even if the family of seminorms does not separate points.

Let (Ω, Σ) be a measurable space and X be a Banach space. We write χ_A for the characteristic function of a set A in Σ. If $m : \Sigma \to X$ is a countably additive vector measure, we write $\|m\|$ for its semivariation and $|m|$ for its variation (see [4, Ch. I]). A measurable (real) function f is integrable with respect to m (m-integrable) if:

(1) it is scalarly integrable, that is, it is integrable with respect to each scalar measure $\langle m, x'\rangle$ defined by $\langle m, x'\rangle(A) := \langle m(A), x'\rangle$ for all $A \in \Sigma$, where $x' \in X'$; and

(2) for every $A \in \Sigma$ there is an element $\int_A f\, dm \in X$ such that

$$\left\langle \int_A f\, dm, x' \right\rangle = \int_A f\, d\langle m, x'\rangle$$

(see [7] for the equivalence of the definition in [1] to that given here).

Let $1 \le p < \infty$. The space $L^p(m)$ of p-integrable functions with respect to m (that is, equivalence classes of measurable functions f which differ on a set of null m-semivariation such that $|f|^p$ is m-integrable) is well known; it has been studied

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in [5, 11, 12]. This space endowed with the almost everywhere order and with the norm given by

$$\|f\|_{L^p(m)} := \sup\left\{\left(\int_\Omega |f|^p \, d|\langle m, x'\rangle|\right)^{1/p} \,\middle|\, x' \in B(X')\right\}$$

for all $f \in L^p(m)$, where $|\langle m, x'\rangle|$ denotes the variation of the scalar measure $\langle m, x'\rangle$, is an order continuous Banach lattice. If $x' \in X'$ and $|\langle m, x'\rangle|$ is a Rybakov measure for m, that is, a scalar control measure for m defined by an element of X' (see [4, Ch. IX]), then $(L^p(m), \|\cdot\|_{L^p(m)})$ is an order continuous Banach function space over $(\Omega, \Sigma, |\langle m, x'\rangle|)$ with weak unit χ_Ω (see 'Köthe function space' in [8, p. 28] for the definition). The relation between the spaces $L^p(m)$ and $L^q(m)$ has been analyzed in [11].

Since $L^p(m) \cdot L^q(m) \subseteq L^1(m)$ (see [11]), it can be proved that each function $g \in L^q(m)$ can be identified with the operator $I_g : L^p(m) \to X$ given by $I_g(f) := \int_\Omega fg \, dm$ (see [5, Proposition 3.1] and [11, Section 3]).

Throughout the paper, $m : \Sigma \to X$ is a countably additive vector measure and $1 < p < \infty$. Such a measure m is *scalarly dominated* by a measure $\widetilde{m} : \Sigma \to X$ if there exists a positive constant K such that

$$|\langle m, x'\rangle|(A) \le K |\langle \widetilde{m}, x'\rangle|(A) \quad \forall A \in \Sigma \ \forall x' \in X'.$$

The following Radon–Nikodym theorem for scalarly dominated measures is given in [9, Theorem 1] and provides an important tool for our work. We give an adapted version in the following lemma.

LEMMA 2.1. *Let m and $\tilde{m}$ be vector measures with range in a Banach space X. The following assertions are equivalent.*

(i) *There exists a bounded measurable function θ such that*

$$m(E) = \int_E \theta \, d\tilde{m} \quad \forall E \in \Sigma.$$

(ii) *m is scalarly dominated by $\tilde{m}$.*

In what follows we characterize the continuous linear operators $G : L^p(m) \to X$ that can be identified with (integration operators defined by) functions of $L^q(m)$ in terms of a domination property. This property is related to the definition of the class of uniformly scalarly integral operators in [12, Definition 3], in which the set of integration operators I_g is always contained.

THEOREM 2.2. *The following assertions are equivalent for an operator $G : L^p(m) \to X$.*

(i) *There is a function $g \in L^q(m)$ such that $G = I_g$, that is, $G(f) = \int fg \, dm$ for every $f \in L^p(m)$.*

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(ii) *There are functions* $g_1, \ldots, g_n$ *in* $L^q(m)$ *such that for all* $x' \in X'$,

$$|\langle G(f), x'\rangle| \leq \sum_{i=1}^{n} \left| \left\langle \int f g_i \, dm, x' \right\rangle \right| \quad \forall f \in L^p(m).$$

(iii) *There is a function* g_0 *in* $L^q(m)$ *such that for all* $x' \in X'$,

$$|\langle G(f), x'\rangle| \leq \int |f g_0| \, d|\langle m, x'\rangle| \quad \forall f \in L^p(m).$$

Moreover, the subspace of all the operators G *of* $L(L^p(m), X)$ *that satisfy* (i), (ii) *or* (iii) *is isometrically isomorphic to* $L^q(m)$.

PROOF. By the representation of the operator G of $L(L^p(m), X)$ as an integral, it is obvious that (i) implies (ii).

The proof that (ii) implies (iii) is a direct consequence of the following inequalities. Let $G : L^p(m) \to X$ be an operator satisfying (ii). For all x' in X' and f in $L^p(m)$,

$$\begin{aligned} |\langle G(f), x'\rangle| &\leq \sum_{i=1}^{n} \left| \left\langle \int f g_i \, dm, x' \right\rangle \right| \leq \sum_{i=1}^{n} \int |f g_i| \, d|\langle m, x'\rangle| \\ &= \int \left(\sum_{i=1}^{n} |g_i| \right) |f| \, d|\langle m, x'\rangle|. \end{aligned}$$

Since $\sum_{i=1}^{n} |g_i| \in L^q(m)$, we obtain (iii).

