FLAT LAGUERRE PLANES OF KLEINEWILLINGHÖFER TYPE E OBTAINED BY CUT AND PASTE

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We provide examples of flat Laguerre planes of Kleinewillinghöfer type E, thus completing the classification of flat Laguerre planes with respect to Laguerre translations in B. Polster and G.F. Steinke, *Results Maths.* (2004). These planes are obtained by a method for constructing a new flat Laguerre plane from three given Laguerre planes devised in B. Polster and G. Steinke, *Canad. Math. Bull.* (1995) but no examples were given there.

1. INTRODUCTION

Kleinewillinghöfer [3] classified Laguerre planes with respect to *central automorphisms*, that is, permutations of the point set of the Laguerre plane such that generators are mapped to generators and circles are mapped to circles, and such that at least one point is fixed and central collineations are induced in the derived projective plane at one of the fixed points. In [10] flat Laguerre planes were considered and their so-called Kleinewillinghöfer types were investigated, that is, the Kleinewillinghöfer types with respect to the full automorphism group. In order to complete the determination of all possible types of flat Laguerre planes with respect to Laguerre translations we provide examples of such planes of Kleinewillinghöfer type E.

In type E one has precisely one tangent pencil of circles for which the group of Laguerre translations is linearly transitive. Hence the flat Laguerre plane has a distinguished point and tangent pencil of circles. The strategy is to use Laguerre planes of translation type, which admit all translations in all possible directions at the distinguished point, and paste different ones of these planes together so that in the end only translations in one direction remain; see [5] or the following section for a description of flat Laguerre planes of translation type.

There are, of course, many ways to combine different flat Laguerre planes to form a new flat Laguerre plane; see [6, 7, 8, 11, 12], or [9, Section 5.3] for a survey. In fact, many of the familiar Laguerre planes can be interpreted in this way. We are interested in one particular and promising method described in [6], which combines sets of circles of

up to three different flat Laguerre planes to circle sets of new Laguerre planes. However, no examples were given in [6] for different flat Laguerre planes satisfying the conditions of the construction.

In order to be more precise, let $\mathcal{L} = (Z, \mathcal{C})$ be a flat Laguerre plane and let $C_0 \in \mathcal{C}$. Consider the collection $\mathcal{C}^1$ of all circles that touch C_0. Note that $\mathcal{C}^1$ includes the circle C_0. Clearly, C_0 separates the cylinder Z into two connected components Z^+ and Z^-. We define $\mathcal{C}^\pm$ to be the collection of all circles that are completely contained in $Z^\pm$. Finally, let $\mathcal{C}^2$ be the set of all circles that intersect C_0 in precisely two points. Obviously, $\mathcal{C}^1 \cup \mathcal{C}^2 \cup \mathcal{C}^+ \cup \mathcal{C}^-$ is a partition of the circle set. With this notation the following result was proved in [6, Proposition 6].

PROPOSITION 1. *Let $\mathcal{L}_i = (Z, \mathcal{C}_i)$, $i = 1, 2, 3$, be three flat Laguerre planes. Suppose that $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3 \supseteq \mathcal{C}^1$ for some circle C_0. Let $\mathcal{C} := \mathcal{C}^1 \cup \mathcal{C}_1^2 \cup \mathcal{C}_2^+ \cup \mathcal{C}_3^-$. Then $\mathcal{L} = (Z, \mathcal{C})$ is a flat Laguerre plane.*

Note that the circles in $\mathcal{C}^2$ cover all of Z and their describing functions can be convex as well as concave so that the usual methods of construction cannot be applied. Furthermore, one can apply the construction in two steps, say first using only $\mathcal{L}_2$ and $\mathcal{L}_3$ (and replacing $\mathcal{L}_1$ by $\mathcal{L}_2$ in the construction) to obtain a flat Laguerre plane $\mathcal{L}'$, and in a second step using $\mathcal{L}_1$ and $\mathcal{L}'$ (thus replacing $\mathcal{L}_2$ and $\mathcal{L}_3$ by $\mathcal{L}'$ in the construction). In that respect it is a matter of debate whether two or three flat Laguerre planes are involved.

Certain semi-classical Laguerre planes pasted along a circle appear to be of this form; see [11]. (The planes $\mathcal{L}(\varphi, id)$ with one of the describing homeomorphisms being the identity.) Here $\mathcal{C}^1$, $\mathcal{C}^+$ and $\mathcal{C}^-$ are as in the classical flat Laguerre plane, but $\mathcal{C}^2$ is not. However, no other embedding of $\mathcal{C}^2$ except the one in the semi-classical planes is known so that these planes do not really qualify as examples for Proposition 1.

