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THE GENERALIZED HOMOLOGY OF PRODUCTS

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Abstract. We construct a spectral sequence that computes the generalized homology $E_*(\prod X^\alpha)$ of a product of spectra. The E_2-term of this spectral sequence consists of the right derived functors of product in the category of E_*E-comodules, and the spectral sequence always converges when E is the Johnson-Wilson theory $E(n)$ and the X^α are L_n-local. We are able to prove some results about the E_2-term of this spectral sequence; in particular, we show that the $E(n)$-homology of a product of $E(n)$-module spectra X^α is just the comodule product of the $E(n)_*X^\alpha$. This spectral sequence is relevant to the chromatic splitting conjecture.

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Introduction. The basic tools of computation in algebraic topology are homology theories. Homology theories preserve coproducts, but can behave very badly on products. There are examples of homology theories E and sets of spectra (generalized spaces) $\{X^\alpha\}$, for which $E_*X^\alpha = 0$ for all i and yet $E_*(\prod_\alpha X^\alpha) \neq 0$. Indeed, we can take $E = H\mathbb{Q}$, rational homology, where we have $(H\mathbb{Q})_*(H\mathbb{Z}/p^k) = 0$ for all k, but

$$(H\mathbb{Q})_*\left(\prod_k H\mathbb{Z}/p^k\right) = \left(\prod_k \mathbb{Z}/p^k\right) \otimes \mathbb{Q} \neq 0$$

since, for example, the element $(1, 1, 1, \ldots)$ is not torsion.

Despite this counterexample, in this paper we build a spectral sequence that converges to $E_*(\prod X^\alpha)$ in good cases. The most important good case is when $E = E(n)$, the Johnson-Wilson theory of great importance in stable homotopy theory. The E_2 term of this spectral sequence is made up of the right derived functors of product applied to $\{E_*X^\alpha\}$. Of course, the product is exact in the category of E_*-modules, so these derived functors are instead taken in the category of E_*E-comodules, where products remain mysterious.

The usefulness of this spectral sequence will depend on our knowledge of its E_2-term. At this point, the author knows very little about the derived functors $\prod^s_{E(n)_*E(n)} M^\alpha$ of product in the category of $E(n)_*E(n)$-comodules. The most important conjecture about them is that $\prod^s_{E(n)_*E(n)} M^\alpha = 0$ for all $s \geq N$ for some N, so that the spectral sequence has a horizontal vanishing line at the E_2 term (we show that the spectral sequence does have a horizontal vanishing line at some E_r term). We expect that N is very close to n itself.

We do prove that derived functors of product can be computed using relatively injective resolutions, such as the cobar complex, rather than honest injective resolutions.

It follows that

$$E(n)_*\Big(\prod X^\alpha\Big) \cong \prod_{E(n)_*E(n)} E(n)_*X^\alpha$$

for a family of $E(n)$-module spectra X^α. We also construct a spectral sequence relating derived functors of product in the category of $E(n)_*E(n)$-comodules to derived functors of product in the category of BP_*BP-comodules. The category of BP_*BP-comodules is easier to cope with since BP_*BP is connective and free over BP_*. These results give the author hope that these derived functors will be understood at some point, though at the moment he does not even understand them in the simple case of $E(1)_*E(1)$-comodules.

The reason for the author's interest in this spectral sequence is the chromatic splitting conjecture [**5**] of Mike Hopkins. Recall that the simplest form of the chromatic splitting conjecture is that $K(n-1)_*L_{K(n)}X$ is a direct sum of two copies of $K(n-1)_*X$, for X a finite p-complete spectrum. Also recall that $L_{K(n)}X$ is a homotopy inverse limit $\operatorname{holim}_I(L_nX \wedge S/I)$ analogous to completion at the ideal $(p, v_1, \dots, v_{n-1})$. This result is due to Hopkins; a precise statement of it can be found in [**8**, Proposition 7.10]. Therefore, if one has a spectral sequence for the $E(n-1)$-homology of a homotopy inverse limit, one might be able to compute $E(n-1)_*(L_{K(n)}X)$ and therefore $K(n-1)_*(L_{K(n)}X)$.

This approach to the chromatic splitting conjecture is due to Mike Hopkins, and is based on the work of Paul Goerss [**4**], who constructed a spectral sequence for the mod p homology of a homotopy inverse limit of spaces. Hopkins suggested this idea to Hal Sadofsky and the author after a talk by Goerss. Sadofsky has constructed a spectral sequence for the $E(n)$-homology of a homotopy inverse limit, as envisioned by Hopkins, and has proved some results about it that are relevant to the chromatic splitting conjecture. Unfortunately, Sadofsky has not yet made a preprint of his work available.

The author decided instead to begin with the simpler case of products, though the methods used in this paper can also be used to construct a version of Sadofsky's spectral sequence. To the author's knowledge, Sadofsky has not considered products. But the author acknowledges his heavy debt to the work of Sadofsky. He also would like to thank Mike Hopkins for his original suggestion, and Paul Goerss for his paper [**4**], without which this paper would never have been written.

1. The modified Adams tower. The first step in constructing a spectral sequence is to resolve the object one is considering. In our case, the resolution we need is called the modified Adams tower and is due to Devinatz and Hopkins [**3**]. The idea is to mimic the usual construction of an injective resolution using E_*-injectives, where E is a well-behaved homology theory. We will have to assume that E is a commutative ring spectrum such that E_*E is flat over E_*; it is well-known [**11**, Proposition 2.2.8] that this implies that (E_*, E_*E) is a flat Hopf algebroid and that E_*X is naturally a left E_*E-comodule for a spectrum X. It also implies that E_*E-comodules form an abelian category [**11**, Theorem A1.1.3] with enough injectives [**11**, Lemma A1.2.2].