To prove that (iii) implies (i), suppose that G satisfies (iii) and define the set function $m_G : \Sigma \to X$ by

$$m_G(A) := G(\chi_A) \quad \forall A \in \Sigma.$$

It is easy to show that m_G is a countably additive vector measure, since $L^p(m)$ is order continuous. Let us define the measure $m_1 : \Sigma \to X$ by $m_1(A) := \int_A g_0 \, dm$ for all $A \in \Sigma$. It clearly satisfies the inequality

$$|\langle G(f), x'\rangle| \leq \int |f| \, d|\langle m_1, x'\rangle| \quad \forall f \in L^p(m).$$

Take a set $A \in \Sigma$; then

$$|\langle m_G(A), x'\rangle| = |\langle G(\chi_A), x'\rangle| \leq \int \chi_A \, d|\langle m_1, x'\rangle| = |\langle m_1, x'\rangle|(A).$$

Hence, m_G is scalarly dominated by m_1. By [9, Theorem 1], there is a bounded measurable function θ such that

$$m_G(A) = G(\chi_A) = \int_A \theta \, dm_1 = \int_A \theta g_0 \, dm$$

for each $A \in \Sigma$. Note that the product θg_0 is also in $L^q(m)$. If $I_{\theta g_0}$ is the integration operator from $L^p(m)$ into X defined by $I_{\theta g_0}(f) = \int f \theta g_0 \, dm$, then $I_{\theta g_0}$

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and G coincide on the set of simple functions. Since this set is dense in $L^p(m)$ we obtain $G(f) = I_{\theta g_0}(f)$ for all f in $L^p(m)$ which gives (i) for $g = \theta g_0$. Finally, the isometry is a consequence of [11, Proposition 8]: for every $g \in L^q(m)$, we have $\|I_g\| = \|g\|_{L^q(m)}$. □

3. Tensor product representations for $L^q(m)$

In this section we provide a representation technique for spaces $L^q(m)$ based on topological tensor products. A first step towards this is in [12], where such an identification is found, although under strong restrictions on the spaces $L^q(m)$; the topologies introduced there are different to those considered here, which lead to a general representation for *any* $L^q(m)$. Indeed we will prove that $L^q(m)$ is always the dual space of a certain topological tensor product. The main tool that we use is Theorem 2.2.

From the technical point of view, it is necessary to define several tensor product topologies. Let us first introduce a *weak* topology for the space $L^q(m)$. For any function $f \in L^p(m)$ and $x' \in X'$, the expression

$$p_{f,x'}(g) := \left|\left\langle \int fg\, dm,\, x' \right\rangle\right|, \quad \forall g \in L^q(m),$$

defines a seminorm on $L^q(m)$. We call the topology generated by this family of seminorms in $L^q(m)$ the m-weak topology, and we denote it by γ. Since the linear functional $f \mapsto \langle \int fg\, dm, x' \rangle$ satisfies

$$\left|\left\langle \int fg\, dm,\, x' \right\rangle\right| \le \|f\|_{L^p(m)} \|g\|_{L^q(m)} \|x'\|,$$

the topology γ is weaker than the weak topology of $L^q(m)$; it has been already studied in [6, 11]. Notice that Theorem 2.2 can be written in terms of the continuity of an operator G with respect to this topology.

We set up the topological framework for the tensor product $L^p(m) \otimes X'$. For $g \in L^q(m)$, we define the seminorm p_g by

$$p_g(z) := \left|\sum_{i=1}^{n} \left\langle \int f_i g\, dm,\, x_i' \right\rangle\right| \quad \text{where } z = \sum_{i=1}^{n} f_i \otimes x_i' \in L^p(m) \otimes X'.$$

The definition is independent of the particular representation of z. Using this family of seminorms we can provide a topology (in general not Hausdorff) on the tensor product $L^p(m) \otimes X'$. We will denote it by τ; it corresponds to the topology generated by the family of seminorms $\{p_g \mid g \in L^q(m)\}$.

Suppose that $g \in L^q(m)$ and define the associated integration map $I_g : L^p(m) \to X$ by $I_g(f) := \int fg\, dm$ for all f in $L^p(m)$. Define the functional

$$\varphi_g : L^p(m) \otimes X' \longrightarrow \mathbb{R}$$

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by $\varphi_g(z) := \sum_{i=1}^n \langle I_g(f_i), x_i' \rangle$, where $z = \sum_{i=1}^n f_i \otimes x_i'$ is any representation of the tensor z in $L^p(m) \otimes X'$; notice again that the definition does not depend on the particular representation of z. The following result shows that this relation provides a procedure to identify the set of q-integrable functions with respect to m with the dual space $(L^p(m) \otimes_\tau X')'$. As usual, we denote by τ_{weak^*} the weak topology generated on a dual space by the elements of the original space.