The aim of this paper is twofold. Firstly, to give examples of three different, in fact, non-isomorphic, flat Laguerre planes $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ satisfying the conditions in Proposition 1 so that the construction can be applied thus validating this paste-and-cut method. This is done in Section 2. The second goal is to obtain flat Laguerre planes of Kleinewillinghöfer type E with respect to Laguerre translations, see [4, 10] for Kleinewillinghöfer types. This is achieved in the last section by determining the Kleinewillinghöfer types of the flat Laguerre planes obtained previously.

2. THE EXAMPLES

A *flat Laguerre plane* $\mathcal{L} = (Z, \mathcal{C})$ is an incidence structure of points and circles whose point set is the cylinder $Z = \mathbb{S}^1 \times \mathbb{R}$ (where the 1-sphere $\mathbb{S}^1$ usually is represented as $\mathbb{R} \cup \{\infty\}$), whose circles $C \in \mathcal{C}$ are graphs of continuous functions $\mathbb{S}^1 \to \mathbb{R}$ such that any three points no two of which are on the same generator $\{c\} \times \mathbb{R}$ of the cylinder can be joined by a unique circle and such that the circles which touch a fixed circle K at $p \in K$

partition the complement in Z of the generator that contains p. For more information on flat Laguerre planes we refer to [**1**, **2**] or [**9**, Chapter 5]. Two points are said to be parallel if they are on the same generator.

The examples we are going to construct are based on the so-called Laguerre planes of translation type; see [**5**] or [**9**, Section 5.3.6]. In order to introduce this kind of planes we need the following definition.

A function f is called *strongly parabolic* if it satisfies the following conditions:

(a) f is differentiable and its derivative is a homeomorphism $\mathbb{R} \to \mathbb{R}$;

(b) f is normalised, that is, $f(0) = f'(0) = 0$ and $f(1) = 1$;

(c) f is twice differentiable on $\mathbb{R} \setminus \{0\}$;

(d) $\log |f'|$ is strictly concave on the open intervals $(-\infty, 0)$ and $(0, +\infty)$;

(e) $\displaystyle\lim_{x \to \pm\infty} \big(f(x+b)/f(x)\big) = 1$ for each $b \in \mathbb{R}$.

Let f and g be two strongly parabolic functions and for $a, b, c \in \mathbb{R}$ let

$$C_{a,b,c} = \begin{cases} \big\{(x, bx + c) \mid x \in \mathbb{R}\big\} \cup \big\{(\infty, 0)\big\} & \text{for } a = 0, \\[2mm] \big\{(x, af(x-b) + c) \mid x \in \mathbb{R}\big\} \cup \big\{(\infty, a)\big\} & \text{for } a > 0, \\[2mm] \big\{(x, ag(x-b) + c) \mid x \in \mathbb{R}\big\} \cup \big\{(\infty, a)\big\} & \text{for } a < 0. \end{cases}$$

Then

$$\mathcal{C}(f, g) = \{C_{a,b,c} \mid a, b, c \in \mathbb{R}\}$$

is the circle set of a flat Laguerre plane $\mathcal{L}(f, g)$ represented on the cylinder Z. We say that a flat Laguerre plane is of *translation type* if it is isomorphic to one of the planes $\mathcal{L}(f, g)$. For simplicity we abbreviate $\mathcal{L}(f, f)$ by $\mathcal{L}(f)$.

Note that a strongly parabolic function f is convex and attains its absolute minimum at $x = 0$. Hence the circles in $\mathcal{L}(f, g)$ that touch $C_{0,0,0}$ are precisely the circles $C_{0,0,c}$ for $c \in \mathbb{R}$ and $C_{a,b,0}$ for $a, b \in \mathbb{R}$, $a \neq 0$, that is,

$$\mathcal{C}^1(f, g) = \{C_{0,0,c} \mid c \in \mathbb{R}\} \cup \{C_{a,b,0} \mid a, b \in \mathbb{R}, a \neq 0\}.$$

Furthermore,

$$\mathcal{C}^2(f, g) = \{C_{a,b,c} \mid a, b, c \in \mathbb{R}, ac < 0\} \cup \{C_{0,b,c} \mid b, c \in \mathbb{R}, b \neq 0\}$$

and

$$\mathcal{C}^+(f, g) = \{C_{a,b,c} \mid a, b, c \in \mathbb{R}, a, c > 0\},$$
$$\mathcal{C}^-(f, g) = \{C_{a,b,c} \mid a, b, c \in \mathbb{R}, a, c < 0\}.$$

Our first goal is to find two Laguerre planes of translation type that share the set $\mathcal{C}^1$ for the circle $C_{0,0,0}$.

