PROBABILITY DISTRIBUTION FUNCTION FOR THE EUCLIDEAN DISTANCE BETWEEN TWO TELEGRAPH PROCESSES

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Abstract

Consider two independent Goldstein–Kac telegraph processes $X_1(t)$ and $X_2(t)$ on the real line $\mathbb{R}$. The processes $X_k(t)$, $k = 1, 2$, describe stochastic motions at finite constant velocities $c_1 > 0$ and $c_2 > 0$ that start at the initial time instant $t = 0$ from the origin of $\mathbb{R}$ and are controlled by two independent homogeneous Poisson processes of rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. We obtain a closed-form expression for the probability distribution function of the Euclidean distance $\rho(t) = |X_1(t) - X_2(t)|$, $t > 0$, between these processes at an arbitrary time instant $t > 0$. Some numerical results are also presented.

Keywords: Telegraph process; telegraph equation; persistent random walk; probability distribution function; Euclidean distance

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1. Introduction

The classical telegraph process $X(t)$ describes the stochastic motion of a particle that moves on the real line $\mathbb{R}$ at some constant finite speed c and alternates between two possible directions of motion (forward and backward) at Poisson-distributed random instants of intensity $\lambda > 0$. This random walk was first introduced in the works of Goldstein [10] and Kac [12] (of which the latter is a reprinting of an earlier 1956 article). The most remarkable fact is that the transition density of $X(t)$ is the fundamental solution to the hyperbolic telegraph equation (which is one of the classical equations of mathematical physics) and, under increasing c and λ, it transforms into the transition density of the standard Brownian motion on $\mathbb{R}$. Thus, the telegraph process can be treated as a finite-velocity counterpart of the one-dimensional Brownian motion. The telegraph process $X(t)$ can also be treated in a more general context of random evolutions (see [20]).

Over several decades the Goldstein–Kac telegraph process and its numerous generalizations have become the subject of extensive research and a great deal of relevant works have been published. The properties of the solution space of the Goldstein–Kac telegraph equation were studied in [2]. The process of one-dimensional random motion at finite speed governed by a Poisson process with a time-dependent parameter was considered in [13]. The relationships between the Goldstein–Kac model and physical processes, including some emerging effects of relativity theory, were examined in [1], [4], and [5]. Formulae for the distributions of the first exit time from a given interval and of the maximum displacement of the telegraph process were obtained in [8], [18], [19], and [20, Section 0.5]. The behavior of the telegraph

process with absorbing and reflecting barriers was examined in [9] and [22]. A one-dimensional stochastic motion with an arbitrary number of velocities and of governing Poisson processes was examined in [15]. The telegraph processes with random velocities were studied in [24]. The behavior of telegraph-type evolutions in inhomogeneous environments were considered in [23]. Probabilistic methods of solving Cauchy problems for the telegraph equation were developed in [11], [12], [14], and [25]. A generalization of the Goldstein–Kac model for the case of a damped telegraph process with logistic stationary distributions was given in [7]. A random motion with velocities alternating at Erlang-distributed random times was studied in [6]. A detailed moment analysis of the telegraph process was carried out in [16]. Explicit formulae for the occupation time distributions of the telegraph process were recently obtained in [3].

In this paper we examine the Euclidean distance between two independent telegraph processes represented by two particles moving randomly at finite speed on the real line $\mathbb{R}$ and whose evolutions are driven by two independent homogeneous Poisson processes. Such a problem is motivated by its great importance in describing various kinds of interactions between the particles arising in physics, chemistry, biology, financial markets, and other fields. For example, in physics and chemistry the particles are atoms or molecules of the substance and their interaction can provoke a physical or chemical reaction. In biology the particles represent biological objects, such as cells, bacteria, animals, etc., and their 'interaction' can mean creating a new cell (or, contrarily, killing the cell), launching an infection mechanism, or founding a new animal population, respectively. In financial markets the moving particles can be interpreted as oscillating exchange rates or stock prices and their 'interaction' can mean gaining or ruining.

Let $X_1(t)$ and $X_2(t)$ be two telegraph processes representing the positions of these particles on $\mathbb{R}$ at an arbitrary time instant $t > 0$. In describing the phenomena of interaction the Euclidean distance between these processes,

$$\rho(t) = |X_1(t) - X_2(t)|, \qquad t > 0, \tag{1.1}$$

is of a special importance. It is quite natural to consider that the particles do not 'feel' each other if $\rho(t)$ is large. In other words, the forces acting between the particles are negligible if the distance $\rho(t)$ is sufficiently large. However, as soon as the distance between the particles becomes less than some given $r > 0$, the particles can start interacting with some positive probability. This means that the occurrence of the random event $\{\rho(t) < r\}$ is the necessary (but, maybe, not sufficient) condition for launching the process of interaction at time instant $t > 0$. Therefore, the distribution $\mathbb{P}\{\rho(t) < r\}$ plays a crucial role in analyzing such processes and is thus the main focus of this research.

The paper is organized as follows. In Section 2 we recall some basic properties of the telegraph process $X(t)$ that we will heavily rely on. In Section 3 we obtain a series representation of the probability of being in an arbitrary subinterval of the support of $X(t)$ at time $t > 0$ and derive a closed-form expression for the probability distribution function of $X(t)$, which to the best of the author's knowledge, have not been obtained in the literature. These results are given in terms of Gauss hypergeometric functions, as well as in terms of Gegenbauer polynomials with noninteger negative upper indices. In Section 4 we formulate and prove our principal result, yielding the closed-form expression for the probability distribution function of the Euclidean distance (1.1) between two independent telegraph processes at an arbitrary time instant $t > 0$. The derivation is based on determining the probability that the particle is located at time t in an r-neighborhood of the second particle. Some approximate numerical results are presented in Section 5. In Appendix A we prove two auxiliary lemmas related to some indefinite integrals of the modified Bessel functions and conditional probabilities that are used in our analysis.

2. Some basic properties of the telegraph process

The telegraph stochastic process describes a particle that starts at the initial time instant $t = 0$ from the origin $x = 0$ of the real line $\mathbb{R}$ and moves with some finite constant speed c. The motion has an initial positive or negative direction with equal probability $\frac{1}{2}$, and is driven by a homogeneous Poisson process $N(t)$ of rate $\lambda > 0$ as follows. Upon a Poisson event, the particle instantaneously takes on the opposite direction and keeps moving with the same speed c until the next Poisson event occurrence, at which time it takes on the opposite direction again independently of its previous motion, and so on. This random motion was first studied by Goldstein [10] and Kac [12], and was thereafter called the *telegraph process*.

Let $X(t)$ denote the particle's position on $\mathbb{R}$ at an arbitrary time instant $t > 0$. Since the speed c is finite, then, at instant $t > 0$, the distribution $\mathbb{P}\{X(t) \in dx\}$ is concentrated in the finite interval $[-ct, ct]$ which is the support of the distribution of $X(t)$. The density $f(x, t)$, $x \in \mathbb{R}$, $t \geq 0$, of the distribution $\mathbb{P}\{X(t) \in dx\}$ has the structure

$$f(x, t) = f^{\mathrm{s}}(x, t) + f^{\mathrm{ac}}(x, t),$$

where $f^{\mathrm{s}}(x, t)$ and $f^{\mathrm{ac}}(x, t)$ are the densities of the singular (with respect to the Lebesgue measure on the line) and the absolutely continuous components of the distribution of $X(t)$, respectively.

The singular component of the distribution is obviously concentrated at the two terminal points $\pm ct$ of the interval $[-ct, ct]$ and corresponds to the case when no one Poisson event occurs until the moment t; hence, the particle does not change its initial direction. Therefore, the probability of being at the terminal points $\pm ct$ at an arbitrary instant $t > 0$ is

$$\mathbb{P}\{X(t) = ct\} = \mathbb{P}\{X(t) = -ct\} = \tfrac{1}{2}\mathrm{e}^{-\lambda t}.$$

The absolutely continuous component of the distribution of $X(t)$ is concentrated in the open interval $(-ct, ct)$ and corresponds to the case when at least one Poisson event occurs by the moment t; hence, the particle changes its initial direction. The probability of this event is

$$\mathbb{P}\{X(t) \in (-ct, ct)\} = 1 - \mathrm{e}^{-\lambda t}. \tag{2.1}$$

The principal result by Goldstein [10] and Kac [12] states that the density $f = f(x, t)$, $x \in [-ct, ct]$, $t > 0$, of the distribution of $X(t)$ satisfies the hyperbolic partial differential equation

$$\frac{\partial^2 f}{\partial t^2} + 2\lambda \frac{\partial f}{\partial t} - c^2 \frac{\partial^2 f}{\partial x^2} = 0 \tag{2.2}$$

(which is referred to as the *telegraph* or *damped wave* equation), and can be found by solving (2.2) with initial conditions

$$f(x, t)\big|_{t=0} = \delta(x), \qquad \frac{\partial f(x, t)}{\partial t}\bigg|_{t=0} = 0,$$

where $\delta(x)$ is the Dirac delta function. This means that the transition density $f(x, t)$ of the process $X(t)$ is the fundamental solution (i.e. Green's function) to the telegraph equation (2.2).