PROPOSITION 3.1. *The map* $\Upsilon : (L^q(m), \gamma) \to ((L^p(m) \otimes_\tau X')', \tau_{\text{weak}^*})$ *given by* $\Upsilon(g) := \varphi_g$, *is an isomorphism.*

PROOF. We start by proving that Υ is well defined and injective. Clearly, if $g \in L^q(m)$, then

$$|\varphi_g(z)| = \left| \sum_{i=1}^n \langle I_g(f_i), x_i' \rangle \right| = p_g(z)$$

for any tensor $z = \sum_{i=1}^n f_i \otimes x_i'$, and then φ_g belongs to $(L^p(m) \otimes_\tau X')'$. It is known that if $h \in L^q(m)$ and $h \neq g$, there are $f \in L^p(m)$ and $x' \in X'$ such that $\langle \int fh\, dm, x' \rangle \neq \langle \int fg\, dm, x' \rangle$ (see [11, 12]), hence the identification $g \mapsto \varphi_g$ given by Υ is injective. Notice that Υ is also linear.

To prove that the map is also surjective, consider a functional ϕ in $(L^p(m) \otimes_\tau X')'$. Since it is continuous with respect to τ, there are functions $g_1, \ldots g_n \in L^q(m)$ such that $|\phi(z)| \leq \sum_{i=1}^n p_{g_i}(z)$ for any tensor $z \in L^p(m) \otimes_\tau X'$. In particular, for a simple tensor $z = f \otimes x'$,

$$|\phi(z)| \leq \sum_{i=1}^n p_{g_i}(f \otimes x') = \sum_{i=1}^n \left| \left\langle \int f g_i\, dm, x' \right\rangle \right|. \tag{3.1}$$

Now fix a p-integrable function f and define the map $F_f : X' \longrightarrow \mathbb{R}$ by $F_f(x') := \phi(f \otimes x')$. Note that F_f is well defined; also, by (3.1),

$$|F_f(x')| = |\phi(f \otimes x')| \leq \sum_{i=1}^n \left| \left\langle \int f g_i\, dm, x' \right\rangle \right|$$

for every $x' \in X'$; since for all $i = 1, \ldots, n$, $\int f g_i\, dm \in X$, it follows that F_f is continuous with respect to the weak* topology of X'. Therefore F_f is an element of the dual space $(X', \tau_{\text{weak}^*})'$ that coincides with X.

Thus, we can define the operator $T_\phi : L^p(m) \to X$ by $T_\phi(f) := F_f$. Note that T_ϕ is linear and $\langle T_\phi(f), x' \rangle = \phi(f \otimes x')$ for all f in $L^p(m)$, and then

$$|\langle T_\phi(f), x' \rangle| = |\phi(f \otimes x')| \leq \sum_{i=1}^n \left| \left\langle \int f g_i\, dm, x' \right\rangle \right|.$$

Therefore, the operator T_ϕ satisfies the inequalities in (ii) of Theorem 2.2. Thus, there is a function g_0 in $L^q(m)$ such that $T_\phi(f) = \int g_0 f\, dm$ for all f in $L^p(m)$. Hence, $\varphi_{g_0}(f \otimes x') = \langle \int g_0 f\, dm, x' \rangle = \phi(f \otimes x')$ for every simple tensor in $L^p(m) \otimes X'$,

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which implies that $\varphi_{g_0} = \phi$, and then Υ is surjective. The fact that Υ is a topological isomorphism is clear because of the definitions of the topologies γ and τ_{weak^*}; the action of the tensors of $L^p(m) \otimes_\tau X'$ on the functionals of its dual space is given by evaluations of a finite set of the functionals that define the topology γ. $\square$

Although Proposition 3.1 provides a representation of the space $L^p(m)$ as the dual of a certain topological linear space, this space is not in general Hausdorff. The following trivial example shows this. Consider the Lebesgue measure space (Ω, Σ, μ) and the vector measure $m_0 : \Sigma \to \ell^2$ given by $m_0(A) := \mu(A)e_1$ for all $A \in \Sigma$, where e_1 is the first element of $\{e_i \mid i \in \mathbb{N}\}$, the canonical basis of ℓ^2. Clearly, if we consider a simple tensor $f \otimes e_i$, where $f \in L^p(m)$ and $i > 1$, then we obtain $p_g(f \otimes e_i) = 0$ for every $g \in L^q(m)$. The same argument can be used for any vector measure m to show that for every $x' \in X'$ that satisfies $\langle \int h\, dm, x' \rangle = 0$ for every $h \in L^1(m)$ and $f \in L^p(m)$, the equality $p_g(f \otimes x') = 0$ holds, and then the induced topology cannot be Hausdorff.

The rest of the paper is devoted to improving the representation of $L^q(m)$ as a dual of a Hausdorff topological vector space. The first step is to construct a (Hausdorff) quotient space preserving the duality properties with respect to $L^q(m)$. As usual, if $g \in L^q(m)$, we define the kernel of p_g as

$$\ker p_g = \{z \in L^p(m) \otimes X' \mid p_g(z) = 0\}.$$

The set $\bigcap_g \ker p_g$, where the intersection is taken over the set of functions g in $L^q(m)$, is a linear subspace of the tensor product. Consider the quotient space (defined algebraically) $(L^p(m) \otimes X')/(\bigcap_g \ker p_g)$. We define in this space the topology $\tilde{\tau}$ generated by the family of quotient seminorms $\{\widetilde{p_g} \mid g \in L^q(m)\}$, that are given by