As a first step in that direction let h be an increasing homeomorphism of $\mathbb{R}$ that fixes 0. We obtain a homeomorphism $\tilde{h}$ of the cylinder Z by defining $\tilde{h}(x, y) = \big(x, h(y)\big)$ for $x \in \mathbb{R} \cup \{\infty\}, y \in \mathbb{R}$. Clearly, $\tilde{h}$ fixes the points on $C_{0,0,0}$. Of course, applying $\tilde{h}$ to $\mathcal{L}(f, g)$ we obtain a flat Laguerre plane $\mathcal{L}^h(f, g)$ which is isomorphic to the original Laguerre plane $\mathcal{L}(f, g)$ and which again contains the circle $C_{0,0,0}$. But then $\mathcal{L}^h(f, g)$ also contains the circles

$$C_{0,0,c}^h = \tilde{h}(C_{0,0,c}) = C_{0,0,h(c)}$$

for $c \in \mathbb{R}$. This shows that $\mathcal{L}(f, g)$ and $\mathcal{L}^h(f, g)$ share one touching pencil of circles. In order to obtain more common touching pencils, however, we have to take special kinds of functions f, g and h.

A special class of strongly parabolic functions are the *skew parabola functions*. These functions are of the form

$$f_{d,r}(x) = \begin{cases} x^d & \text{for } x \geqslant 0, \\ r|x|^d & \text{for } x \leqslant 0, \end{cases}$$

where $d > 1$ and $r > 0$. Note that the graph of the skew parabola function with $d = 2$, $r = 1$ is a Euclidean parabola and that $\mathcal{L}(f_{2,1})$ is just the *classical flat Laguerre plane*, that is, the geometry of all intersections of the cylinder with non-vertical planes in $\mathbb{R}^3$.

We start from a flat Laguerre plane $\mathcal{L}(f_{d,r})$ of translation type over a skew parabola function $f_{d,r}$ and let the homeomorphism $h = h_k$ where $k > 0$ be given by

$$h_k(x) = x|x|^{k-1}.$$

Note that h_k is multiplicative and that

$$h_k \circ f_{d,r} = f_{kd,h_k(r)}$$

is again a skew parabola function provided that $kd > 1$. In this case, the circles of $\mathcal{L}^{h_k}(f_{d,r})$ are

$$C_{a,b,c}^{h_k} = \begin{cases} \Big\{ (x, h_k(bx + c)) \mid x \in \mathbb{R} \Big\} \cup \Big\{ (\infty, 0) \Big\} & \text{for } a = 0, \\ \Big\{ \big(x, h_k(a)h_k\big(f_{d,r}(x - b) + c\big)\big) \mid x \in \mathbb{R} \Big\} \cup \Big\{ (\infty, h_k(a)) \Big\} & \text{for } a \neq 0, \end{cases}$$

where $a, b, c \in \mathbb{R}$. In particular,

$$C_{0,0,c}^{h_k} = \overline{C}_{0,0,h_k(c)} = \Big\{ (x, h_k(c)) \mid x \in \mathbb{R} \Big\} \cup \Big\{ (\infty, 0) \Big\}$$
$$C_{a,b,0}^{h_k} = \overline{C}_{a,b,0} \quad = \Big\{ (x, h_k(a)f_{kd,h_k(r)}(x - b)) \mid x \in \mathbb{R} \Big\} \cup \Big\{ (\infty, h_k(a)) \Big\}$$

where the bar refers to the circles of $\mathcal{L}(h_k \circ f_{d,r}) = \mathcal{L}(f_{kd,h_k(r)})$. Hence we proved the following result.

LEMMA 2. *The flat Laguerre planes $\mathcal{L}(f_{kd,h_k(r)})$ and $\mathcal{L}^{h_k}(f_{d,r})$ of translation type where $k,d,r > 0$ and $d, kd > 1$ have in common the set $\mathcal{C}^1$ of all circles touching the circle $C_0 = \mathbb{S}^1 \times \{0\}$.*

In order to obtain a situation as in Proposition 1 all we have to do is to choose k_i, d_i, r_i, $i = 1, 2, 3$, such that $k_i d_i = d_1$ and $h_{k_i}(r_i) = r_1$. This is easily achieved: Let $d_1, d_2, d_3 > 1$ and $r_1 > 0$ and define $k_i = d_1/d_i$ and $r_i = r_1^{k_i}$ for $i = 1, 2, 3$. We then use the skew parabola functions $f_i = f_{d_i, r_i}$ and the multiplicative homeomorphisms h_{k_i} for $i = 1, 2, 3$. Then the flat Laguerre planes $\mathcal{L}^{h_{k_i}}(f_i)$ for $i = 1, 2, 3$ have the set $\mathcal{C}^1$ of circles in common.

Applying Proposition 1 we thus obtain the following Laguerre planes.

