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MOMENTS OF RANDOM SUMS AND ROBBINS' PROBLEM OF OPTIMAL STOPPING

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Abstract

Robbins' problem of optimal stopping is that of minimising the expected *rank* of an observation chosen by some nonanticipating stopping rule. We settle a conjecture regarding the *value* of the stopped variable under the rule that yields the minimal expected rank, by embedding the problem in a much more general context of selection problems with the nonanticipation constraint lifted, and with the payoff growing like a power function of the rank.

Keywords: Stopping time; Robbins' problem of minimising the expected rank; Poisson embedding; random sum

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Let $X_1, \ldots, X_n$ be independent random variables sampled sequentially from the uniform [0, 1] distribution, and let $Y_1 < \cdots < Y_n$ be their order statistics. The rank R_j of the variable X_j is defined by setting $R_j = k$ on the event $X_j = Y_k$. Robbins' problem of optimal stopping [3] is that of minimising the expected rank $\mathrm{E}\, R_\tau$ over all stopping times τ that assume values in $\{1, \ldots, n\}$ and are adapted to the natural filtration of the sequence $X_1, \ldots, X_n$. Let τ_n be the optimal stopping time. The minimum expected rank $\mathrm{E}\, R_{\tau_n}$ increases as n grows, and converges to some finite limit v whose exact value is unknown. The closest known upper bound is slightly less than $\frac{7}{3}$. Finding v or even improving the existing rough bounds remains a challenge. A major source of difficulty is that the optimal stopping time τ_n is a very complicated function of the sample. It seems that τ_n has not been computed for $n > 3$. Moreover, for large n, there is no simplification, and the complexity of the optimal stopping time persists in the '$n = \infty$' limiting form of the problem [6].

In a recent paper Bruss and Swan [4] stressed that it is not even known if $\limsup_n n\, \mathrm{E}\, X_{\tau_n}$ is finite. They mentioned that the property was first conjectured in [2]. Although the conjecture emerged in connection with attempts to bound v by comparison with the much simpler problem of minimising $\mathrm{E}\, X_\tau$ (or minor variations of the problem), it seems that the question is of independent interest as a relation between the stopped sample value and its rank. In this note we settle the conjecture by proving a considerably more general assertion.

Proposition 1. *Fix $p > 0$. For $n = 1, 2, \ldots$, let σ_n be a random variable with range $\{1, \ldots, n\}$ and arbitrary joint distribution with $X_1, \ldots, X_n$. Then*

$$\limsup_n \mathrm{E}[R_{\sigma_n}]^p < \infty \quad \Longrightarrow \quad \limsup_n n^p\, \mathrm{E}[X_{\sigma_n}]^p < \infty.$$

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In particular, $\lim_{n\to\infty} n^p\,\mathrm{E}[X_{\tau_n}]^p < \infty$ *for* τ_n*, the stopping time minimising* $\mathrm{E}[R_\tau]^p$ *over all stopping times adapted to* $X_1, \ldots, X_n$*.*

We defer the proof to the end of the note. The idea is to bound X_{σ_n} by exploiting properties of a random walk with negative drift.

Let $\xi, \xi_1, \xi_2, \ldots$ be independent and identically distributed (i.i.d.) nonnegative random variables with $\mu = \mathrm{E}\,\xi \in (0, \infty)$. Let $S_k := \xi_1 + \cdots + \xi_k$, and, for $\lambda > \mu$, let $M_\lambda := \sup_{k\geq 0}(S_k - \lambda k)$.

Proposition 2. *For* $p > 0$*,*

$$\mathrm{E}\,\xi^{p+1} < \infty \quad \Longleftrightarrow \quad \mathrm{E}\,M_\lambda^p < \infty.$$

Proof. The moment condition on ξ is equivalent to $\mathrm{E}[(\xi - \lambda)^+]^{p+1} < \infty$, and the result follows from Lemma 3.5 of [1].

Corollary 1. *Suppose that* $\mathrm{E}\,\xi^{p+1} < \infty$*, and let* σ *be a nonnegative integer random variable with* $\mathrm{E}\,\sigma^p < \infty$*. Then* $\mathrm{E}\,S_\sigma^p < \infty$*.*

Proof. The proof follows from $S_\sigma^p \leq (M_\lambda + \lambda\sigma)^p \leq c_p(M_\lambda^p + \lambda^p\sigma^p)$, where $c_p := 2^{p-1} \vee 1$.