$$\widetilde{p_g}([z]) := \inf_{v \in [z]} p_g(v) = \inf_{\sum_{i=1}^n f_i \otimes x_i' \in [z]} \left| \sum_{i=1}^n \left\langle \int f_i g\, dm, x_i' \right\rangle \right|,$$

where the elements of the equivalence classes $[z]$ of z are elements of the tensor product $L^p(m) \otimes X'$. Note that $\widetilde{p_g}([z])$ can be computed directly by

$$\widetilde{p_g}([z]) = \left| \sum_{i=1}^n \left\langle \int f_i g\, dm, x_i' \right\rangle \right|$$

for any $\sum_{i=1}^n f_i \otimes x_i' \in [z]$, since the quotient is defined using the family of seminorms $\{p_g\}$. The next result, together with Proposition 3.1, provides a representation of the space $L^q(m)$ as the dual space of a *Hausdorff* topological vector space.

PROPOSITION 3.2. *The map*

$$Q : ((L^p(m) \otimes_\tau X')', \tau_{\text{weak}^*}) \to \left(\left((L^p(m) \otimes X') \Big/ \left(\bigcap_g \ker p_g \right), \tilde{\tau} \right)', \tau_{\text{weak}^*} \right)$$

given by $Q(\phi) = \tilde{\phi}$, where $\tilde{\phi}([z]) = \phi(z)$ for each tensor z in $L^p(m) \otimes X'$, is a linear isomorphism.

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PROOF. We give the straightforward proof for the sake of completeness. Let ϕ be a functional in the dual space $(L^p(m) \otimes_\tau X')'$. By the continuity of ϕ with respect to τ, for every tensor z in $L^p(m) \otimes X'$ there are n q-integrable functions $g_1, \ldots, g_n$ such that

$$|\phi(z)| \leq \sum_{i=1}^{n} p_{g_i}(z). \tag{3.2}$$

Define the linear map $\tilde{\phi}$ from $(L^p(m) \otimes X')/(\bigcap_g \ker p_g)$ into $\mathbb{R}$ by $\tilde{\phi}([z]) := \phi(z)$, for all $z \in [z] \in (L^p(m) \otimes X')/(\bigcap_g \ker p_g)$. If $[z_1] = [z_2]$, then $p_g(z_1 - z_2) = 0$ for each $g \in L^q(m)$, and so $\phi(z_1 - z_2) = 0$ by (3.2). Thus $\tilde{\phi}([z_1]) = \tilde{\phi}([z_2])$ by the linearity of ϕ. Obviously $\tilde{\phi}$ is continuous with respect to the topology $\tilde{\tau}$, since $|\tilde{\phi}([z])| \leq \sum_{i=1}^{n} \tilde{p}_{g_i}([z])$ for every $[z]$ by (3.2). Therefore $Q(\phi) := \tilde{\phi}$ is well defined and injective.

To see that it is also surjective, consider a ($\tilde{\tau}$-continuous) functional $\tilde{\phi} : (L^p(m) \otimes X')/(\bigcap_g \ker p_g) \to \mathbb{R}$ and define the map $\phi : L^p(m) \otimes X' \to \mathbb{R}$ by $\phi(z) := \tilde{\phi}([z])$. Direct computations like those in the previous part of the proof show that ϕ belongs to the space $(L^p(m) \otimes X', \tau)'$; clearly $Q(\phi) = \tilde{\phi}$. By the arguments used above, the equivalence between the weak* topologies of both spaces is also clear. □

In what follows we introduce the uniform topology associated with τ in the tensor product $L^p(m) \otimes X'$ in order to find a representation of $L^q(m)$ as the dual space of a *normed* space. We denote by τ_u the topology generated by the seminorm

$$u(z) = \sup_{\|g\|_{L^q(m)} \leq 1} \left| \sum_{i=1}^{n} \left\langle \int f_i g \, dm, x_i' \right\rangle \right|,$$

where $z = \sum_{i=1}^{n} f_i \otimes x_i'$ is an element of $L^p(m) \otimes X'$. For a functional ϕ in $(L^p(m) \otimes X', \tau_u)'$, we define

$$\|\phi\|_u := \sup |\phi(z)|,$$

where the supremum is computed over all tensors $z \in L^p(m) \otimes X'$ satisfying $u(z) \leq 1$.

Clearly $\ker u = \bigcap_g \ker p_g$, where the intersection is defined over all the integrable functions in $L^q(m)$; as in the previous case, we will deal with the quotient space $(L^p(m) \otimes X')/\ker u$. In this case, we also define the quotient topology $\tau_{\tilde{u}}$ generated by the seminorm $\tilde{u}([z]) := u(z)$, for $z \in L^p(m) \otimes X'$. The corresponding norm on the dual of the quotient space is given by

$$\|\tilde{\phi}\|_{\tilde{u}} := \sup_{\tilde{u}([z]) \leq 1} |\tilde{\phi}([z])| \quad \forall \tilde{\phi} \in ((L^p(m) \otimes X')/\ker u, \tau_{\tilde{u}})',$$

where the elements $[z]$ belong to $(L^p(m) \otimes X')/\ker u$.

We omit the proof of the next proposition, which follows along the lines of the proof of Proposition 3.2.