The explicit form of the density $f(x, t)$ is given by the formula (see, for instance, [17, Section 2.5]) or [20, Section 0.4]

$$
\begin{aligned}
f(x, t) = {} & \frac{e^{-\lambda t}}{2}[\delta(ct - x) + \delta(ct + x)] \\
& + \frac{\lambda e^{-\lambda t}}{2c}\left[I_0\left(\frac{\lambda}{c}\sqrt{c^2 t^2 - x^2}\right)\right. \\
& \left. + \frac{ct}{\sqrt{c^2 t^2 - x^2}}I_1\left(\frac{\lambda}{c}\sqrt{c^2 t^2 - x^2}\right)\right]\Theta(ct - |x|),
\end{aligned}
\tag{2.3}
$$

where $I_0(z)$ and $I_1(z)$ are the modified Bessel functions of the zero and first orders, respectively (that is, the Bessel functions with imaginary argument), given by

$$
I_0(z) = \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k}, \qquad I_1(z) = \sum_{k=0}^{\infty}\frac{1}{k!\,(k+1)!}\left(\frac{z}{2}\right)^{2k+1},
$$

and $\Theta(x)$ is the Heaviside step function

$$
\Theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}
$$

The first term in (2.3), $f^s(x, t) = \frac{1}{2}e^{-\lambda t}[\delta(ct - x) + \delta(ct + x)]$, is the singular part of the density of the distribution of $X(t)$ concentrated at the two terminal points $\pm ct$ of the interval $[-ct, ct]$, while the second term in (2.3),

$$
\begin{aligned}
f^{ac}(x, t) = {} & \frac{\lambda e^{-\lambda t}}{2c}\left[I_0\left(\frac{\lambda}{c}\sqrt{c^2 t^2 - x^2}\right)\right. \\
& \left. + \frac{ct}{\sqrt{c^2 t^2 - x^2}}I_1\left(\frac{\lambda}{c}\sqrt{c^2 t^2 - x^2}\right)\right]\Theta(ct - |x|),
\end{aligned}
\tag{2.4}
$$

represents the density of the absolutely continuous part of the distribution of $X(t)$ concentrated in the open interval $(-ct, ct)$.

Other important properties of the telegraph random processes can be found in the recently published book [17].

3. Distribution function of the telegraph process

Consider a telegraph process $X(t)$ describing the stochastic motion of a particle that starts at the initial time instant $t = 0$ from the origin $x = 0$ of the real line $\mathbb{R}$ and moves with some finite constant speed $c > 0$ whose evolution is driven by a homogeneous Poisson process $N(t)$ of rate $\lambda > 0$, as described above.

As noted above, at an arbitrary time instant $t > 0$ the process $X(t)$ is concentrated in the interval $[-ct, ct]$. Let $a, b \in \mathbb{R}$, $a < b$, be arbitrary points of $\mathbb{R}$ such that the intervals (a, b) and $(-ct, ct)$ have a nonempty intersection, that is, $(a, b) \cap (-ct, ct) \neq \varnothing$. We are interested in the probability $\mathbb{P}\{X(t) \in (a, b) \cap (-ct, ct)\}$ that the process $X(t)$ at time instant $t > 0$ is located in the subinterval $(a, b) \cap (-ct, ct) \subseteq (-ct, ct)$. This result is presented in the following proposition.

Proposition 3.1. *For an arbitrary time instant $t > 0$ and an arbitrary open interval $(a, b) \subset \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$, such that $(a, b) \cap (-ct, ct) \neq \varnothing$,*

$$\mathbb{P}\{X(t) \in (a, b) \cap (-ct, ct)\} = \frac{\lambda e^{-\lambda t}}{2c} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda t}{2}\right)^{2k} \left(1 + \frac{\lambda t}{2k + 2}\right)$$
$$\times \left[\beta F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{\beta^2}{c^2 t^2}\right) - \alpha F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{\alpha^2}{c^2 t^2}\right)\right], \quad (3.1)$$

where

$$\alpha = \max\{-ct, a\}, \qquad \beta = \min\{ct, b\} \quad (3.2)$$

and

$$F(\xi, \eta; \zeta; z) = {}_2F_1(\xi, \eta; \zeta; z) = \sum_{k=0}^{\infty} \frac{(\xi)_k (\eta)_k}{(\zeta)_k} \frac{z^k}{k!}$$

is the Gauss hypergeometric function.

Proof. By integrating density (2.4) and applying formulae (A.4) and (A.5) given in Appendix A, we obtain

$$\mathbb{P}\{X(t) \in (a, b) \cap (-ct, ct)\}$$
$$= \frac{\lambda e^{-\lambda t}}{2c} \left[\int_\alpha^\beta I_0\left(\frac{\lambda}{c}\sqrt{c^2 t^2 - x^2}\right) dx + ct \int_\alpha^\beta \frac{I_1(\lambda \sqrt{c^2 t^2 - x^2}/c)}{\sqrt{c^2 t^2 - x^2}} dx\right]$$
$$= \frac{\lambda e^{-\lambda t}}{2c} \left[x \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda t}{2}\right)^{2k} F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{x^2}{c^2 t^2}\right)\right.$$
$$\left. + x \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} \left(\frac{\lambda t}{2}\right)^{2k+1} F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{x^2}{c^2 t^2}\right)\right]\Bigg|_{x=\alpha}^{x=\beta}$$
$$= \frac{\lambda e^{-\lambda t}}{2c} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda t}{2}\right)^{2k} \left(1 + \frac{\lambda t}{2k+2}\right)$$
$$\times \left[\beta F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{\beta^2}{c^2 t^2}\right) - \alpha F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{\alpha^2}{c^2 t^2}\right)\right],$$

proving (3.1).

Remark 3.1. Let $x \in (-ct, ct)$ be an arbitrary interior point of the open interval $(-ct, ct)$, and let $r > 0$ be an arbitrary positive number such that $(x - r, x + r) \subseteq (-ct, ct)$. Then, according to (3.1), we obtain the following formula for the probability of being in the subinterval $(x - r, x + r) \subseteq (-ct, ct)$ of radius r centered at the point x:

$$\mathbb{P}\{X(t) \in (x - r, x + r)\}$$
$$= \frac{\lambda e^{-\lambda t}}{2c} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda t}{2}\right)^{2k} \left(1 + \frac{\lambda t}{2k + 2}\right)$$
$$\times \left[(x + r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(x + r)^2}{c^2 t^2}\right) - (x - r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(x - r)^2}{c^2 t^2}\right)\right] \quad (3.3)$$

for $-ct \leq x - r < x + r \leq ct$.

By setting $x = 0$ in (3.3) we obtain

$$\mathbb{P}\{X(t) \in (-r, r)\} = \frac{\lambda r e^{-\lambda t}}{c} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda t}{2}\right)^{2k} \left(1 + \frac{\lambda t}{2k + 2}\right) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{r^2}{c^2 t^2}\right), \quad (3.4)$$

yielding the probability of being in the symmetric (with respect to the starting point $x = 0$) subinterval $(-r, r) \subseteq (-ct, ct)$.

For further analysis, we need

$$F\left(-k, \frac{1}{2}; \frac{3}{2}; 1\right) = \frac{(2k)!!}{(2k + 1)!!} = \frac{2^k k!}{(2k + 1)!!}, \qquad k \geq 0, \quad (3.5)$$

which is the particular case of the more general relation (see [21, Formula 163 p. 465])

$$F\left(-k, \frac{1}{2}; \frac{3}{2}; z\right) = -\frac{2^k k!}{(2k + 1)!! \sqrt{z}} C_{2k+1}^{-k-1/2}(\sqrt{z}), \qquad k \geq 0, \quad (3.6)$$

where the $C_n^{\nu}(z)$ are the Gegenbauer polynomials with negative noninteger upper indices.

Setting $r = ct$ in (3.4) and applying (3.5), we obtain

$$\begin{aligned}
\mathbb{P}\{X(t) \in (-ct, ct)\} &= \lambda t e^{-\lambda t} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda t}{2}\right)^{2k} \left(1 + \frac{\lambda t}{2k + 2}\right) F\left(-k, \frac{1}{2}; \frac{3}{2}; 1\right) \\
&= \lambda t e^{-\lambda t} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda t}{2}\right)^{2k} \left(1 + \frac{\lambda t}{2k + 2}\right) \frac{2^k k!}{(2k + 1)!!} \\
&= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k+1}}{2^k k! \, (2k + 1)!!} \left(1 + \frac{\lambda t}{2k + 2}\right) \\
&= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k+1}}{(2k + 1)!} \left(1 + \frac{\lambda t}{2k + 2}\right) \\
&= e^{-\lambda t} \left[\sum_{k=0}^{\infty} \frac{(\lambda t)^{2k+1}}{(2k + 1)!} + \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k+2}}{(2k + 2)!}\right] \\
&= e^{-\lambda t} [\sinh(\lambda t) + \cosh(\lambda t) - 1] \\
&= e^{-\lambda t} (e^{\lambda t} - 1) \\
&= 1 - e^{-\lambda t},
\end{aligned}$$

which is (2.1).

