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SIMULTANEOUS ITERATION BY ENTIRE OR RATIONAL FUNCTIONS AND THEIR INVERSES

I. N. BAKER and ZALMAN RUBINSTEIN

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Abstract

For a non-constant entire or rational function f normalized by $f(0) = 0, f'(0) = 1, f''(0) \neq 0$, which is not a Möbius transformation, the existence of a sequence $\{z_n\}_{n=-\infty}^{n=+\infty}$ is established which has the properties $f(z_n) = z_{n+1}$, $\lim_{n\to\infty} z_n = \lim_{n\to-\infty} z_n = 0$. The result certainly implies $f(0) = 0, |f'(0)| = 1$, so these conditions cannot be omitted. The condition $f''(0) \neq 0$ can be replaced by $f^{(k)}(0) \neq 0$ for some $k \geqslant 2$.

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Introduction

The theory of fixpoints and iterations of functions is of continuing interest both from the theoretical and numerical points of view. For example the volume [7] is entirely devoted to this topic. Iterations of entire functions were first studied seriously in [5] and more recently for example in [1], [6, chapter 2] and [8]. Iteration of polynomials and rational functions was treated in [4] with occasional more recent contributions such as [3].

Although iteration of the inverse functions of polynomials is discussed in the fundamental paper [4], the results seem to be relatively unknown. In the present paper some of these results are used to solve a problem which was raised recently by A. Shields and C. Pearcy (oral communication). They asked whether there exists a polynomial f and an infinite sequence $\{z_n\}_{n=-\infty}^{\infty}$ of complex numbers such that $\lim_{n\to\infty} z_n = \lim_{n\to-\infty} z_n = 0$ and $z_{n+1} = f(z_n)$ for all n.

We give a positive solution to this problem not only for polynomials but for entire or rational functions also. The second author first found the example

$f(z) = z + z^2$ by elementary arguments. This example is now included in the more general:

THEOREM. *Let f be a non-constant entire or rational function which is not a Moebius transformation. Suppose that $f(0) = 0$, $f'(0) = 1$ and $f''(0) \neq 0$. Then there exists a sequence $\{z_n\}_{n=-\infty}^{\infty}$ of complex numbers such that*

(i) $f(z_n) = z_{n+1}$ *and*

(ii) $\lim_{n\to+\infty} z_n = \lim_{n\to-\infty} z_n = 0$.

REMARK. The condition $f''(0) \neq 0$ is not essential and can be replaced by $f^{(k)}(0) \neq 0$ for some k at the cost of complicating the description required for the proof. Without loss of generality we may assume that $f''(0) > 0$ since a scale change $z_n \to t_n = \lambda^{-1} z_n$, λ constant, replaces $f(t)$ by $\lambda^{-1} f(\lambda t)$. Denote by f^n the nth iterate of f, $n \geqslant 0$, and let $\mathfrak{F} = \mathfrak{F}(f)$ be the set of those points z in whose neighborhood the sequence $\{f^n\}$ is not a normal family. It was proved by Fatou in [4, 5] *that for functions f which are entire or rational but not Möbius transformations*:

I. *$\mathfrak{F}(f)$ is a non-empty perfect set.*

II. *Given any $\beta \in \mathfrak{F}$ then if z is not one of the at most two exceptional values which depend only on f and not on β, there is a sequence of integers n_k and a sequence of complex numbers z_k such that*

$$n_k \to \infty, \quad z_k \to \beta, \quad f^{n_k}(z_k) = z.$$

Note also that under the assumptions of the theorem $0 \in \mathfrak{F}(f)$. Indeed we have $f^n(z) = z + n a_2 z^2 + \cdots$, $n \geqslant 0$, valid in a neighborhood of the origin. If a sequence f^{n_k} is uniformly convergent to g in a neighborhood of 0 we must have $g(0) = 0$ so that g is analytic at 0 and $[f^{n_k}(z)]'' \to g''(z)$ at $z = 0$, which conflicts with $n_k a_2 \to \infty$.

Note also the following

LEMMA [2, Lemma 9, case $m = 1$]. *If $g(z) = z + a_2 z^2 + \cdots$, $a_2 = \sigma e^{i\alpha}$, $\sigma > 0$, α real, is analytic at $z = 0$, then for a given θ with $0 < \theta < \pi/2$ and for all sufficiently small $\rho > 0$ the iterates g^n of g are defined in*

$$D(\theta, \rho) = \{z\colon 0 < |z| < \rho,\ -\gamma + \theta - \pi < \arg z < -\gamma - \theta + \pi\},$$

where $\gamma = \alpha + \pi$. Further, $g^n(z) \to 0$ locally uniformly in $D(\theta, \rho)$. This means that for $z \in D(\theta, \rho)$ all the values $g^n(z)$ lie in $D(\theta, \rho)$. It turns out that the $g^n(z)$ approach 0 *from the direction* $\arg z = -\gamma$.

PROOF OF THE THEOREM. By assumption the expansion of f at 0 is $f(z) = z + \sigma z^2 + \cdots$, $\sigma > 0$. Choose a fixed θ in $0 < \theta < \pi/2$ and apply the lemma with

$\alpha = 0$. There is $\rho > 0$ such that in

$$D(\theta, \rho) = \{z : 0 < |z| < \rho, \theta < \arg z < 2\pi - \theta\}$$

we have $f^n(z) \to 0$ locally uniformly. Thus $D(\rho, \theta)$ belongs to the complement of $\mathfrak{F}(f)$.

The inverse function f^{-1} has a branch F whose expansion near 0 is

$$F(z) = z - \sigma z^2 + \cdots,$$

which converges for say $|z| < \delta$. Applying the lemma to F with $\alpha = \pi$ shows that for sufficiently small positive ρ' the iterates F^n, $n \geqslant 1$ of F are defined in

$$D'(\theta, \rho') = \{z : 0 < |z| < \rho', \theta - \pi < \arg z < \pi - \theta\}.$$

Again $F^n(z) \to 0$ as $n \to \infty$.

Take any z_0 in $D(\theta, \rho)$ which is not exceptional in the sense of II. Since $0 \in \mathfrak{F}$ and $\mathfrak{F}$ is perfect there are points in $\mathfrak{F} - \{0\}$ arbitrarily near 0 and these points are not in $D(\theta, \rho)$. Thus every $D'(\theta, \rho')$ must contain such points. By II there is therefore an integer $k > 0$ and a point z_{-k} in $D'(\theta, \rho')$ such that $f^k(z_{-k}) = z_0$. Put

$$z_n = \begin{cases} f^m(z_{-k}) & \text{if } n \geqslant -k \text{ so that } m = n + k \geqslant 0. \\ F^m(z_{-k}) & \text{if } n < -k \text{ so that } m = -(n+k) \geqslant 0. \end{cases}$$

In particular $z_0 = f^k(z_{-k})$ which agrees with our earlier notation. It is immediate that $f(z_n) = z_{n+1}$ for all integers n. Further

$$\lim_{n \to -\infty} z_n = \lim_{n \to \infty} F^m(z_{-k}) = 0 \quad \text{since } z_{-k} \in D'(\theta, \rho'),$$

$$\lim_{n \to \infty} z_n = \lim_{m \to \infty} f^{n+k}(z_{-k}) = \lim_{n \to \infty} f^n(z_0) = 0 \quad \text{since } z_0 \in D(\theta, \rho).$$

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Department of Mathematics
Imperial College
London S.W. 7, U.K.

University of Michigan
Ann Arbor, Michigan 48109

University of Haifa
Haifa, Israel