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PROPOSITION 3.3. *The function*

$$Q_u : ((L^p(m) \otimes_{\tau_u} X')', \|\cdot\|_u) \longrightarrow ((L^p(m) \otimes X'/\ker u, \tau_{\tilde{u}})', \|\cdot\|_{\tilde{u}}),$$

defined by $Q_u(\phi) = \tilde{\phi}$, *where* $\tilde{\phi}([z]) = \phi(z)$ *for each tensor* z *in* $L^p(m) \otimes X'$, *is an isometric isomorphism.*

We give an easy example of the representation procedure developed here.

EXAMPLE 1. Let $1 < r, p < \infty$, and s, q be the corresponding conjugate exponents, and let $([0, 1], \Sigma, \mu)$ be the Lebesgue measure space. We define the vector measure $m : \Sigma \to L^r(\mu)$ as $m(A) := \chi_A$, for all $A \in \Sigma$. It is easy to see that $L^p(m) = L^{pr}(\mu)$, while $L^q(m) = L^{qr}(\mu)$, and $(L^r(\mu))' = L^s(\mu)$. Notice also that for every function k in $L^1(m)$, $\int k \, dm = k$. Take a tensor $z = \sum_{i=1}^n f_i \otimes h_i$ in $L^p(m) \otimes (L^r(\mu))' = L^{pr}(\mu) \otimes L^s(\mu)$. Then

$$\begin{aligned} u(z) &= \sup_{g \in B(L^q(m))} \left| \sum_{i=1}^n \left\langle \int f_i g \, dm, h_i \right\rangle \right| \\ &= \sup_{g \in B(L^{qr}(\mu))} \left| \sum_{i=1}^n \int \left(\int f_i g \, dm \right) h_i \, d\mu \right| \\ &= \sup_{g \in B(L^{qr}(\mu))} \left| \int g \sum_{i=1}^n f_i h_i \, d\mu \right|. \end{aligned}$$

Since

$$\frac{1}{pr} + \frac{1}{s} = \left(1 - \frac{1}{q}\right)\frac{1}{r} + \frac{1}{s} = 1 - \frac{1}{qr},$$

$\sum_{i=1}^n f_i h_i \in (L^{qr}(\mu))'$ and $u(z) = \|\sum_{i=1}^n f_i h_i\|_{(L^{qr}(\mu))'}$. Observe that

$$\ker u = \left\{ z = \sum_{i=1}^n f_i \otimes h_i \in L^p(m) \otimes L^s(\mu) : \sum_{i=1}^n f_i h_i = 0 \ \mu\text{-almost everywhere} \right\}.$$

Therefore, the space $((L^p(m) \otimes L^s(\mu))/\ker u, \tau_u)$ can be identified isometrically with $(L^q(m))' = L^t(\mu)$, where $1/qr + 1/t = 1$, and the formulae for u provide an equivalent representation of the norm of $L^t(\mu)$.

The following theorem is the main result of this paper. We show that the key to obtaining a satisfactory generalization of the duality results for classical L^p-spaces (scalar measure) is a certain compactness assumption for the unit ball of $L^q(m)$. The theorem gives a description of a suitable normed predual of the space $L^p(m)$, and consequently of the dual space $(L^p(m))'$.

THEOREM 3.4. *The two spaces* $(((L^p(m) \otimes X')/\ker u, \tau_{\tilde{u}})', \|\cdot\|_{\tilde{u}})$ *and* $(L^q(m), \|\cdot\|_{L^q(m)})$ *are isometrically isomorphic if and only if the unit ball of* $L^q(m)$ *is* m*-weakly compact.*

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PROOF. We start by showing the direct implication. If $L^q(m)$ is the topological dual of $((L^p(m) \otimes X')/\ker u, \tau_{\tilde{u}})$ then this space defines the weak* topology on bounded sets of $L^q(m)$. By Alaoglu's theorem the unit ball of $L^q(m)$ is weakly* compact; since the weak* topology coincides with the m-weak topology of $L^q(m)$ on its unit ball, $B_{L^q(m)}$ is m-weakly compact.

To prove the converse, first notice that the function space $L^q(m)$ can be identified with a subspace of $((L^p(m) \otimes X')/\ker u, \tau_{\tilde{u}})'$ that coincides with $(L^p(m) \otimes X', \tau_u)'$ by Proposition 3.3. The inclusion is given by the identification explained before Proposition 3.1, that is, by the map

$$i : L^q(m) \to (L^p(m) \otimes X', \tau_{\tilde{u}})',$$

where $i(g) := \varphi_g$ for g in $L^q(m)$ with

$$\varphi_g\bigg(\sum_{i=1}^n f_i \otimes x_i'\bigg) := \sum_{i=1}^n \bigg\langle \int f_i g \, dm, x_i' \bigg\rangle.$$