We first apply Corollary 1 to a Poisson embedded, limiting form of the stopping problem with continuous time [6]. Let $\xi_1, \xi_2, \ldots$ be i.i.d. rate-one exponential variables, let S_k be as above, and let $T_1, T_2, \ldots$ be i.i.d. uniform [0, 1] random times, independent of the ξ_js. The points (T_k, S_k) are the atoms of a homogeneous planar Poisson process $\mathcal{P}$ in $[0, 1] \times [0, \infty)$. To introduce the dynamics, consider an observer whose information at time $t \in [0, 1]$ is the (infinite) configuration of points of $\mathcal{P}$ within the strip $[0, t] \times [0, \infty)$, that is, $\{(T_k, S_k) \colon T_k \leq t\}$. The rank of point (T_k, S_k) is defined as $R_{T_k} = k$, meaning that S_k is the kth smallest value among $S_1, S_2, \ldots$. The piece of information added at time T_k is the point (T_k, S_k), but not the rank R_{T_k}.

Suppose that the objective of the observer is to minimise $\mathrm{E}[R_\tau]^p$ over stopping times τ that assume values in the random set $\{T_1, T_2, \ldots\}$ and are adapted to the information flow of the observer. For the optimal stopping time τ_∞, it is known from the previous studies that $\mathrm{E}[R_{\tau_\infty}]^p < \infty$ (see [5] and [6]). Taking $\sigma = R_{\tau_\infty}$, we have $\mathrm{E}[S_\sigma]^p < \infty$. The case $p = 1$ corresponds to the infinite version of Robbins' problem of minimising the expected rank.

Returning to our main task, we wish to apply the above to a finite sample of fixed size. To this end, we shall use the familiar realisation of uniform order statistics through sums of exponential variables, as

$$(Y_k,\ 1 \leq k \leq n) \stackrel{\mathrm{D}}{=} \left(\frac{S_k}{S_n},\ 1 \leq k \leq n\right).$$

Introducing the event $A_n := \{n/S_n > 1 + \varepsilon\}$, we can estimate, for $1 \leq k \leq n$,

$$n^p Y_k^p = n^p Y_k^p\, \mathbf{1}_{A_n} + n^p Y_k^p\, \mathbf{1}_{A_n^c} \leq n^p\, \mathbf{1}_{A_n} + (1+\varepsilon)^p S_k^p \leq n^p\, \mathbf{1}_{A_n} + c_p(1+\varepsilon)^p(M_\lambda^p + \lambda^p k^p),$$

where we have used $S_k \leq M_\lambda + \lambda k$. Using a large deviation bound for the probability of A_n and sending $\varepsilon \to 0$, we conclude that, for any random variable σ_n with values in $\{1, \ldots, n\}$,

$$\limsup_n n^p\,\mathrm{E}[Y_{\sigma_n}]^p \leq c_p\lambda^p \limsup_n \mathrm{E}\,\sigma_n^p + c_p\,\mathrm{E}\,M_\lambda^p.$$

Finally, taking $\sigma_n = R_{\tau_n}$, Proposition 1 follows from

$$\limsup_n n^p \, \mathrm{E}[X_{\tau_n}]^p \le c_p \lambda^p \limsup_n \mathrm{E}[R_{\tau_n}]^p + c_p \, \mathrm{E}\, M_\lambda^p < \infty,$$

since $\mathrm{E}[R_{\tau_n}]^p$ converges to a finite limit (see [5] and [6]).

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References

[1] Alsmeyer, G. and Iksanov, A. (2009). A log-type moment result for perpetuities and its application to martingales in supercritical branching random walks. *Electron. J. Prob.* **14,** 289–312.
[2] Assaf, D. and Samuel-Cahn, E. (1996). The secretary problem: minimizing the expected rank with i.i.d. random variables. *Adv. Appl. Prob.* **28,** 828–852.
[3] Bruss, F. T. (2005). What is known about Robbins' problem? *J. Appl. Prob.* **42,** 108–120.
[4] Bruss, F. T. and Swan, Y. C. (2009). A continuous-time approach to Robbins' problem of minimizing the expected rank. *J. Appl. Prob.* **46,** 1–18.
[5] Gianini, J. and Samuels, S. M. (1976). The infinite secretary problem. *Ann. Prob.* **3,** 418–432.
[6] Gnedin, A. V. (2007). Optimal stopping with rank-dependent loss. *J. Appl. Prob.* **44,** 996–1011.