PROPOSITION 3. *Let $d_1, d_2, d_3 > 1$ and $r_1 > 0$ and let $\mathcal{C}(d_1, r_1; d_2, d_3)$ be the collection of all sets of the form*

$$
C_{a,b,c} = \begin{cases}
\big\{(x, bx + c) \mid x \in \mathbb{R}\big\} \cup \big\{(\infty, 0)\big\} & \text{for } a = 0, \\
\Big\{\Big(x, a\big(f_1(x - b) + c\big)\Big) \mid x \in \mathbb{R}\Big\} \cup \big\{(\infty, a)\big\} & \text{for } a \neq 0,\ c \leqslant 0, \\
\Big\{\Big(x, a\big(f_2(x - b) + c\big)^{d_1/d_2}\Big) \mid x \in \mathbb{R}\Big\} \cup \big\{(\infty, a)\big\} & \text{for } a, c > 0, \\
\Big\{\Big(x, a\big(f_3(x - b) + c\big)^{d_1/d_3}\Big) \mid x \in \mathbb{R}\Big\} \cup \big\{(\infty, a)\big\} & \text{for } a < 0 < c,
\end{cases}
$$

where $a, b, c \in \mathbb{R}$ and $f_i = f_{d_i, r_i}$, $r_i = r_1^{d_i/d_1}$ for $i = 1, 2, 3$. Then $\mathcal{C}(d_1, r_1; d_2, d_3)$ is the circle set of a flat Laguerre plane $\mathcal{L}(d_1, r_1; d_2, d_3)$ represented on the cylinder $Z = (\mathbb{R} \cup \{\infty\}) \times \mathbb{R}$.

For example, for $d_1 = 2$ and $r_1 = 1$ we obtain $r_i = 1$ and $f_i(x) = f_{d_i,1}(x) = |x|^{d_i}$ for $i = 2, 3$. Then $\mathcal{C}^1 \cup \mathcal{C}^2$ is as in the classical flat Laguerre plane; these circles are graphs of polynomials of degree at most 2. Depending on the values of d_2 and d_3 we replace the sets $\mathcal{C}^{\pm}$ of the classical flat Laguerre plane by the corresponding circle sets of other Laguerre planes. For example, if $d_2 = 4$ and $d_3 = 2$ we obtain the following circles

$$
\big\{(x, ax^2 + bx + c) \mid x \in \mathbb{R}\big\} \cup \big\{(\infty, a)\big\} \quad \text{for } a \leqslant 0 \text{ or } ac \leqslant 0,
$$
$$
\big\{(x, a\sqrt{(x - b)^4 + c}) \mid x \in \mathbb{R}\big\} \cup \big\{(\infty, a)\big\} \quad \text{for } a, c > 0.
$$

In particular, this shows that there is a flat Laguerre plane that agrees with the classical flat Laguerre plane in all circles but those that are completely above a given circle (the circle $\mathbb{S}^1 \times \{0\}$ in the above example). Note that from the flat Laguerre plane above we can obtain different embeddings of the circle set $\mathcal{C}^2$ in the semi-classical Laguerre planes $\mathcal{L}(\varphi, id)$. Furthermore, by using these semi-classical Laguerre planes on $\mathcal{C}^2$ instead of the classical flat Laguerre plane we can generalise the planes $\mathcal{L}(2, 1; d_2, d_3)$ from Proposition 3. This shows that Proposition 1 can, in fact, be applied to a great variety of flat Laguerre planes.

The Laguerre planes $\mathcal{L}(f_i)$ involved in the construction of the planes $\mathcal{L}(d_1, r_1; d_2, d_3)$ as in Proposition 3 can be chosen to be mutually non-isomorphic. For example, this is the

case if the skew parabola functions f_i are not equivalent under affine transformations. As mentioned in the introduction one can however obtain the same flat Laguerre planes from Proposition 3 by using just two planes, the flat Laguerre planes $\mathcal{L}(f_1)$ and $\mathcal{L}^h(f_2, f_3)$ where h is the homeomorphism given by $h(x) = x^{d_1/d_2}$ for $x \geqslant 0$ and $h(x) = -|x|^{d_1/d_3}$ for $x < 0$. This indicates that by using the more general setting of flat Laguerre planes of translation type with two different skew parabola functions we can generalise Proposition 3 slightly. This involves a plane $\mathcal{L}(f_{d,r}, f_{d',r'})$ and the isomorphic model $\mathcal{L}^{h_{k,l}}(f_{d_+,r_+}, f_{d_-,r_-})$ of the Laguerre plane $\mathcal{L}(f_{d_+,r_+}, f_{d_-,r_-})$ of translation type where $h_{k,l}$ is the homeomorphism of $\mathbb{R}$ given by

$$h_{k,l}(x) = \begin{cases} x^k & \text{for } x \geqslant 0, \\ -|x|^l & \text{for } x < 0, \end{cases}$$

for $k, l > 0$. The circles of $\mathcal{L}(f_{d_+,r_+}, f_{d_-,r_-})$ touching $C_{0,0,0}$ are the sets

$$C_{0,0,c} = \big\{ (x, c) \mid x \in \mathbb{R} \big\} \cup \big\{ (\infty, 0) \big\},$$
$$C_{a,b,0} = \big\{ \big(x, a f_{d_+,r_+}(x - b)\big) \mid x \in \mathbb{R} \big\} \cup \big\{ (\infty, a) \big\}, \quad \text{where } a > 0,$$
$$C_{a,b,0} = \big\{ \big(x, a f_{d_-,r_-}(x - b)\big) \mid x \in \mathbb{R} \big\} \cup \big\{ (\infty, a) \big\}, \quad \text{where } a < 0.$$