Clearly i is well defined, and direct computations show that it is continuous. To prove that it is an isomorphism, we take $\tilde{\phi} \in ((L^p(m) \otimes X')/\ker u, \tau_{\tilde{u}})'$ and we must prove that $\tilde{\phi}$ belongs to $((L^p(m) \otimes X')/\ker u, \tilde{\tau})'$, which can be identified with $L^q(m)$ by Propositions 3.1 and 3.2; our aim is to show that any functional $\tilde{\phi} : (L^p(m) \otimes X')/\ker u \to \mathbb{R}$ that is continuous with respect to $\tau_{\tilde{u}}$ is also continuous with respect to the topology $\tilde{\tau}$. Thus, we search for a function g_0 in $L^q(m)$ such that

$$|\tilde{\phi}([z])| \le \|\tilde{\phi}\|_{\tilde{u}} \cdot \widetilde{p_{g_0}}([z])$$

for every $[z] \in (L^p(m) \otimes X')/\ker u$. In order to find this element it is necessary to use a separation argument; we choose one based on Ky Fan's lemma (see [3, p. 491]). For a fixed $z = \sum_{i=1}^n f_i \otimes x_i'$ in $L^p(m) \otimes X'$, we define the function $\Phi_z : B(L^q(m)) \to \mathbb{R}$ as follows:

$$\begin{aligned}\Phi_z(g) &:= \tilde{\phi}([z]) - \|\tilde{\phi}\|_{\tilde{u}} \varphi_g(z) \\ &= \sum_{i=1}^n \phi(f_i \otimes x_i') - \|\tilde{\phi}\|_{\tilde{u}} \bigg(\sum_{i=1}^n \bigg\langle \int f_i g \, dm, x_i' \bigg\rangle\bigg),\end{aligned}$$

where ϕ is a functional satisfying $Q(\phi) = \tilde{\phi}$ provided by Proposition 3.2, and $\sum_{i=1}^n f_i \otimes x_i'$ is any representation of z (note that the definition of the function Φ_z is independent of the particular representations of $[z]$ and of $\tilde{\phi}$). Thus let $\mathcal{F}$ be the family of functions Φ_z for z in $L^p(m) \otimes X'$. We need to prove that $\mathcal{F}$ satisfies all the hypotheses of Ky Fan's lemma.

All the functions Φ_z are defined on the unit ball of $L^q(m)$, which is compact with respect to the m-weak topology by assumption. Remark that the space $L^q(m)$ with the m-weak topology is a Hausdorff space.

The family of functions $\mathcal{F}$ is concave; if z_1 and z_2 are in $L^p(m) \otimes X'$ then for any α in [0, 1], there is an element z_0 in $L^p(m) \otimes X'$ such that $\alpha\Phi_{z_1} + (1-\alpha)\Phi_{z_2} = \Phi_{z_0}$; take $z_0 = \alpha z_1 + (1-\alpha) z_2$.

Let us now show that for every tensor z in $L^p(m) \otimes X'$, the function Φ_z is convex. By the linearity of Φ it is enough to prove that this holds for a simple tensor $z = f \otimes x'$. Suppose that $g_1, g_2 \in B_{L^q(m)}$ and $\alpha \in [0, 1]$. Then

$$\begin{aligned}\Phi_z(\alpha g_1 + (1-\alpha)g_2) &= \phi(f \otimes x') - \|\tilde{\phi}\|_{\tilde{u}}\Big\langle \int f(\alpha g_1 + (1-\alpha)g_2)\, dm, x' \Big\rangle \\ &= \alpha\Phi_z(g_1) + (1-\alpha)\Phi_z(g_2).\end{aligned}$$

Moreover, by construction, Φ_z is continuous with respect to the m-weak topology of $L^p(m)$ for all z in $L^p(m) \otimes X'$.

Finally, we must prove that for all z in the tensor product $L^p(m) \otimes X'$, there is a function g_z in the unit ball of $L^q(m)$ such that $\Phi_z(g_z) \le 0$; this is a consequence of the fact that Φ_z is a continuous function defined on a compact set. Indeed,

$$\begin{aligned}\phi(z) := \tilde{\phi}([z]) \le |\tilde{\phi}([z])| &\le \|\tilde{\phi}\|_{\tilde{u}} \cdot \tilde{u}([z]) \\ &= \|\tilde{\phi}\|_{\tilde{u}} \cdot \sup_{\|g\|_{L^q(m)} \le 1} \bigg(\Big\langle \sum_{i=1}^{n} f_i g\, dm, x'_i \Big\rangle \bigg)\end{aligned}$$

and this supremum is attained for some g_z in the unit ball of $L^q(m)$. Then for all z in $L^p(m) \otimes X'$, there exists g_z in $B_{L^q(m)}$ such that

$$\tilde{\phi}\bigg(\sum_{i=1}^{n} f_i \otimes x'_i\bigg) \le \|\tilde{\phi}\|_{\tilde{u}} \bigg(\sum_{i=1}^{n} \Big\langle \int f_i g_z\, dm, x'_i \Big\rangle\bigg).$$

By Ky Fan's lemma, we conclude that there is g_0 in $B(L^q(m))$ such that for all $z = \sum_{i=1}^{n} f_i \otimes x'_i \in L^p(m) \otimes X'$,

$$\tilde{\phi}\bigg(\sum_{i=1}^{n} f_i \otimes x'_i\bigg) \le \|\tilde{\phi}\|_{\tilde{u}} \bigg(\sum_{i=1}^{n} \Big\langle \int f_i g_0\, dm, x'_i \Big\rangle\bigg). \tag{3.3}$$