From Proposition 3.1 we can extract the explicit form of the probability distribution function of $X(t)$.

Proposition 3.2. *The probability distribution function of the telegraph process $X(t)$ has the form*

$$\mathbb{P}\{X(t) < x\}$$

$$= \begin{cases} 0, & x \in (-\infty, -ct], \\ \dfrac{1}{2} + \dfrac{\lambda x e^{-\lambda t}}{2c} \sum_{k=0}^{\infty} \dfrac{1}{(k!)^2} \left(\dfrac{\lambda t}{2}\right)^{2k} \left(1 + \dfrac{\lambda t}{2k+2}\right) \\ \qquad\qquad \times F\left(-k, \dfrac{1}{2}; \dfrac{3}{2}; \dfrac{x^2}{c^2 t^2}\right), & x \in (-ct, ct], \\ 1, & x \in (ct, +\infty). \end{cases} \quad (3.7)$$

Proof. According to Proposition 3.1, for arbitrary $x \in (-ct, ct)$, we have

$$\mathbb{P}\{X(t) \in (-ct, x)\}$$

$$= \dfrac{\lambda e^{-\lambda t}}{2c} \sum_{k=0}^{\infty} \dfrac{1}{(k!)^2} \left(\dfrac{\lambda t}{2}\right)^{2k} \left(1 + \dfrac{\lambda t}{2k+2}\right)$$

$$\times \left[x F\left(-k, \dfrac{1}{2}; \dfrac{3}{2}; \dfrac{x^2}{c^2 t^2}\right) + ct F\left(-k, \dfrac{1}{2}; \dfrac{3}{2}; 1\right) \right]$$

$$= \dfrac{\lambda x e^{-\lambda t}}{2c} \sum_{k=0}^{\infty} \dfrac{1}{(k!)^2} \left(\dfrac{\lambda t}{2}\right)^{2k} \left(1 + \dfrac{\lambda t}{2k+2}\right) F\left(-k, \dfrac{1}{2}; \dfrac{3}{2}; \dfrac{x^2}{c^2 t^2}\right)$$

$$+ e^{-\lambda t} \sum_{k=0}^{\infty} \dfrac{1}{(k!)^2} \left(\dfrac{\lambda t}{2}\right)^{2k+1} \left(1 + \dfrac{\lambda t}{2k+2}\right) F\left(-k, \dfrac{1}{2}; \dfrac{3}{2}; 1\right).$$

We separately consider the second term of this expression. Applying (3.5) we obtain

$$e^{-\lambda t} \sum_{k=0}^{\infty} \dfrac{1}{(k!)^2} \left(\dfrac{\lambda t}{2}\right)^{2k+1} \left(1 + \dfrac{\lambda t}{2k+2}\right) F\left(-k, \dfrac{1}{2}; \dfrac{3}{2}; 1\right)$$

$$= e^{-\lambda t} \sum_{k=0}^{\infty} \dfrac{2^k}{k!\,(2k+1)!!} \left(\dfrac{\lambda t}{2}\right)^{2k+1} \left(1 + \dfrac{\lambda t}{2k+2}\right)$$

$$= e^{-\lambda t} \sum_{k=0}^{\infty} \dfrac{2^{2k}}{2^k k!\,(2k+1)!!} \left(\dfrac{\lambda t}{2}\right)^{2k+1} \left(1 + \dfrac{\lambda t}{2k+2}\right)$$

$$= \dfrac{e^{-\lambda t}}{2} \sum_{k=0}^{\infty} \dfrac{(\lambda t)^{2k+1}}{(2k+1)!} \left(1 + \dfrac{\lambda t}{2k+2}\right)$$

$$= \dfrac{e^{-\lambda t}}{2} \left[\sum_{k=0}^{\infty} \dfrac{(\lambda t)^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty} \dfrac{(\lambda t)^{2k+2}}{(2k+2)!} \right]$$

$$= \dfrac{e^{-\lambda t}}{2} [\sinh(\lambda t) + \cosh(\lambda t) - 1]$$

$$= \dfrac{1}{2} - \dfrac{e^{-\lambda t}}{2}.$$

Therefore, for arbitrary $x \in (-ct, ct]$, we obtain

$$
\begin{aligned}
&\mathbb{P}\{X(t) < x\} \\
&\quad = \mathbb{P}\{X(t) = -ct\} + \mathbb{P}\{X(t) \in (-ct, x)\} \\
&\quad = \frac{e^{-\lambda t}}{2} + \frac{\lambda x e^{-\lambda t}}{2c} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda t}{2}\right)^{2k} \left(1 + \frac{\lambda t}{2k+2}\right) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{x^2}{c^2 t^2}\right) + \frac{1}{2} - \frac{e^{-\lambda t}}{2} \\
&\quad = \frac{1}{2} + \frac{\lambda x e^{-\lambda t}}{2c} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda t}{2}\right)^{2k} \left(1 + \frac{\lambda t}{2k+2}\right) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{x^2}{c^2 t^2}\right).
\end{aligned}
$$

This completes the proof.

The shape of probability distribution function (3.7) at time instant $t = 2$ in the interval $(-2, 2]$ (for parameters $c = 1$ and $\lambda = 1.5$) is plotted in Figure 1.

Remark 3.2. In view of (3.6), formulae (3.1) and (3.7) can also be represented in terms of Gegenbauer polynomials, i.e.

$$
\begin{aligned}
&\mathbb{P}\{X(t) \in (a, b) \cap (-ct, ct)\} \\
&\quad = \frac{e^{-\lambda t}}{2} \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k+1}}{(2k+1)!} \left(1 + \frac{\lambda t}{2k+2}\right) \\
&\qquad \times \left[\operatorname{sgn}(\alpha) C_{2k+1}^{-k-1/2}\left(\frac{|\alpha|}{ct}\right) - \operatorname{sgn}(\beta) C_{2k+1}^{-k-1/2}\left(\frac{|\beta|}{ct}\right)\right],
\end{aligned}
$$

where α and β are given in (3.2), and

$$
\mathbb{P}\{X(t) < x\} \\
= \begin{cases}
0, & x \in (-\infty, -ct], \\
\dfrac{1}{2} - \dfrac{e^{-\lambda t}}{2} \operatorname{sgn}(x) \displaystyle\sum_{k=0}^{\infty} \frac{(\lambda t)^{2k+1}}{(2k+1)!} \left(1 + \frac{\lambda t}{2k+2}\right) \\
\qquad\qquad \times C_{2k+1}^{-k-1/2}\left(\dfrac{|x|}{ct}\right), & x \in (-ct, ct], \\
1, & x \in (ct, +\infty).
\end{cases}
$$

Remark 3.3. We see that distribution function (3.7) has discontinuities at the points $\pm ct$ determined by the singularities concentrated at these two points. It is easy to check that distribution function (3.7) produces the expected equalities:

$$
\lim_{\varepsilon \to 0+0} \mathbb{P}\{X(t) < -ct + \varepsilon\} = \frac{e^{-\lambda t}}{2}, \qquad \mathbb{P}\{X(t) < ct\} = 1 - \frac{e^{-\lambda t}}{2}.
$$

This means that probability distribution function (3.7) is left continuous and has jumps at the terminal points $\pm ct$ of the same amplitude $e^{-\lambda t}/2$.
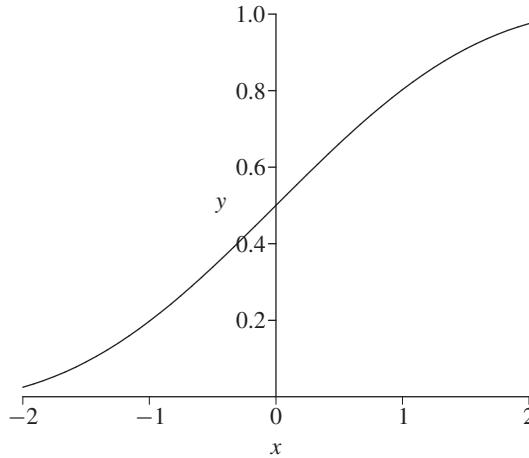
FIGURE 1: The shape of the probability distribution function (3.7) at time instant $t = 2$ (for $c = 1$ and $\lambda = 1.5$).

4. Euclidean distance between two telegraph processes

Consider two independent telegraph processes $X_1(t)$ and $X_2(t)$ that describe the stochastic motions of two particles (as described in Section 2) with finite speeds $c_1 > 0$ and $c_2 > 0$, driven by two independent Poisson processes $N_1(t)$ and $N_2(t)$ of rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. We suppose that at the initial time instant $t = 0$ both the processes $X_1(t)$ and $X_2(t)$ simultaneously start from the origin $x = 0$ of the real line $\mathbb{R}$. For the sake of definiteness, we also suppose that $c_1 \geq c_2$ (otherwise, we could merely change the numeration of the processes).