Applying $\widetilde{h}_{k,l}$ we then obtain

$$C_{0,0,c}^{h_{k,l}} = \Big\{ \big(x, h_{k,l}(c)\big) \mid x \in \mathbb{R} \Big\} \cup \big\{ (\infty, 0) \big\},$$
$$C_{a,b,0}^{h_{k,l}} = \Big\{ \big(x, a^k f_{kd_+,r_+^k}(x - b)\big) \mid x \in \mathbb{R} \Big\} \cup \big\{ (\infty, a^k) \big\}, \qquad \text{where } a > 0$$
$$C_{a,b,0}^{h_{k,l}} = \Big\{ \big(x, -|a|^l \big(f_{ld_-,r_-^l}(x - b)\big)\big) \mid x \in \mathbb{R} \Big\} \cup \Big\{ \big(\infty, -|a|^l\big) \Big\}, \quad \text{where } a < 0$$

We therefore have to match f_{kd_+, r_+^k} and f_{ld_-, r_-^l} with skew-parabola functions $f_{d,r}$ and $f_{d',r'}$ from the other flat Laguerre plane. Again this can be easily achieved.

PROPOSITION 4. *Let $d, d', d_+, d_- > 1$ and $r, r' > 0$ and let $\mathcal{C}(d_+, d_-; d, r; d', r')$ be the collection of all sets of the form*

$$\begin{array}{ll} \big\{ (x, bx + c) \mid x \in \mathbb{R} \big\} \cup \big\{ (\infty, 0) \big\} & \text{for } b, c \in \mathbb{R}, \\[4pt] \Big\{ \big(x, a(f_{d,r}(x - b) + c)\big) \mid x \in \mathbb{R} \Big\} \cup \big\{ (\infty, a) \big\} & \text{for } a, b, c \in \mathbb{R}, \ a > 0, c \leqslant 0, \\[4pt] \Big\{ \big(x, a(f_{d',r'}(x - b) + c)\big) \mid x \in \mathbb{R} \Big\} \cup \big\{ (\infty, a) \big\} & \text{for } a, b, c \in \mathbb{R}, \ a < 0, c \leqslant 0, \\[4pt] \Big\{ \big(x, a(f_{d_+,r_+}(x - b) + c)^{d/d_+}\big) \mid x \in \mathbb{R} \Big\} \cup \big\{ (\infty, a) \big\} & \text{for } a, b, c \in \mathbb{R}, \ a, c > 0, \\[4pt] \Big\{ \big(x, a(f_{d_-,r_-}(x - b) + c)^{d'/d_-}\big) \mid x \in \mathbb{R} \Big\} \cup \big\{ (\infty, a) \big\} & \text{for } a, b, c \in \mathbb{R}, \ a < 0 < c \end{array}$$

where $r_+ = r^{d_+/d}$, $r_- = (r')^{d_-/d'}$. Then $\mathcal{C}(d_+, d_-; d, r; d', r')$ is the circle set of a flat Laguerre plane $\mathcal{L}(d_+, d_-; d, r; d', r')$ represented on the cylinder $Z = \big(\mathbb{R} \cup \{\infty\}\big) \times \mathbb{R}$.

3. KLEINWILLINGHÖFER TYPES

Similar to the Lenz–Barlotti classification of projective planes and the Hering classification of Möbius planes Kleinewillinghöfer classified Laguerre planes with respect to

linearly transitive groups of central automorphisms; see [4]. She obtained a multitude of types depending on the kinds of central automorphisms involved. In particular, with respect to Laguerre translations Kleinewillinghöfer obtained 11 types of Laguerre planes, labelled A to K; see [4, Satz 2] or [9, 5.5.2]. A *Laguerre translation* of a Laguerre plane $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely the points of one generator and induces a translation in the derived affine plane at one of its fixed points. The *derived affine plane* at a point p comprises all points of $\mathcal{L}$ that are not parallel to p and has lines the circles passing through p (minus the point p) and the generators not containing p. Let G and C be the generator and a circle that contains p and let $B(p, C)$ denote the *tangent pencil with support p*, that is, $B(p, C)$ consists of all circles that touch the circle C at the point p. In the derived affine plane at p the tangent pencil represents a parallel class of lines and we can look at translations in this direction. Then a $(G, B(p, C))$-*translation* of $\mathcal{L}$ is a Laguerre translation that fixes each point on G and each circle in $B(p, C)$ globally.