Thus $\tilde{\phi}$ is continuous with respect to $\tilde{\tau}$ and so $\tilde{\phi} \in ((L^p(m) \otimes X')/\ker u, \tilde{\tau})'$; the identification is clearly bijective, since this space is isomorphic to $L^q(m)$ by Propositions 3.1 and 3.2. A direct computation using inequality (3.3) shows that the function $g'_0 := g_0\|\tilde{\phi}\|_{\tilde{u}} \in L^q(m)$ can be identified with $\tilde{\phi}$ and clearly $\|g'_0\|_{L^q(m)} \le \|\tilde{\phi}\|_{\tilde{u}}$. The converse inequality follows by a simple calculation: if $\tilde{\phi}_{g'_0}$ and $\phi_{g'_0}$ are the functionals defined by g'_0 and $z = \sum_{i=1}^{n} f_i \otimes x'_i \in L^p(m) \otimes X'$, then

$$\begin{aligned}|\tilde{\phi}([z])| &= |\tilde{\phi}_{g'_0}([z])| = |\phi_{g'_0}(z)| \\ &= \bigg| \sum_{i=1}^{n} \Big\langle \int f_i \|g'_0\|_{L^q(m)} \frac{g'_0}{\|g'_0\|_{L^q(m)}}\, dm, x'_i \Big\rangle \bigg|\end{aligned}$$

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$$\begin{aligned} &\leq \|g_0'\|_{L^q(m)} \cdot \sup_{\|h\|_{L^q(m)} \leq 1} \left| \sum_{i=1}^{n} \left\langle \int f_i h \, dm, x_i' \right\rangle \right| \\ &= \|g_0'\|_{L^q(m)} \cdot u(z) = \|g_0'\|_{L^q(m)} \cdot \tilde{u}([z]). \end{aligned}$$

This proves the isometry and completes the proof. □

Since the m-weak topology is weaker than the weak topology of the space $L^p(m)$, the compactness property required in Theorem 3.4 is satisfied if the space $L^q(m)$ is *reflexive*; some results regarding reflexivity of this space may be found in [5]. In fact, from [6], it is known that the space $L^q(m)$ is reflexive if and only if its unit ball is compact for the m-weak topology.

In the following corollary, we isolate a result on the duality of the space $L^q(m)$ that was implicitly shown in the proof of Theorem 3.4; in particular, this theorem gives a sufficient and necessary condition for the topological duals of the space $(L^p(m) \otimes X')/\ker u$ with the topologies $\tilde{\tau}$ and $\tau_{\tilde{u}}$ to coincide. This assertion is the natural 'vector measure' version of one of the main results of the duality theory of Banach spaces: the dual of a Banach space with the norm topology coincides with the dual of the space with the weak topology.

COROLLARY 3.5. *The following assertions are equivalent.*

(i) *The unit ball of $L^q(m)$ is compact with respect to the m-weak topology.*
(ii) $((L^p(m) \otimes X')/\ker u, \tilde{\tau})' = ((L^p(m) \otimes X')/\ker u, \tau_{\tilde{u}})'$.
(iii) $(L^p(m) \otimes X', \tau)' = (L^p(m) \otimes X', \tau_u)'$.

Let us finish by illustrating our procedure with two examples. In the first, we obtain an alternative formula to define the norm in the dual of $L^q(m)$ of a vector measure m over an Orlicz space. In the second, we characterize the dual of $L^q(m_V)$ of a measure m_V induced by a kernel operator.

EXAMPLE 2. Let (Ω, Σ, μ) be a measure space, and $L^0(\mu)$ be the space of (classes of μ-a.e. equal) measurable functions. Take a Young's function Φ with the Δ_2-property (see [2] for basic definitions about Orlicz spaces). We define the vector measure $m : \Sigma \to L^\Phi(\mu)$ by $m(A) = \chi_A$. Since $L^\Phi(\mu)$ is order continuous, the equality $L^1(m) = L^\Phi(\mu)$ holds, and then

$$\begin{aligned} L^p(m) &= \{f \in L^0(\mu) : |f|^p \in L^1(m)\} \\ &= \{f \in L^0(\mu) : |f|^p \in L^\Phi(\mu)\} \\ &= \{f \in L^0(\mu) : \Phi(|f|^p) \in L^1(\mu)\}. \end{aligned}$$

Notice that the function $\Phi \circ \pi : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\Phi \circ \pi(t) = \Phi(t^p)$ is a Young's function, and that the Δ_2-property for Φ implies the Δ_2-property for $\Phi \circ \pi$ since there exists b such that, for all $t > 0$,

$$\Phi \circ \pi(2t) = \Phi(2^p t^p) \leq b^{\lfloor p \rfloor + 1} \Phi(2^{p - \lfloor p \rfloor - 1} t^p) \leq b^{\lfloor p \rfloor + 1} \Phi \circ \pi(t),$$