We are interested in the Euclidean distance

$$\rho(t) = |X_1(t) - X_2(t)|, \qquad t > 0, \tag{4.1}$$

between the above processes at time instant $t > 0$.

It is clear that $0 \leq \rho(t) \leq (c_1 + c_2)t$, that is, the interval $[0, (c_1 + c_2)t]$ is the support of the distribution $\mathbb{P}\{\rho(t) < r\}$ of process (4.1). The distribution of $\rho(t)$, $t > 0$, consists of two components. The singular part of the distribution is concentrated at two points, $(c_1 - c_2)t$ and $(c_1 + c_2)t$, of the support. For arbitrary $t > 0$, the process $\rho(t)$ is located at the point $(c_1 - c_2)t$ if and only if both the particles take the same initial direction (the probability of this event is $\frac{1}{2}$), and no one Poisson event occurs till time instant t (the probability of this event is $e^{-(\lambda_1 + \lambda_2)t}$). Similarly, $\rho(t)$ is located at the point $(c_1 + c_2)t$ if and only if the particles take different initial directions (the probability of this event is $\frac{1}{2}$), and no one Poisson event occurs till time instant t (the probability of this event is $e^{-(\lambda_1 + \lambda_2)t}$). Thus, we have

$$\mathbb{P}\{\rho(t) = (c_1 - c_2)t\} = \tfrac{1}{2}e^{-(\lambda_1 + \lambda_2)t}, \qquad \mathbb{P}\{\rho(t) = (c_1 + c_2)t\} = \tfrac{1}{2}e^{-(\lambda_1 + \lambda_2)t}, \qquad t > 0.$$

Therefore, the singular part, $\varphi^{\mathrm{s}}(r, t)$, of the density $\varphi(r, t)$ of the distribution $\mathbb{P}\{\rho(t) < r\}$ is the generalized function

$$\varphi^{\mathrm{s}}(r, t) = \tfrac{1}{2}e^{-(\lambda_1 + \lambda_2)t}[\delta(r - (c_1 - c_2)t) + \delta(r - (c_1 + c_2)t)], \qquad r \in \mathbb{R},\ t > 0,$$

where $\delta(x)$ is the Dirac delta function.

The remaining part of the distribution is concentrated in the area

$$M_t = (0, (c_1 - c_2)t) \cup ((c_1 - c_2)t, (c_1 + c_2)t), \qquad t > 0$$

(note that if $c_1 = c_2 = c$ then M_t transforms into the interval $(0, 2ct)$). This is the support of the absolutely continuous part of the distribution $\mathbb{P}\{\rho(t) < r\}$, corresponding to the case when at least one Poisson event occurs before time instant $t > 0$.

Our goal is to obtain an explicit formula for the probability distribution function

$$\Phi(r, t) = \mathbb{P}\{\rho(t) < r\}, \qquad r \in \mathbb{R}, \, t > 0, \tag{4.2}$$

of the Euclidean distance $\rho(t)$. The form of this distribution function is somewhat different for the cases $c_1 = c_2$ and $c_1 > c_2$ due to the fact that if $c_1 = c_2$ then the singularity point $(c_1 - c_2)t = 0$ and this is the *terminal* point, while in the case $c_1 > c_2$ this point is an *interior* point of the support. This is why in the following theorem we derive the probability distribution function in the more difficult case $c_1 > c_2$. Similar results concerning the more simple case $c_1 = c_2$ will be given separately at the end of this section.

Theorem 4.1. *Under the condition $c_1 > c_2$, probability distribution function (4.2) has the form*

$$\Phi(r, t) = \begin{cases} 0 & \text{if } r \in (-\infty, 0], \\ G(r, t) & \text{if } r \in (0, (c_1 - c_2)t], \\ Q(r, t) & \text{if } r \in ((c_1 - c_2)t, (c_1 + c_2)t], \\ 1 & \text{if } r \in ((c_1 + c_2)t, +\infty), \end{cases} \qquad r \in \mathbb{R}, \, t > 0, \, c_1 > c_2, \tag{4.3}$$

where the functions $G(r, t)$ and $Q(r, t)$ are given by

$$\begin{aligned} G(r, t) = {} & \frac{\lambda_1 e^{-(\lambda_1 + \lambda_2)t}}{2c_1} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_1 t}{2}\right)^{2k} \left(1 + \frac{\lambda_1 t}{2k+2}\right) \\ & \times \left[(c_2 t + r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(c_2 t + r)^2}{c_1^2 t^2}\right) \right. \\ & \qquad \left. - (c_2 t - r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(c_2 t - r)^2}{c_1^2 t^2}\right) \right] \\ & + \frac{\lambda_1 \lambda_2 e^{-(\lambda_1 + \lambda_2)t}}{4c_1 c_2} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_1 t}{2}\right)^{2k} \left(1 + \frac{\lambda_1 t}{2k+2}\right) \mathcal{I}_k(r, t), \end{aligned} \tag{4.4}$$

$$\begin{aligned} Q(r, t) = {} & \tfrac{1}{2}[(1 - e^{-\lambda_1 t})e^{-\lambda_2 t} + (1 - e^{-\lambda_2 t})e^{-\lambda_1 t} + e^{-(\lambda_1 + \lambda_2)t}] \\ & - \frac{\lambda_1 (c_2 t - r)e^{-(\lambda_1 + \lambda_2)t}}{2c_1} \\ & \times \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_1 t}{2}\right)^{2k} \left(1 + \frac{\lambda_1 t}{2k+2}\right) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(c_2 t - r)^2}{c_1^2 t^2}\right) \\ & - \frac{\lambda_2 (c_1 t - r)e^{-(\lambda_1 + \lambda_2)t}}{2c_2} \end{aligned}$$

$$\times \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_2 t}{2}\right)^{2k} \left(1 + \frac{\lambda_2 t}{2k+2}\right) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(c_1 t - r)^2}{c_2^2 t^2}\right)$$

$$+ \frac{\lambda_1 \lambda_2 e^{-(\lambda_1 + \lambda_2)t}}{4 c_1 c_2} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_1 t}{2}\right)^{2k} \left(1 + \frac{\lambda_1 t}{2k+2}\right) \mathcal{I}_k(r, t), \qquad (4.5)$$

with the integral factor

$$\mathcal{I}_k(r, t) = \int_{-c_2 t}^{c_2 t} \left[\beta(x, r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(\beta(x, r))^2}{c_1^2 t^2}\right)\right.$$

$$\left. - \alpha(x, r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(\alpha(x, r))^2}{c_1^2 t^2}\right)\right]$$

$$\times \left[I_0\left(\frac{\lambda_2}{c_2}\sqrt{c_2^2 t^2 - x^2}\right) + \frac{c_2 t}{\sqrt{c_2^2 t^2 - x^2}} I_1\left(\frac{\lambda_2}{c_2}\sqrt{c_2^2 t^2 - x^2}\right)\right] dx, \quad (4.6)$$

where the variables $\alpha(x, r)$ and $\beta(x, r)$ are defined by

$$\alpha(x, r) = \max\{-c_1 t, x - r\}, \qquad \beta(x, r) = \min\{c_1 t, x + r\}, \qquad x \in (-c_2 t, c_2 t), \ r \in M_t.$$

Proof. For probability distribution function (4.2), we have

$$\Phi(r, t) = e^{-(\lambda_1 + \lambda_2)t} \mathbb{P}\{\rho(t) < r \mid N_1(t) = 0, \ N_2(t) = 0\}$$

$$+ (1 - e^{-\lambda_1 t}) e^{-\lambda_2 t} \mathbb{P}\{\rho(t) < r \mid N_1(t) \geq 1, \ N_2(t) = 0\}$$

$$+ e^{-\lambda_1 t}(1 - e^{-\lambda_2 t}) \mathbb{P}\{\rho(t) < r \mid N_1(t) = 0, \ N_2(t) \geq 1\}$$

$$+ (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t}) \mathbb{P}\{\rho(t) < r \mid N_1(t) \geq 1, \ N_2(t) \geq 1\}. \qquad (4.7)$$

Let us evaluate the conditional probabilities on the right-hand side of (4.7) separately. Obviously, the first conditional probability is

$$\mathbb{P}\{\rho(t) < r \mid N_1(t) = 0, \ N_2(t) = 0\} = \begin{cases} 0 & \text{if } r \in (-\infty, (c_1 - c_2)t], \\ \frac{1}{2} & \text{if } r \in ((c_1 - c_2)t, (c_1 + c_2)t], \\ 1 & \text{if } r \in ((c_1 + c_2)t, +\infty). \end{cases} \qquad (4.8)$$

Evaluation of $\mathbb{P}\{\rho(t) < r \mid N_1(t) \geq 1, \ N_2(t) = 0\}$. We note that the following equalities for random events hold:

$$\{N_1(t) \geq 1\} = \{X_1(t) \in (-c_1 t, c_1 t)\}, \qquad \{N_2(t) = 0\} = \{X_2(t) = -c_2 t\} + \{X_2(t) = c_2 t\}.$$