Of the 11 Kleinewillinghöfer types with respect to Laguerre translations one type, type E, has precisely one tangent pencil $B(p, C)$ with support p for which the automorphism group is $(G, B(p, C))$-transitive, that is, the group of all $(G, B(p, C))$-translations acts transitively on $C \setminus \{p\}$. We shall see that this type can be realised with the flat Laguerre planes $\mathcal{L}(d_1, r_1; d_2, d_3)$ from Proposition 3. We keep the notation for circles from Proposition 3. Note that for $t \in \mathbb{R}$ the transformations

$$(x, y) \mapsto \begin{cases} (x + t, y) & \text{for } x \in \mathbb{R} \\ (\infty, y) & \text{for } x = \infty \end{cases}$$

are $\left(G_\infty, B\big((\infty, 0), C_{0,0,0}\big)\right)$-translations of $\mathcal{L}(d_1, r_1; d_2, d_3)$ where G_∞ denotes the generator that contains $(\infty, 0)$, and the collection of all these Laguerre translations is transitive on $C_{0,0,0} \setminus \{(\infty, 0)\}$. So, for type E, we just have to make sure that the automorphism group of $\mathcal{L}(d_1, r_1; d_2, d_3)$ is not $(G, B(p, C))$-transitive for any other tangent pencil $B(p, C)$. Note that, in general, the point $(\infty, 0)$ is fixed by all automorphisms of a flat Laguerre plane of translation type. Indeed, $(\infty, 0)$ is fixed by each automorphism in a flat Laguerre plane $\mathcal{L}(f, g)$ of translation type unless $f(x) = g(x) = x^2$; see [5, Theorems 5 and 9]. Furthermore, two Laguerre planes $\mathcal{L}(f_{d_1, r_1})$ and $\mathcal{L}(f_{d_2, r_2})$ are isomorphic if and only if $d_1 = d_2$ and either $r_1 = r_2$ or $r_1 r_2 = 1$; compare [5, Theorem 10]. Also note that the circle C_0 in the construction of the pasted Laguerre planes in Proposition 1 is distinguished. Therefore, in general, we can expect that the flag $\big((\infty, 0), C_{0,0,0}\big)$ is fixed by each automorphism of a flat Laguerre plane $\mathcal{L}(d_1, r_1; d_2, d_3)$.

LEMMA 5. *A flat Laguerre plane $\mathcal{L}(d_1, r_1; d_2, d_3)$ admits the automorphisms*

$$(x, y) \mapsto \begin{cases} (sx + t, ry) & \text{for } x \in \mathbb{R} \\ (\infty, rs^{-d_1}y) & \text{for } x = \infty \end{cases}$$

where $r, s, t \in \mathbb{R}$, $r, s > 0$, so that the automorphism group of $\mathcal{L}(d_1, r_1; d_2, d_3)$ is at least 3-dimensional. Furthermore, the transformation

$$\tau : (x, y) \mapsto \begin{cases} (x, y+1) & \text{for } x \in \mathbb{R} \\ (\infty, y) & \text{for } x = \infty \end{cases}$$

is an automorphism of $\mathcal{L}(d_1, r_1; d_2, d_3)$ if and only if $d_1 = d_2 = d_3$. In this case the plane $\mathcal{L}(d_1, r_1; d_2, d_3) = \mathcal{L}(f_{d_1, r_1})$ is is of translation type.

PROOF: For the first statement note that a skew parabola function $f_{d,r}$ is semi-multiplicative, that is, $f_{d,r}(sx) = s^d f_{d,r}(x) = f_{d,r}(s) f_{d,r}(x)$ for all $s, x \in \mathbb{R}$, $s > 0$.

For the second statement consider the circle

$$\tau(C_{1,0,0}) = \Big\{ \big(x, f_1(x) + 1\big) \mid x \in \mathbb{R} \Big\} \cup \big\{ (\infty, 1) \big\}.$$

This circle in $\mathcal{L}(d_1, r_1; d_2, d_3)$ has to be of the form $C_{1,b,c}$ for some $b, c \in \mathbb{R}$, $c > 0$. But $f_1(x) + 1$ assumes its minimum when $x = 0$, giving a minimum of 1, and $(f_2(x-b)+c)^{d_1/d_2}$ assumes its minimum when $x = b$, giving a minimum of c^{d_1/d_2}. Hence $b = 0$ and $c = 1$. We then have that $f_1(x) + 1 = (f_2(x) + 1)^{d_1/d_2}$ for all $x \in \mathbb{R}$. In particular, for $x = 1$, one obtains $2 = 2^{d_1/d_2}$ so that $d_1 = d_2$. One similarly finds that $d_1 = d_3$ by considering the circle $\tau^{-1}(C_{-1,0,0})$.   ☐

LEMMA 6. *The derived affine plane of a flat Laguerre plane $\mathcal{L}(d_1, r_1; d_2, d_3)$ at the point $(0,0)$ is Desarguesian if and only if $d_1 = 2$ and $r_1 = 1$.*

