A BROUWER TYPE COINCIDENCE THEOREM

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1. Introduction. Brouwer's celebrated fixed point theorem states that every map of a closed n-cell into itself has a fixed point.

A similar theorem is here proved for coincidences between a pair of maps $(f, g): I^n \to I^n$, where I^n denotes a closed n-cell (i.e. a homeomorph of the n-ball) and a coincidence is a point $x \in I^n$ for which $f(x) = g(x)$. That two arbitrary maps $(f, g): I^n \to I^n$ need not have a coincidence is shown by the pair $f: I^n \to y_0$, $g: I^n \to y_1$, where $y_0, y_1 \in I^n$ and $y_0 \neq y_1$. More generally, one can immediately construct a map g so that (f, g) is coincidence free if f is not surjective.

Therefore some restriction has to be imposed on the pair (f, g) for a 'Brouwer type' coincidence theorem. We prove

THEOREM 1. If $f: I^n \to I^n$ maps the boundary of I^n essentially onto itself, then every pair $(f, g): I^n \to I^n$ has a coincidence.

It includes Brouwer's fixed point theorem as the special case where f is the identity. A standard proof of Brouwer's result can be adapted to yield Theorem 1, but we use a different demonstration which renders in the fixed point case a probably new - although not a simpler - proof of Brouwer's theorem.

Holsztyński [1] has recently called a map $f: X \to Y$ 'universal for all maps of X into Y' if for all $g: X \to Y$ there

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exists an $x \in X$ such that $f(x) = g(x)$. Using his terminology, our theorem can be formulated as

THEOREM 1'. If $f: I^n \to I^n$ maps the boundary of I^n essentially onto itself, then f is universal for all maps of I^n into itself. The condition that f maps the boundary $\dot{I}^n$ of I^n essentially onto itself is sufficient, but not necessary for f to be universal, as is shown by the following example: Let $I^n = \{x = (x_1, x_2, \ldots, x_n) \in R^n; -1 \le x_i \le 1 \text{ for } i = 1, 2, \ldots, n\}$, and define $f: I^n \to I^n$ by

$$f(x_1, x_2, \ldots, x_n) = (1 + 2x_1, x_2, \ldots, x_n) \text{ if } -1 \le x_1 \le 0,$$

$$= (1 - 2x_1, x_2, \ldots, x_n) \text{ if } 0 \le x_1 \le 1.$$

Then f maps $\dot{I}^n$ onto itself with degree zero, but is universal for all maps of I^n into itself in consequence of Brouwer's fixed point theorem.

2. <u>A Lemma.</u> A pair of maps $(f, g): I^n \to I^n$ determines a product map $f \times g: I^n \to I^n \times I^n$, and a coincidence of (f, g) is a point $x \in I^n$ for which $(f \times g)(x) \subset \Delta$, where Δ denotes the diagonal of $I^n \times I^n$. We establish first some results about the homotopy of the deleted product $I^n \times I^n - \Delta$ needed in the proof of Theorem 1. The symbol $X - A$ is used for the difference $X - (A \cap X)$, where not necessarily $A \subseteq X$. The n-cell I^n is described in terms of a homeomorphic n-ball

$$\{x = (x_1, x_2, \ldots, x_n) \in R^n; \; x_1^2 + x_2^2 + \ldots + x_n^2 \le 1\},$$

and $c = (0, 0, \ldots, 0)$ is its centre.

Consider the commutative diagram

$$(1) \qquad \begin{array}{ccc} \pi_{n-1}(\dot{I}^n \times c) & \xrightarrow{k_*} & \pi_{n-1}(I^n \times I^n - \Delta) \\ & {\scriptstyle i_*} \searrow \quad \nearrow {\scriptstyle j_*} & \\ & \pi_{n-1}(\dot{I}^n \times I^n - \Delta) & \end{array}$$

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in which i_*, j_*, and k_* are induced by injections.

LEMMA. The homomorphisms in (1) are isomorphisms for all $n > 1$.

Proof. i) Consider the homotopy

$$p_t : \dot{I}^n \times I^n - \Delta \to \dot{I}^n \times I^n - \Delta$$

given by

$$p_t(y_1, \lambda c + (1 - \lambda)y_2) = (y_1, (t + \lambda - \lambda t)c + (1-t)(1-\lambda)y_2),$$

where

$y_1, y_2 \in \dot{I}^n$, $y_1 \neq y_2$, $0 \leq \lambda \leq 1$, $0 \leq t \leq 1$. Then p_0 is the identity and $p = p_1$ is a deformation retraction of $\dot{I}^n \times I^n - \Delta$ onto $\dot{I}^n \times c$.

Hence i_* is the inverse of the isomorphism

(2) $$p_* : \pi_{n-1}(\dot{I}^n \times I^n - \Delta) \to \pi_{n-1}(\dot{I}^n \times c).$$

ii) Consider the homotopy

$$q_t : I^n \times I^n - \Delta \to I^n \times I^n - \Delta$$

given by

$$q_t(\lambda x + (1 - \lambda)y, x) = (\lambda(1 - t)x + (1 - \lambda + \lambda_t)y, x),$$

where

$x \in I^n$, $y \in \dot{I}^n$, $y \neq x$, $0 \leq \lambda < 1$, $0 \leq t \leq 1$. Then q_0 is the identity and $q = q_1$ is a deformation retraction of $I^n \times I^n - \Delta$ onto $\dot{I}^n \times I^n - \Delta$. Hence j_* is the inverse of the isomorphism

$$q_* : \pi_{n-1}(I^n \times I^n - \Delta) \to \pi_{n-1}(\dot{I}^n \times I^n - \Delta).$$

iii) As i_* and j_* are isomorphisms, so is k_*.

3. Proof of Theorem 1. If $n = 1$, the theorem follows

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quickly from a direct argument using the graph of $f \times g$.

Assume now that $n > 1$, and let $(f, g): I^n \to I^n$ be a pair of maps such that $f|\dot{I}^n$ is an essential map onto $\dot{I}^n$. We prove that it cannot be coincidence free.

If (f, g) has a coincidence on $\dot{I}^n$, then there is nothing left to show. Otherwise (f, g) determines a map

$$f \times g: I^n, \dot{I}^n \to I^n \times I^n, \dot{I}^n \times I^n - \Delta,$$

and the restriction $f \times g|\dot{I}^n$ defines an element $\alpha \in \pi_{n-1}(\dot{I}^n \times I^n - \Delta)$.

Assume by way of contradiction that (f, g) is coincidence free, so that $f \times g$ is in fact a map

$$f \times g: I^n, \dot{I}^n \to I^n \times I^n - \Delta, \dot{I}^n \times I^n - \Delta.$$

Then $j_* \alpha \in \pi_{n-1}(I^n \times I^n - \Delta)$ is the zero element, as the map $j \circ (f \times g|\dot{I}^n)$ has an extension over I^n.

But it follows from the construction of p_* in (2) that $p_* \alpha \in \pi_{n-1}(\dot{I}^n \times c)$ is the element determined by $f|\dot{I}^n: \dot{I}^n \to \dot{I}^n$, and hence non-zero as $f|\dot{I}^n$ is essential. The lemma asserts that $k_* p_* \alpha = k_* i_*^{-1} \alpha = j_* \alpha$ and that k_* is an isomorphism. This yields a contradiction, and therefore (f, g) must have a coincidence.

REFERENCE

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