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where $\lfloor p \rfloor = \max\{n \in \mathbb{Z} \mid n \leq p\}$. Therefore $L^p(m) = L^{\Phi\circ\pi}(\mu)$. Let Ψ be the conjugate Young's function of Φ. Since $L^\Phi(\mu)$ is order continuous, $(L^\Phi(\mu))' = L^\Psi(\mu)$. Take $z = \sum_{i=1}^n f_i \otimes h_i \in L^p(m) \otimes L^\Psi(\mu)$. Then

$$\begin{aligned} u(z) &= \sup_{g\in B(L^q(m))} \left| \sum_{i=1}^n \left\langle \int f_i g \, dm, h_i \right\rangle \right| \\ &= \sup_{g\in B(L^q(m))} \left| \int g \left(\sum_{i=1}^n \int f_i h_i \right) d\mu \right| \\ &= \left\| \sum_{i=1}^n f_i h_i \right\|^O_{(L^q(m))'}, \end{aligned}$$

where $(L^q(m))'$ is again an Orlicz space and $\|\cdot\|^O_{(L^q(m))'}$ is the corresponding Orlicz norm. Assume now that $L^q(m)$ is reflexive. Since a Banach space Z is reflexive if and only if Z' is reflexive (see [13, II.A.14]), it is a consequence of Theorem 3.4 that there is an isometric isomorphism between the spaces $(L^q(m), \|\cdot\|^O)'$ and $((L^p(m) \otimes L^\Psi(\mu))/\ker u, \tau_{\tilde{u}})$. Thus we can represent the elements of a dense subset the dual space of $L^q(m)$ as equivalence classes of elements $\sum_{i=1}^n f_i \otimes h_i \in L^p(m) \otimes L^\Psi(m)$.

EXAMPLE 3. Fix $1 < p < \infty$ and $1 < r < \infty$ and let q and v be their respective conjugate exponents. Let $([0, 1], \Sigma, \mu)$ be the Lebesgue measure space and $V : L^r(\mu) \to L^r(\mu)$ the kernel operator defined by

$$V(f)(t) := \int_0^t f(s) K(s, t) \, ds,$$

where $K : [0, 1] \times [0, 1] \to \mathbb{R}^+$ is a bounded integrable function. We define the vector measure $m_V : \Sigma \to L^r(\mu)$ by $m_V(A) := V(\chi_A)$. Notice that for $\phi = \sum_{i=1}^n a_i \chi_{A_i}$,

$$\int \phi \, dm_V = \sum_{i=1}^n a_i m_V(A_i) = V(\phi).$$

For $0 \leq f \in L^1(m)$, there is a sequence $(\phi_n) \in \mathcal{S}(\Sigma)$ such that $\phi_n \uparrow f$. By the order continuity of $L^1(m_V)$, $\phi_n \to f$ in $L^1(m_V)$, and then $\int \phi_n \, dm_V \to \int f \, dm_V$ in $L^r(\mu)$. There is a subsequence $(\Phi_{n_k})_k$ such that $0 \leq \phi_{n_k} \uparrow f$ and

$$\begin{aligned} \int f \, dm_V &= \lim_k \int \phi_{n_k} \, dm_V = \lim_k V(\phi_{n_k}) \\ &= \lim_k \int_0^t \phi_{n_k}(s) K(s, t) \, ds. \end{aligned}$$

Fix $t \in [0, 1]$. Now $0 \leq \phi_{n_k}(s) K(s, t) \uparrow f(s) K(s, t)$, since the kernel $K(s, t)$ is positive in its first variable. A direct application of the monotone convergence theorem

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yields the formula

$$\int_0^t f(s)K(s,t)\,ds = \lim_k \int_0^t \phi_{n_k}(s)K(s,t)\,ds.$$

Since every function $f \in L^1(m_V)$ may be written as a difference of positive functions, $\int f\,dm_V = V(f)$ for all $f \in L^1(m_V)$. From Fubini's theorem, for any representation of z as $\sum_{i=1}^n f_i \otimes h_i$ in $L^p(m) \otimes L^r(\mu)' = L^p(m_V) \otimes L^v(\mu)$,

$$\begin{aligned}
u(z) &= \sup_{g\in B(L^q(m_V))} \left| \sum_{i=1}^n \left\langle \int_0^1 f_i g\,dm_V, h_i \right\rangle \right| \\
&= \sup_{g\in B(L^q(m_V))} \left| \sum_{i=1}^n \int_0^1 \left(\int_0^1 f_i g\,dm_V \right) h_i\,d\mu \right| \\
&= \sup_{g\in B(L^q(m_V))} \left| \sum_{i=1}^n \int_0^1 V(f_i g) h_i\,d\mu \right| \\
&= \sup_{g\in B(L^q(m_V))} \left| \sum_{i=1}^n \int_0^1 \left(\int_0^t f_i(s)g(s)K(s,t)\,d\mu(s) \right) h_i(t)\,d\mu(t) \right| \\
&= \sup_{g\in B(L^q(m_V))} \left| \sum_{i=1}^n \int_0^1 \left(\int_0^1 f_i(s)g(s)K(s,t)\chi_{[0,t]}(s)\,d\mu(s) \right) h_i(t)\,d\mu(t) \right| \\
&= \sup_{g\in B(L^q(m_V))} \left| \sum_{i=1}^n \int_0^1 g(s) f_i(s) \left(\int_s^1 K(s,t)h_i(t)\,d\mu(t) \right) d\mu(s) \right| \\
&= \sup_{g\in B(L^q(m_V))} \left| \int_0^1 g(s) \sum_{i=1}^n \phi_i\,d\mu(s) \right| \\
&= \left\| \sum_{i=1}^n \phi_i \right\|_{L^q(m_V)'}
\end{aligned}$$

where $\phi_i(s) = f_i(s) \int_s^1 K(s,t)h_i(t)\,d\mu(t)$. Then we obtain a representation of a dense subset of the elements of the predual space $L^q(m_V)$ as equivalence classes of functions defined by means of elements of $L^p(m_V)$ and $L^v(\mu)$.

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