PROOF: Let $f = f_{d_1, r_1}$. Circles in $\mathcal{L}(d_1, r_1; d_2, d_3)$ that pass through $(0,0)$ are of the form $C_{0,m,0}$ for $m \in \mathbb{R}$ or $C_{a,-b,-f(b)}$ for $a, b \in \mathbb{R}$, $a \neq 0$. Let $L_{0,m}$ and $L_{a,b}$ denote the corresponding lines in the derived affine plane $\mathcal{A}$ of $\mathcal{L}(d_1, r_1; d_2, d_3)$ at $(0,0)$. Then two lines $L_{a,b}$ and $L_{a',b'}$ for $a, a' \neq 0$ are parallel if and only if $a f'(b) = a' f'(b')$ where f' denotes the derivative of f, that is,

$$f'(x) = \begin{cases} d_1 x^{d_1 - 1} & \text{for } x \geqslant 0, \\ -r_1 d_1 |x|^{d_1 - 1} & \text{for } x < 0. \end{cases}$$

(The corresponding circles $C_{a,-b,-f(b)}$ and $C_{a',-b',-f(b')}$ touch analytically at $(0,0)$.) Two lines $L_{0,m}$ and $L_{0,m'}$ are parallel if and only if $m = m'$ and, finally, $L_{0,m}$ and $L_{a,b}$ are parallel if and only if $m = a f'(b)$.

Note that the plane $\mathcal{A}$ is the same as the derived affine plane of the flat Laguerre plane $\mathcal{L}(f)$ of translation type at $(0,0)$. Thus, if $d_1 = 2$ and $r_1 = 1$, then $\mathcal{L}(f)$ is classical and $\mathcal{A}$ is Desarguesian.

Conversely, assume that $\mathcal{A}$ is Desarguesian. We first show that $r = 1$. To this end consider the perspective triangles with respective vertices $(-1, r_1)$, $(1, 1)$, $(\infty, 0)$ and $(-1, 0)$, $(1, 0)$, (∞, a) where $a = -(1 + r_1^{1/d_1})^{d_1 - 1}/(d_1 r_1)$. The lines through $(-1, r_1)$,

$(1,1)$ and $(-1,0)$, $(1,0)$ are $L_{1,0}$ and $L_{0,0}$ and thus are parallel in $\mathcal{A}$. The lines through $(1,1)$, $(\infty,0)$ and $(1,0)$, (∞,a) are $L_{0,1}$ and $L_{a,b}$ where $b = -1/(1 + r_1^{1/d_1})$ and thus are again parallel. By Desargues' theorem the lines through $(-1,r_1)$, $(\infty,0)$ and $(-1,0)$, (∞,a) must also be parallel. The former line is $L_{0,-r_1}$ and the latter is $L_{a,b'}$ where $b' = r_1^{1/d_1}/(1 + r_1^{1/d_1}) > 0$. Being parallel then implies that $af'(b') = -r_1$ and we obtain $(1 + r_1^{1/d_1})^{d_1-1}d_1 r_1^{(d_1-1)/d_1} = d_1 r_1^2(1 + r_1^{1/d_1})^{d_1-1}$, that is, $r_1^{(d_1-1)/d_1} = r_1^2$ or $r_1^{(d+1)/d} = 1$. Hence $r_1 = 1$.

Let Φ be the translation of $\mathcal{A}$ that takes the point (∞,a) to the point $(\infty,0)$ where $a = -2^{d_1-1}/d_1$ is as above. Then Φ induces a homeomorphism on the parallel bundle of lines $\{L_{c,0} \mid c \in \mathbb{R}\}$ of the form $\Phi(L_{c,0}) = L_{\varphi(c),0}$ where φ is a fixed-point-free homeomorphism of $\mathbb{R}$. From the perspective triangles above we see that $\varphi(a) = 0$ and $\varphi(0) = 1$. Furthermore, we have $\Phi(\infty,y) = (\infty,\varphi(y))$ and $\Phi(1,y) = (1,\varphi(y))$ for all $y \in \mathbb{R}$. From $\Phi(L_{0,f'(u)}) = L_{1,u}$ for $u \in \mathbb{R}$ we obtain that $\varphi(f'(u)) = f(1+u) - f(u)$. In particular, if u is such that $f'(u) = a$ (such a u exists because f' is a homeomorphism of $\mathbb{R}$) one finds that $0 = \varphi(a) = f(1+u) - f(u)$. Buth then $u = -1/2$ and thus $-2^{d_1-1}/d_1 = a = f'(u) = -d_1/2^{d_1-1}$. Hence $2^{d_1-1_1} = d_1$ and $d_1 = 2$. (Note that we always assume $d_1 > 1$.)   □

PROPOSITION 7. *A flat Laguerre plane $\mathcal{L}(d_1,r_1;d_2,d_3)$ is of Kleinewillinghöfer type E unless $d_1 = d_2 = d_3$ or $(d_1,r_1) = (2,1)$.*

The distinguished tangent pencil is the pencil of all circles that touch $C_{0,0,0}$ at $(\infty,0)$. The transformations $(x,y) \mapsto (x+t,y)$ for $t \in \mathbb{R}$ extend to automorphisms of the Laguerre plane and form a transitive group of $\left(G_\infty, B((\infty,0), C_{0,0,0})\right)$-translations.