Then, according to (A.8) given in Appendix A we have

$$\mathbb{P}\{\rho(t) < r \mid N_1(t) \geq 1, \ N_2(t) = 0\}$$

$$= \mathbb{P}\{\rho(t) < r \mid \{X_1(t) \in (-c_1 t, c_1 t)\} \cap (\{X_2(t) = -c_2 t\} + \{X_2(t) = c_2 t\})\}$$

$$= \frac{1}{2}[\mathbb{P}\{\rho(t) < r \mid \{X_1(t) \in (-c_1 t, c_1 t)\} \cap \{X_2(t) = -c_2 t\}\}$$

$$+ \mathbb{P}\{\rho(t) < r \mid \{X_1(t) \in (-c_1 t, c_1 t)\} \cap \{X_2(t) = c_2 t\}\}]$$

$$= \frac{1}{2} \left[\frac{\mathbb{P}\{\{X_1(t) \in (X_2(t) - r, X_2(t) + r)\} \cap \{X_1(t) \in (-c_1t, c_1t)\} \cap \{X_2(t) = -c_2t\}\}}{\mathbb{P}\{X_1(t) \in (-c_1t, c_1t)\} \mathbb{P}\{X_2(t) = -c_2t\}} \right.$$
$$\left. + \frac{\mathbb{P}\{\{X_1(t) \in (X_2(t) - r, X_2(t) + r)\} \cap \{X_1(t) \in (-c_1t, c_1t)\} \cap \{X_2(t) = c_2t\}\}}{\mathbb{P}\{X_1(t) \in (-c_1t, c_1t)\} \mathbb{P}\{X_2(t) = c_2t\}} \right]$$

$$= \frac{1}{2(1 - e^{-\lambda_1 t})} [\mathbb{P}\{\{X_1(t) \in (-c_2t - r, -c_2t + r)\} \cap \{X_1(t) \in (-c_1t, c_1t)\}\}$$
$$+ \mathbb{P}\{\{X_1(t) \in (c_2t - r, c_2t + r)\} \cap \{X_1(t) \in (-c_1t, c_1t)\}\}]$$

$$= \frac{1}{2(1 - e^{-\lambda_1 t})} [\mathbb{P}\{X_1(t) \in (\alpha, -c_2t + r)\} + \mathbb{P}\{X_1(t) \in (c_2t - r, \beta)\}],$$

where

$$\alpha = \max\{-c_1t, -c_2t - r\}, \qquad \beta = \min\{c_1t, c_2t + r\}.$$

Applying (3.1) we obtain

$$\mathbb{P}\{\rho(t) < r \mid N_1(t) \geq 1, N_2(t) = 0\}$$

$$= \frac{1}{2(1 - e^{-\lambda_1 t})}$$

$$\times \left\{ \frac{\lambda_1 e^{-\lambda_1 t}}{2c_1} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_1 t}{2}\right)^{2k} \left(1 + \frac{\lambda_1 t}{2k + 2}\right) \right.$$
$$\times \left[\beta F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{\beta^2}{c_1^2 t^2}\right) - (c_2t - r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(c_2t - r)^2}{c_1^2 t^2}\right) \right]$$

$$+ \frac{\lambda_1 e^{-\lambda_1 t}}{2c_1} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_1 t}{2}\right)^{2k} \left(1 + \frac{\lambda_1 t}{2k + 2}\right)$$
$$\times \left[(-c_2t + r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(-c_2t + r)^2}{c_1^2 t^2}\right) \right.$$
$$\left. \left. - \alpha F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{\alpha^2}{c_1^2 t^2}\right) \right] \right\}. \tag{4.9}$$

It is easy to check that

$$\beta = \begin{cases} c_2t + r & \text{if } r \in (0, (c_1 - c_2)t], \\ c_1t & \text{if } r \in ((c_1 - c_2)t, (c_1 + c_2)t], \end{cases}$$

$$\alpha = \begin{cases} -c_2t - r & \text{if } r \in (0, (c_1 - c_2)t], \\ -c_1t & \text{if } r \in ((c_1 - c_2)t, (c_1 + c_2)t]. \end{cases}$$

From these formulae we see that $\alpha = -\beta$ independently of r. Therefore, (4.9) becomes

$$\mathbb{P}\{\rho(t) < r \mid N_1(t) \geq 1, N_2(t) = 0\}$$
$$= \frac{\lambda_1 e^{-\lambda_1 t}}{2c_1(1 - e^{-\lambda_1 t})} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_1 t}{2}\right)^{2k} \left(1 + \frac{\lambda_1 t}{2k + 2}\right)$$
$$\times \left[\beta F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{\beta^2}{c_1^2 t^2}\right) \right.$$
$$\left. - (c_2t - r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(c_2t - r)^2}{c_1^2 t^2}\right) \right]. \tag{4.10}$$

If $r \in (0, (c_1 - c_2)t]$ then $\beta = c_2t + r$ and, therefore, (4.10) becomes

$$
\begin{aligned}
\mathbb{P}\{\rho(t) < r \mid N_1(t) \geq 1,\, N_2(t) = 0\} \\
= \frac{\lambda_1 e^{-\lambda_1 t}}{2c_1(1 - e^{-\lambda_1 t})} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_1 t}{2}\right)^{2k} \left(1 + \frac{\lambda_1 t}{2k+2}\right) \\
\times \left[(c_2t + r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(c_2t + r)^2}{c_1^2 t^2}\right) \right. \\
\left. - (c_2t - r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(c_2t - r)^2}{c_1^2 t^2}\right) \right]
\end{aligned}
$$

if $r \in (0, (c_1 - c_2)t]$. \hfill (4.11)

If $r \in ((c_1 - c_2)t, (c_1 + c_2)t]$ then $\beta = c_1 t$ and formula (4.10) becomes

$$
\begin{aligned}
\mathbb{P}\{\rho(t) < r \mid N_1(t) \geq 1,\, N_2(t) = 0\} \\
= \frac{1}{1 - e^{-\lambda_1 t}} \left\{ \frac{\lambda_1 e^{-\lambda_1 t}}{2c_1} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_1 t}{2}\right)^{2k} \left(1 + \frac{\lambda_1 t}{2k+2}\right) \right. \\
\times \left[c_1 t\, F\left(-k, \frac{1}{2}; \frac{3}{2}; 1\right) \right. \\
\left. \left. - (c_2t - r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(c_2t - r)^2}{c_1^2 t^2}\right) \right] \right\}
\end{aligned}
$$

if $r \in ((c_1 - c_2)t, (c_1 + c_2)t]$. \hfill (4.12)

Formula (4.12) can be simplified. In view of (3.5), we can easily show that

$$
\frac{\lambda_1 e^{-\lambda_1 t}}{2c_1} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_1 t}{2}\right)^{2k} \left(1 + \frac{\lambda_1 t}{2k+2}\right) c_1 t\, F\left(-k, \frac{1}{2}; \frac{3}{2}; 1\right) = \frac{1 - e^{-\lambda_1 t}}{2},
$$

and, therefore, (4.12) takes the form

$$
\begin{aligned}
\mathbb{P}\{\rho(t) < r \mid N_1(t) \geq 1,\, N_2(t) = 0\} \\
= \frac{1}{2} - \frac{\lambda_1 (c_2t - r) e^{-\lambda_1 t}}{2c_1(1 - e^{-\lambda_1 t})} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_1 t}{2}\right)^{2k} \left(1 + \frac{\lambda_1 t}{2k+2}\right) \\
\times F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(c_2t - r)^2}{c_1^2 t^2}\right)
\end{aligned}
$$

if $r \in ((c_1 - c_2)t, (c_1 + c_2)t]$. \hfill (4.13)

Evaluation of $\mathbb{P}\{\rho(t) < r \mid N_1(t) = 0,\, N_2(t) \geq 1\}$. It is obvious that the following relation holds:

$$
\mathbb{P}\{\rho(t) < r \mid N_1(t) = 0, N_2(t) \geq 1\} = 0 \quad \text{if } r \in (0, (c_1 - c_2)t]. \tag{4.14}
$$

Now let $r \in ((c_1 - c_2)t, (c_1 + c_2)t]$. Since

$$
\begin{aligned}
\{N_1(t) = 0\} &= \{X_1(t) = -c_1 t\} + \{X_1(t) = c_1 t\}, \\
\{N_2(t) \geq 1\} &= \{X_2(t) \in (-c_2 t, c_2 t)\},
\end{aligned}
$$

then, similarly as above, we can show that

$$\mathbb{P}\{\rho(t) < r \mid N_1(t) = 0, \ N_2(t) \geq 1\}$$
$$= \frac{1}{2(1 - e^{-\lambda_2 t})}[\mathbb{P}\{X_2(t) \in (-c_2 t, -c_1 t + r)\} + \mathbb{P}\{X_2(t) \in (c_1 t - r, c_2 t)\}].$$

