A REMARK CONCERNING GRAVES' CLOSURE CRITERION

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In a paper recently published in this journal [**1**], R. E. Graves proved a closure criterion for orthonormal sets of functions. A refined form of it may be stated as follows:

THEOREM A. *Let p be a function whose zeros have Lebesgue measure zero, such that for each $x \in (a, b)$, $p \in L_2$ on* $\min (c, x) < t < \max (c, x)$, *where $a \leqslant c \leqslant b$. ($a$, b, and c may be infinite.) Let w be measurable, almost everywhere positive, and such that*

$$w(x) \int_c^x |p(t)|^2 dt \in L_1$$

on (a, b). Then for any family $\{\phi_n\}$, orthonormal in (a, b),

$$\sum_{n=1}^{\infty} \int_a^b \left| \int_c^x p(t)\phi_n(t)\,dt \right|^2 w(x)\,dx \leqslant \int_a^b \left| \int_c^x |p(t)|^2\,dt \right| w(x)\,dx,$$

where equality holds if and only if $\{\phi_n\}$ is closed in L_2 on (a, b).

In Graves' version of the theorem, the zeros and discontinuities of p were assumed to have Jordan content zero.

The proof of Theorem A is quite similar to the one given in [**1**]; we merely replace Theorem III of [**1**] by Theorem B below, whose proof is actually simpler than that of Theorem III.

THEOREM B. *If $p \in L_2$ on every compact sub-interval of (a, b) and if $p(t)$ is different from zero almost everywhere on (a, b), then the set of functions of the form*

$$f(t) = \sum_{k=1}^{m} c_k\, p(t)\, \chi_{(a_k, b_k)}(t) \qquad (a < a_k < b_k < b) \tag{1}$$

is dense in L_2 on (a, b).

Here χ_E denotes the characteristic function of the set E.

Proof. Suppose $a < \alpha < \beta < b$, let $g \in L_2$ on (α, β), and suppose $g(t) = 0$ outside (α, β). It suffices to approximate functions of this type in the L_2-norm by functions of the form (1).

We shall do this by showing that the set of functions

$$p(t)\, \chi_{(\alpha, \gamma)}(t) \qquad (\alpha < \gamma < \beta) \tag{2}$$

is complete in L_2 on (α, β). Let $h \in L_2$ on (α, β), and suppose

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$$\int_{\alpha}^{\beta} h(t)p(t)\ \chi_{(\alpha,\gamma)}(t)\,dt = 0 \qquad (\alpha < \gamma < \beta),$$

that is,

$$\int_{\alpha}^{\gamma} h(t)p(t)\,dt = 0 \qquad (\alpha < \gamma < \beta).$$

It follows that $h(t)p(t) = 0$ almost everywhere, so that $h(t) = 0$ almost everywhere.

Hence the set of functions (2) is complete in L_2 on (α, β). Theorem B follows, since closure is equivalent to completeness in L_2.

Reference

1. R. E. Graves, *A closure criterion for orthogonal functions*, Can. J. Math., *4* (1952), 198-203.

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