PROOF: We assume that $(d_1,r_1) \neq (2,1)$ and that not all d_i are equal. Since the automorphism group of $\mathcal{L}(d_1,r_1;d_2,d_3)$ is $(G, B(p,C))$-transitive for at least one tangent pencil $B(p,C)$, the type of the Laguerre plane is either E, G, H, or K, see [9, Proposition 5.5.8], or [10, Proposition 4.8]. In types G or K, all translations of the derived affine plane at $(\infty,0)$ extend to Laguerre translations. In particular, the transformation τ from Lemma 5 is an automorphism of $\mathcal{L}(d_1,r_1;d_2,d_3)$. But this implies $d_1 = d_2 = d_3$ by Lemma 5. This shows that types G and K are not possible.

In type H there is a unique circle C such that the automorphism group is $(G_p, B(p,C))$-transitive for all $p \in C$ where G_p denotes the generator containing p. Since our Laguerre planes are $\left(G_\infty, B((\infty,0), C_{0,0,0})\right)$-transitive and because $C_{0,0,0}$ is the only circle fixed under the automorphisms from Lemma 5, we see that $C = C_{0,0,0}$ in type H. But in type H the automorphism group of $\mathcal{L}(d_1,r_1;d_2,d_3)$ is transitive on $C_{0,0,0}$. In particular, the derived affine plane of $\mathcal{L}(d_1,r_1;d_2,d_3)$ at $(0,0)$ is isomorphic to the derived affine plane at $(\infty,0)$, that is, this plane is Desarguesian. Lemma 6 then implies $d_1 = 2$ and $r_1 = 1$. This shows that type H is not possible either. Hence, under our assumptions, a flat Laguerre plane $\mathcal{L}(d_1,r_1;d_2,d_3)$ must be of Kleinewillinghöfer type E.   □

Kleinewillinghöfer further considered Laguerre homologies and Laguerre homotheties. A *Laguerre homology* of a Laguerre plane $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely the points of one circle. One speaks of a *C-homology* if C is the distinguished circle. Of the six possible types of flat Laguerre planes with respect to Laguerre homologies only types I and II can occur in combination with type E, see [**10**, Theorem 6.1]. In types I and II the set of all circles for which the automorphism group of $\mathcal{L}$ is linearly transitive with respect to Laguerre homologies, that is, the group of all C-homologies is transitive on each generator minus its point of intersection with C, is empty or consists of a single circle, respectively.

A *Laguerre homothety* of $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that fixes two non-parallel points and induces a homothety in the derived affine plane at each of these two fixed points. One then speaks of a *$\{p,q\}$-homothety* if p, q are the two fixed points. Of the seven possible types of flat Laguerre planes with respect to Laguerre homologies only types 1 and 4 can occur in combination with type E, see [**10**, Theorem 6.1]. In types 1 and 4 the set of all unordered pairs of distinct points for which the automorphism group of $\mathcal{L}$ is linearly transitive with respect to Laguerre homotheties is empty or consists of the set $\{\{p,q\} \mid q \in C \setminus \{p\}\}$ for a circle C and a point $p \in C$, respectively.

As in the proof of Lemma 5 it follows by considering the images of the circles $C_{1,0,1}$ and $C_{1,0,0}$ that the transformations

$$\sigma_1 : (x,y) \mapsto (x,-y)$$

and

$$\sigma_2 : (x,y) \mapsto (-x,-y)$$

are automorphisms of $\mathcal{L}(d_1,r_1;d_2,d_3)$ if and only if $d_2 = d_3$ or $r_1 = 1$, $d_2 = d_3$, respectively. In both cases the transformations $(x,y) \mapsto (x,ry)$ for $r \in \mathbb{R}$, $r \neq 0$, form a transitive group of $C_{0,0,0}$-homologies.

PROPOSITION 8. *Assume that $(d_1,r_1) \neq (2,1)$ and that d_1,d_2,d_3 are not all the same. Then a flat Laguerre plane $\mathcal{L}(d_1,r_1;d_2,d_3)$ is of Kleinewillinghöfer type*

(a)   *I.E.1 if and only if $d_2 \neq d_3$;*

(b)   *II.E.1 if and only if $d_2 = d_3$ and $r_1 \neq 1$;*

(c)   *II.E.4 if and only if $d_2 = d_3$ and $r_1 = 1$.*

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