Applying (3.1) we obtain

$$\mathbb{P}\{\rho(t) < r \mid N_1(t) = 0, \ N_2(t) \geq 1\}$$
$$= \frac{1}{2(1 - e^{-\lambda_2 t})}\left\{\frac{\lambda_2 e^{-\lambda_2 t}}{2c_2}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{\lambda_2 t}{2}\right)^{2k}\left(1 + \frac{\lambda_2 t}{2k+2}\right)\right.$$
$$\times\left[(-c_1 t + r)F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(-c_1 t + r)^2}{c_2^2 t^2}\right)\right.$$
$$\left.+ c_2 t F\left(-k, \frac{1}{2}; \frac{3}{2}; 1\right)\right]$$
$$+ \frac{\lambda_2 e^{-\lambda_2 t}}{2c_2}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{\lambda_2 t}{2}\right)^{2k}\left(1 + \frac{\lambda_2 t}{2k+2}\right)$$
$$\left.\times\left[c_2 t F\left(-k, \frac{1}{2}; \frac{3}{2}; 1\right) - (c_1 t - r)F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(c_1 t - r)^2}{c_2^2 t^2}\right)\right]\right\}$$
$$= \frac{1}{1 - e^{-\lambda_2 t}}\left\{\frac{\lambda_2 e^{-\lambda_2 t}}{2c_2}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{\lambda_2 t}{2}\right)^{2k}\left(1 + \frac{\lambda_2 t}{2k+2}\right)\right.$$
$$\times\left[c_2 t F\left(-k, \frac{1}{2}; \frac{3}{2}; 1\right)\right.$$
$$\left.\left.- (c_1 t - r)F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(c_1 t - r)^2}{c_2^2 t^2}\right)\right]\right\}.$$

Taking into account the fact that

$$\frac{\lambda_2 e^{-\lambda_2 t}}{2c_2}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{\lambda_2 t}{2}\right)^{2k}\left(1 + \frac{\lambda_2 t}{2k+2}\right)c_2 t F\left(-k, \frac{1}{2}; \frac{3}{2}; 1\right) = \frac{1 - e^{-\lambda_2 t}}{2}$$

we finally obtain

$$\mathbb{P}\{\rho(t) < r \mid N_1(t) = 0, \ N_2(t) \geq 1\}$$
$$= \frac{1}{2} - \frac{\lambda_2(c_1 t - r)e^{-\lambda_2 t}}{2c_2(1 - e^{-\lambda_2 t})}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{\lambda_2 t}{2}\right)^{2k}\left(1 + \frac{\lambda_2 t}{2k+2}\right)$$
$$\times F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(c_1 t - r)^2}{c_2^2 t^2}\right)$$

$$\text{if } r \in ((c_1 - c_2)t, (c_1 + c_2)t]. \tag{4.15}$$

Evaluation of $\mathbb{P}\{\rho(t) < r \mid N_1(t) \geq 1, \ N_2(t) \geq 1\}$. Since

$$\{N_1(t) \geq 1\} = \{X_1(t) \in (-c_1 t, c_1 t)\}, \qquad \{N_2(t) \geq 1\} = \{X_2(t) \in (-c_2 t, c_2 t)\},$$

then, for the fourth conditional probability on the right-hand side of (4.7), we have

$$\mathbb{P}\{\rho(t) < r \mid N_1(t) \geq 1, \ N_2(t) \geq 1\}$$
$$= \frac{\mathbb{P}\{\{\rho(t) < r\} \cap \{X_1(t) \in (-c_1t, c_1t)\} \cap \{X_2(t) \in (-c_2t, c_2t)\}\}}{\mathbb{P}\{X_1(t) \in (-c_1t, c_1t)\}\mathbb{P}\{X_2(t) \in (-c_2t, c_2t)\}}$$
$$= \frac{1}{(1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})}$$
$$\times \mathbb{P}\{\{X_1(t) \in (X_2(t) - r, X_2(t) + r)\} \cap \{X_1(t) \in (-c_1t, c_1t)\}$$
$$\cap \{X_2(t) \in (-c_2t, c_2t)\}\}$$
$$= \frac{1}{(1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})}$$
$$\times \mathbb{P}\{\{X_1(t) \in (\max\{X_2(t) - r, -c_1t\}, \min\{X_2(t) + r, c_1t\})\}$$
$$\cap \{X_2(t) \in (-c_2t, c_2t)\}\}$$
$$= \frac{1}{(1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})}$$
$$\times \int_{-c_2t}^{c_2t} \mathbb{P}\{X_1(t) \in (\alpha(x, r), \beta(x, r)) \mid X_2(t) = x\}\mathbb{P}\{X_2(t) \in dx\},$$

where

$$\alpha(x, r) = \max\{x - r, -c_1t\}, \qquad \beta(x, r) = \min\{x + r, c_1t\}.$$

In view of (3.1) and (2.4), we obtain

$$\mathbb{P}\{\rho(t) < r \mid N_1(t) \geq 1, \ N_2(t) \geq 1\}$$
$$= \frac{1}{(1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})}$$
$$\times \int_{-c_2t}^{c_2t} \left\{ \frac{\lambda_1 e^{-\lambda_1 t}}{2c_1} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_1 t}{2}\right)^{2k} \left(1 + \frac{\lambda_1 t}{2k+2}\right) \right.$$
$$\times \left[\beta(x, r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(\beta(x, r))^2}{c_1^2 t^2}\right) \right.$$
$$\left. \left. - \alpha(x, r) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(\alpha(x, r))^2}{c_1^2 t^2}\right) \right] \right\} f_2^{ac}(x, t) \, dx$$
$$= \frac{\lambda_1 \lambda_2 e^{-(\lambda_1 + \lambda_2)t}}{4c_1 c_2 (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{\lambda_1 t}{2}\right)^{2k} \left(1 + \frac{\lambda_1 t}{2k+2}\right) \mathscr{I}_k(r, t), \quad (4.16)$$

where the integral factor $\mathscr{I}_k(r, t)$ is defined by (4.6) and $f_2^{ac}(x, t)$ is the density of the absolutely continuous part of the distribution of the telegraph process $X_2(t)$ given by (2.4).

Substituting (4.8), (4.11), (4.14), and (4.16) into (4.7) we obtain the $G(r, t)$ term in distribution function (4.3) defined in the interval $r \in (0, (c_1 - c_2)t]$ and given by (4.4). Similarly, by substituting (4.8), (4.13), (4.15), and (4.16) into (4.7) we obtain the $Q(r, t)$ term in distribution function (4.3) defined in the interval $r \in ((c_1 - c_2)t, (c_1 + c_2)t]$ and given by (4.5). This completes the proof of the theorem.

Remark 4.1. We can easily see that if $r \in (0, (c_1 - c_2)t]$ then the variables $\alpha(x, r)$ and $\beta(x, r)$ take the values

$$\alpha(x, r) = x - r, \qquad \beta(x, r) = x + r, \quad \text{for } r \in (0, (c_1 - c_2)t],$$

independently of $x \in (-c_2 t, c_2 t)$. In this case the integral factor $\mathcal{l}_k(r, t)$ can be rewritten in a slightly more explicit form. In contrast, if $r \in ((c_1 - c_2)t, (c_1 + c_2)t]$ then each of these variables can take both possible values.

Remark 4.2. Taking into account that, for any $x \in (-c_2 t, c_2 t)$,

$$\alpha(x, 0) = \beta(x, 0) = x, \qquad \alpha(x, (c_1 + c_2)t) = -c_1 t, \qquad \beta(x, (c_1 + c_2)t) = c_1 t,$$
$$\alpha(x, (c_1 - c_2)t) = x - (c_1 - c_2)t, \qquad \beta(x, (c_1 - c_2)t) = x + (c_1 - c_2)t,$$

we can easily prove the following limiting relations:

$$\lim_{r \to 0+0} G(r, t) = 0, \qquad \lim_{r \to (c_1+c_2)t-0} Q(r, t) = 1 - \tfrac{1}{2} e^{-(\lambda_1 + \lambda_2)t}, \tag{4.17}$$
$$\lim_{r \to (c_1-c_2)t+0} Q(r, t) - \lim_{r \to (c_1-c_2)t-0} G(r, t) = \tfrac{1}{2} e^{-(\lambda_1 + \lambda_2)t}.$$

Formulae (4.17) show that probability distribution function (4.3) is left continuous with jumps of the same amplitude $e^{-(\lambda_1 + \lambda_2)t}/2$ at the singularity points $(c_1 \pm c_2)t$. This is in agreement with the structure of the distribution of the process $\rho(t)$ described above.

Remark 4.3. The crucial point to note when using probability distribution function (4.3) is the possibility of computing the integral term $\mathcal{l}_k(r, t)$ given in (4.6). By means of tedious computations and by applying formulae (A.6) and (A.7) given in Appendix A we can obtain a series representation of integral $\mathcal{l}_k(r, t)$; however, it has an extremely complicated and cumbersome form and is therefore omitted here. This is why for practical purposes it is more convenient to use just the integral form of factor $\mathcal{l}_k(r, t)$, which is easily computable on a personal computer (for more details, see Section 5 where we numerically evaluate the formulae obtained in Theorem 4.1).

We end this section by presenting a result related to the more simple case of equal velocities. Suppose that both the telegraph processes $X_1(t)$ and $X_2(t)$ have the same speed $c_1 = c_2 = c$. In this case the support of distribution (4.2) is the closed interval $[0, 2ct]$. The singular component of distribution has the density (as a generalized function)

$$\varphi^{\text{s}}(r, t) = \frac{e^{-(\lambda_1 + \lambda_2)t}}{2} [\delta(r) + \delta(r - 2ct)], \qquad r \in \mathbb{R}, \ t > 0,$$

concentrated at the terminal points 0 and $2ct$, while the open interval $(0, 2ct)$ is the support of the absolutely continuous part of distribution (4.2). The form of probability distribution function (4.2) for the case of equal velocities is presented in the following theorem.

Theorem 4.2. *Under the condition $c_1 = c_2 = c$, probability distribution function (4.2) has the form*

$$\Phi(r, t) = \begin{cases} 0 & \text{if } r \in (-\infty, 0], \\ H(r, t) & \text{if } r \in (0, 2ct], \\ 1 & \text{if } r \in (2ct, +\infty), \end{cases} \qquad r \in \mathbb{R}, \ t > 0, \tag{4.18}$$

where the function $H(r, t)$ is given by

$$H(r, t) = \tfrac{1}{2}[(1 - e^{-\lambda_1 t})e^{-\lambda_2 t} + e^{-\lambda_1 t}(1 - e^{-\lambda_2 t}) + e^{-(\lambda_1 + \lambda_2)t}]$$

$$- e^{-(\lambda_1 + \lambda_2)t}\left(1 - \frac{r}{ct}\right)$$

$$\times \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left[\left(\frac{\lambda_1 t}{2}\right)^{2k+1}\left(1 + \frac{\lambda_1 t}{2k+2}\right) + \left(\frac{\lambda_2 t}{2}\right)^{2k+1}\left(1 + \frac{\lambda_2 t}{2k+2}\right)\right]$$

$$\times F\left(-k, \frac{1}{2}; \frac{3}{2}; \left(1 - \frac{r}{ct}\right)^2\right)$$

$$+ \frac{\lambda_1 \lambda_2}{4c^2} e^{-(\lambda_1 + \lambda_2)t} \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{\lambda_1 t}{2}\right)^{2k}\left(1 + \frac{\lambda_1 t}{2k+2}\right)\mathcal{J}_k(r, t)$$

with the integral factor

$$\mathcal{J}_k(r, t) = \int_{-ct}^{ct}\left[\beta(x, r)F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(\beta(x, r))^2}{c^2 t^2}\right) - \alpha(x, r)F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{(\alpha(x, r))^2}{c^2 t^2}\right)\right]$$

$$\times \left[I_0\left(\frac{\lambda_2}{c}\sqrt{c^2 t^2 - x^2}\right) + \frac{ct}{\sqrt{c^2 t^2 - x^2}} I_1\left(\frac{\lambda_2}{c}\sqrt{c^2 t^2 - x^2}\right)\right]dx, \quad (4.19)$$

where the variables $\alpha(x, r)$ and $\beta(x, r)$ are defined by

$$\alpha(x, r) = \max\{-ct, x - r\}, \qquad \beta(x, r) = \min\{ct, x + r\}, \qquad x \in (-ct, ct), \ r \in (0, 2ct).$$

   Proof. The proof is similar to that of Theorem 4.1 and is therefore omitted.

Remark 4.4. The results obtained in Theorems 4.1 and 4.2 may be useful for analysing the distribution of the difference between two independent telegraph processes $X_1(t)$ and $X_2(t)$. While the distribution of the difference (as well as of the sum) is given by a respective convolution, the evaluation of such a convolution is a very difficult (and, maybe, impracticable) problem owing to a fairly complicated form to the probability law of the telegraph process (see (2.3) for its density or (3.7) for its distribution function). Let $F_D(r, t) = \mathbb{P}\{D(t) < r\}$ denote the probability distribution function of the difference $D(t) = X_1(t) - X_2(t)$. The interval $[-(c_1 + c_2)t, (c_1 + c_2)t]$ is the support of the distribution of $D(t)$ with the two singularity points $\pm(c_1 + c_2)t$ in the case $c_1 \neq c_2$ and the three singularity points $0, \pm 2ct$ in the case of equal velocities $c_1 = c_2 = c$.

   Then the distribution function $F_D(r, t)$ of the difference $D(t)$, and the distribution function $\Phi(r, t)$ of the Euclidean distance $\rho(t)$ between $X_1(t)$ and $X_2(t)$ are connected through the functional relation

$$F_D(r, t) - F_D(-r, t) - \mathbb{P}\{D(t) = -r\} = \Phi(r, t), \qquad r \in \mathbb{R}, \ t > 0.$$

Note that the term $\mathbb{P}\{D(t) = -r\}$ takes a nonzero value if and only if r is the singular point of the distribution of process $D(t)$. For regular r, this term vanishes.

5. Some numerical results

   While probability distribution functions (4.3) and (4.18) have fairly complicated analytical forms, they can, nevertheless, be approximately evaluated with good accuracy using a standard

TABLE 1: Values of $G(r, 3)$ on the subinterval $r \in (0, 6]$.

r	$G(r, 3)$	r	$G(r, 3)$	r	$G(r, 3)$
0.2	0.0271	2.2	0.2916	4.2	0.5269
0.4	0.0541	2.4	0.3168	4.4	0.5480
0.6	0.0811	2.6	0.3417	4.6	0.5686
0.8	0.1080	2.8	0.3663	4.8	0.5888
1.0	0.1348	3.0	0.3905	5.0	0.6083
1.2	0.1614	3.2	0.4143	5.2	0.6274
1.4	0.1879	3.4	0.4377	5.4	0.6459
1.6	0.2142	3.6	0.4607	5.6	0.6639
1.8	0.2402	3.8	0.4832	5.8	0.6813
2.0	0.2660	4.0	0.5053	6.0	0.6983

TABLE 2: Values of $Q(r, 3)$ on the subinterval $r \in (6, 12]$.

r	$Q(r, 3)$	r	$Q(r, 3)$	r	$Q(r, 3)$
6.2	0.7146	8.2	0.8472	10.2	0.9294
6.4	0.7302	8.4	0.8576	10.4	0.9350
6.6	0.7455	8.6	0.8674	10.6	0.9405
6.8	0.7601	8.8	0.8768	10.8	0.9461
7.0	0.7741	9.0	0.8855	11.0	0.9510
7.2	0.7877	9.2	0.8945	11.2	0.9554
7.4	0.8006	9.4	0.9019	11.4	0.9589
7.6	0.8131	9.6	0.9093	11.6	0.9631
7.8	0.8250	9.8	0.9170	11.8	0.9673
8.0	0.8364	10.0	0.9233	12.0	0.9704

numerical package (such as MATHEMATICA® or MAPLE®) on a personal computer. As noted in Remark 4.3, the crucial point is the evaluation of the integral factors $\mathcal{I}_k(r, t)$ given in (4.6) (for $c_1 > c_2$) and $\mathcal{J}_k(r, t)$ given in (4.19) (for $c_1 = c_2 = c$).

To approximately evaluate the series of the functions given in (4.4) and (4.5), we do not need to compute the integral term in (4.6) for all $k \geq 0$. We note that each series contains the factor $1/(k!)^2$, providing very fast convergence. In fact, we can see that, if we take only five terms of each series in the functions $G(r, t)$ and $Q(r, t)$, their approximate values stabilize at the fourth digit.

Let us set

$$\lambda_1 = 2, \qquad \lambda_2 = 1, \qquad c_1 = 4, \qquad c_2 = 2, \qquad t = 3 \qquad (5.1)$$

in our model. In this case, the support of the distribution is the interval $[0, 18]$ with the two singularity points $r = 6$ (the interior point of the support) and $r = 18$ (the terminal point). The results of numerical analysis on probability distribution function (4.3) with parameters (5.1) for $G(r, 3)$ defined on the subinterval $r \in (0, 6]$ and $Q(r, 3)$ defined on the subinterval $r \in (6, 12]$ are respectively given in Tables 1 and 2.

Note that in evaluating these functions we take only seven terms in the series. Also, note that, although $Q(r, 3)$ is defined on the whole interval $(6, 18]$, we consider it only over $(6, 12]$ because it has very small increments over $(12, 18]$.

We now consider the behavior of the probability distribution function $\Phi(r, 3)$ in the neighborhoods of singularity points. As noted above, for the parameters given in (5.1), $\Phi(r, 3)$ has

two singularity points, namely, $r = 6$ and $r = 18$. At the first (interior) point $r = 6$, (4.4) and (4.5) yield the values

$$G(6, 3) \approx 0.698\,298, \qquad \lim_{r \to 6+0} Q(r, 3) \approx 0.698\,360,$$

and, therefore, their difference is

$$\lim_{r \to 6+0} Q(r, 3) - G(6, 3) \approx 0.698\,360 - 0.698\,298 = 0.000\,062.$$

We see that this difference is equal to the value of the jump amplitude at this singularity point: $\mathrm{e}^{-9}/2 \approx 0.000\,062$.

Similarly, at the second (terminal) singularity point $r = 18$, (4.5) yields the value $Q(18, 3) \approx 0.999\,938$ and, therefore, the difference is

$$1 - Q(18, 3) \approx 1 - 0.999\,938 = 0.000\,062.$$

This is equal to the value of the jump amplitude at this singularity point: $\mathrm{e}^{-9}/2 \approx 0.000\,062$.

Note that in evaluating $G(r, 3)$ and $Q(r, 3)$ at the singularity points $r = 6$ and $r = 18$ we take 15 terms in each series because we need more accuracy in this case.

Suppose that every time the particles close in the distance less than $r = 0.6$, they can begin to interact with probability 0.3. The probability of interaction starting at time instant $t = 3$ is

$$\mathbb{P}\{\rho(3) < 0.6\} \cdot 0.3 = G(0.6, 3) \cdot 0.3 = 0.0811 \cdot 0.3 = 0.02433.$$

Here we have used the value of $G(r, 3)$ for $r = 0.6$ given in Table 1.

Remark 5.1. The model considered in this paper can generate some other interesting problems. The obtained results enable us to compute the probability of starting the interaction at an arbitrary time instant $t > 0$. However, for practical needs, it is more important to evaluate the probability of interaction starting *before* some fixed time point. Let $T > 0$ be an arbitrary time instant, and let k_T^r denote the random variable counting how many times during the time interval $(0, T)$ the distance between the particles was less than some given $r > 0$. The distribution of the nonnegative integer-valued random variable k_T^r is of special importance because it would enable us to evaluate the probability of interaction starting before time T.

Appendix A

In this appendix we give two auxiliary lemmas concerning some indefinite integrals of the modified Bessel functions and a useful formula related to conditional probabilities.

Lemma A.1. *For arbitrary $q \geq 0$, $p > 0$ the following formulae hold:*

$$\int x^n I_0\left(q\sqrt{p^2 - x^2}\right) \mathrm{d}x$$
$$= \frac{x^{n+1}}{n+1} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{pq}{2}\right)^{2k} F\left(-k, \frac{n+1}{2}; \frac{n+3}{2}; \frac{x^2}{p^2}\right) + \psi_1, \tag{A.1}$$

$$\int x^n \frac{I_1\left(q\sqrt{p^2 - x^2}\right)}{\sqrt{p^2 - x^2}} \mathrm{d}x$$
$$= \frac{x^{n+1}}{p(n+1)} \sum_{k=0}^{\infty} \frac{1}{k!\,(k+1)!} \left(\frac{pq}{2}\right)^{2k+1} F\left(-k, \frac{n+1}{2}; \frac{n+3}{2}; \frac{x^2}{p^2}\right) + \psi_2, \tag{A.2}$$

for $n \geq 0$ and $|x| \leq p$, where ψ_1 and ψ_2 are arbitrary functions not depending on x.

Proof. Let us check (A.1). First we show that the series on the right-hand side of (A.1) converges uniformly with respect to $x \in [-p, p]$. To prove this, we need the following uniform (in z) estimate:

$$\left| F\left(-k, \frac{n+1}{2}; \frac{n+3}{2}; z\right) \right| \le 2^k, \qquad |z| \le 1, \, n \ge 0, \, k \ge 0. \tag{A.3}$$

Using the well-known formulae for the Pochhammer symbol,

$$(-k)_s = \frac{(-1)^s k!}{(k-s)!}, \quad k \ge 0, \, 0 \le s \le k, \qquad \frac{(a)_s}{(a+1)_s} = \frac{a}{a+s}, \quad a > 0, \, s \ge 0,$$

we obtain (for $|z| \le 1$, $n \ge 0$, and $k \ge 0$)

$$\begin{aligned}
\left| F\left(-k, \frac{n+1}{2}; \frac{n+3}{2}; z\right) \right| &= \left| \sum_{s=0}^{k} \frac{(-k)_s ((n+1)/2)_s}{((n+1)/2+1)_s} \frac{z^s}{s!} \right| \\
&= \left| \sum_{s=0}^{k} (-1)^s \frac{k!}{s!\,(k-s)!} \frac{n+1}{n+2s+1} z^s \right| \\
&\le \sum_{s=0}^{k} \frac{k!}{s!\,(k-s)!} \\
&= 2^k,
\end{aligned}$$

proving (A.3). Now applying estimate (A.3), we obtain

$$\left| \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{pq}{2}\right)^{2k} F\left(-k, \frac{n+1}{2}; \frac{n+3}{2}; \frac{x^2}{p^2}\right) \right| \le \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{pq}{2}\right)^{2k} 2^k = I_0(pq\sqrt{2}) < \infty,$$

proving the uniform convergence in $x \in [-p, p]$ of the series in (A.1). From this fact, it follows that one may separately differentiate each term of the series on the right-hand side of (A.1). Thus, differentiating the expression on the right-hand side of (A.1) with respect to x we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^{n+1}}{n+1} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{pq}{2}\right)^{2k} F\left(-k, \frac{n+1}{2}; \frac{n+3}{2}; \frac{x^2}{p^2}\right) \right] \\
&= \frac{1}{n+1} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{pq}{2}\right)^{2k} \frac{\mathrm{d}}{\mathrm{d}x}\left[x^{n+1} F\left(-k, \frac{n+1}{2}; \frac{n+3}{2}; \frac{x^2}{p^2}\right) \right] \\
&= \frac{1}{n+1} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{pq}{2}\right)^{2k} \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{s=0}^{k} \frac{(-1)^s k!\,(n+1)}{s!\,(k-s)!\,(n+2s+1)} \frac{x^{n+2s+1}}{p^{2s}} \right] \\
&= x^n \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{pq}{2}\right)^{2k} \left[\sum_{s=0}^{k} (-1)^s \binom{k}{s} \left(\frac{x^2}{p^2}\right)^s \right]
\end{aligned}$$

$$= x^n \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{pq}{2}\right)^{2k} \left(1 - \frac{x^2}{p^2}\right)^k$$

$$= x^n \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{q}{2}\right)^{2k} (p^2 - x^2)^k$$

$$= x^n \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{q\sqrt{p^2 - x^2}}{2}\right)^{2k}$$

$$= x^n I_0(q\sqrt{p^2 - x^2}),$$

yielding the integrand on the left-hand side of (A.1). Formula (A.2) can be checked in the same manner. This completes the proof.

By setting $n = 0$ in (A.1) and (A.2), we obtain

$$\int I_0(q\sqrt{p^2 - x^2}) \, dx$$
$$= x \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{pq}{2}\right)^{2k} F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{x^2}{p^2}\right) + \psi_1, \qquad |x| \le p, \tag{A.4}$$

$$\int \frac{I_1(q\sqrt{p^2 - x^2})}{\sqrt{p^2 - x^2}} \, dx$$
$$= \frac{x}{p} \sum_{k=0}^{\infty} \frac{1}{k!\,(k+1)!} \left(\frac{pq}{2}\right)^{2k+1} F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{x^2}{p^2}\right) + \psi_2, \qquad |x| \le p. \tag{A.5}$$

Applying Lemma A.1 we obtain, for arbitrary real a,

$$\int (a \pm x)^n I_0(q\sqrt{p^2 - x^2}) \, dx$$
$$= \sum_{m=0}^{n} (\pm 1)^m \binom{n}{m} a^{n-m} \frac{x^{m+1}}{m+1} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{pq}{2}\right)^{2k} F\left(-k, \frac{m+1}{2}; \frac{m+3}{2}; \frac{x^2}{p^2}\right) + \psi_1, \tag{A.6}$$

$$\int (a \pm x)^n \frac{I_1(q\sqrt{p^2 - x^2})}{\sqrt{p^2 - x^2}} \, dx$$
$$= \frac{1}{p} \sum_{m=0}^{n} (\pm 1)^m \binom{n}{m} a^{n-m} \frac{x^{m+1}}{m+1} \sum_{k=0}^{\infty} \frac{1}{k!\,(k+1)!} \left(\frac{pq}{2}\right)^{2k+1}$$
$$\times F\left(-k, \frac{m+1}{2}; \frac{m+3}{2}; \frac{x^2}{p^2}\right) + \psi_2,$$

for $n \ge 0$ and $|x| \le p$, \hfill (A.7)

The next lemma yields a useful formula for the probabilities conditioned on pairwise independent random events that has been used in the proof of Theorem 4.1.

Lemma A.2. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space, and let* $A, B, C, D \in \mathcal{F}$ *be random events such that* B *is independent of* C *and* D, $C \cap D = \varnothing$, $P(C) = P(D) \neq 0$, *and* $P(B) \neq 0$. *Then*

$$P(A \mid B(C + D)) = \tfrac{1}{2}[P(A \mid BC) + P(A \mid BD)]. \tag{A.8}$$

Proof. The proof straightforwardly follows by applying well-known formulae of elementary probability theory connecting the conditional and joint probabilities of random events.

Acknowledgements

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