The following definition is taken from [**3**].

Definition 1.1. Let E be a commutative ring spectrum such that E_*E is flat over E_*. Define a functor D from injective E_*E-comodules to the stable homotopy category $\mathcal{S}$ as follows. Given an injective E_*E-comodule I, consider the functor D_I from spectra

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to abelian groups defined by

$$D_I(X) = \mathrm{Hom}_{E_*E}(E_*X, I).$$

Then D_I is a cohomology functor, so there is a unique spectrum $D(I)$ such that there is a natural isomorphism

$$D_I(X) \cong [X, D(I)].$$

The hypotheses we have given on E are sufficient to define $D(I)$, but apparently insufficient to compute $E_*D(I)$. For this we need some form of the following definition; this particular form comes from [**6**].

Definition 1.2. A ring spectrum E is called *topologically flat* if E is the minimal weak colimit of a filtered diagram of finite spectra X_i such that E_*X_i is a finitely generated projective E_*-module.

Minimal weak colimits are discussed in [**7**, Section 2.2]. Adams [**1**, Section III.13] proves that many standard spectra such as BP are topologically flat; in addition, any Landweber exact commutative ring spectrum over BP or MU is topologically flat [**6**, Theorem 1.4.9]. Note that if E is topologically flat, then E_*E is flat over E_*, since it is the colimit of projective modules.

The following theorem is a translation of Theorem 1.5 of [**3**] to this terminology.

Theorem 1.3. *Suppose E is a topologically flat commutative ring spectrum, and I is an injective E_*E-comodule. Then there is a natural isomorphism $E_*D(I) \cong I$.*

We can now describe the modified Adams tower. Let E be a topologically flat commutative ring spectrum, and suppose we have a spectrum X. Let $C = E_*X$, and choose an injective resolution

$$0 \to C \xrightarrow{\eta} I_0 \xrightarrow{\tau_0} I_1 \xrightarrow{\tau_1} \cdots$$

of C in the category of E_*E-comodules. Let $\eta_s\colon C_s \to I_s$ denote the kernel of τ_s, so that $\eta_0 = \eta$.

As explained in [**3**, Section 1], we can use this resolution of C to build a tower over X with good properties. More precisely, we have the following lemma, which is easily proved by induction on n.

Lemma 1.4. *Let E be a topologically flat commutative ring spectrum, let X be a spectrum, and choose an injective resolution of E_*X as above. Then there is a tower*

$$\begin{array}{ccccccc} X = X_0 & \xleftarrow{\;g_0\;} & X_1 & \xleftarrow{\;g_1\;} & X_2 & \xleftarrow{\;g_2\;} & \cdots \\ & & \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle f_1} & & \\ & & K_0 & & K_1 & & \end{array}$$

over X satisfying the following properties.

(a) $K_s = \Sigma^{-s}D(I_s)$.
(b) *X_{s+1} is the fiber of f_s.*
(c) $E_*X_s \cong \Sigma^{-s}C_s$.
(d) *The map f_s is induced by the inclusion $C_s \to I_s$.*

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(e) $E_*g_s = 0$, *and the boundary map* $K_s \to \Sigma X_{s+1}$ *induces the surjection* $\Sigma^{-s} I_s \to \Sigma^{-s} C_{s+1}$ *on* E_**-homology.*

We call this tower the *modified Adams tower* for X based on E-homology. Of course, it actually depends on the injective resolution as well. We obtain a spectral sequence by applying $[Z, -]$ for any Z to get the modified Adams spectral sequence of Devinatz [**3**]; its E_2-term is $\text{Ext}^{**}_{E_*E}(E_*X, E_*Y)$, it is independent of the choice of resolution from the E_2 page on, and in good cases it converges to $[Z, L_E X]_*$.

2. Products of comodules. In order to understand the spectral sequence for products of spectra, we need to know a little about products of comodules. So suppose (A, Γ) is a flat Hopf algebroid. As mentioned above, basic facts about the category of Γ-comodules can be found in [**11**, Appendix 1], though he does not discuss products. A more in-depth look at the global structure of the category of Γ-comodules, including products, can be found in [**6**].

The main point of interest here is that the forgetful functor to A-modules does not preserve products. It is easiest to understand this when Γ is free over A. In this case, every element m in a Γ-comodule M has a diagonal of the form $\sum \gamma_i \otimes m_i$, where γ_i runs through a basis of Γ as a right A-module, and all but finitely many of the m_i are zero. In the A-module product $\prod_\alpha M^\alpha$ of comodules M^α, there may well be elements whose diagonal would have to be infinitely long. In fact, when Γ is projective over A, the comodule product $\prod_\Gamma M^\alpha$ is the submodule of $\prod M^\alpha$ consisting of those elements whose diagonal lands in

$$\Gamma \otimes_A \prod M^\alpha \subseteq \prod (\Gamma \otimes_A M^\alpha).$$

To construct the product when Γ is only assumed to be flat over A, one first checks that

$$\prod_\Gamma (\Gamma \otimes_A N^\alpha) \cong \Gamma \otimes_A \left(\prod N^\alpha \right)$$

for A-modules N^α, where $\Gamma \otimes_A P$ denotes the extended Γ-comodule, in which Γ coacts only on the Γ factor. One then constructs $\prod_\Gamma f^\alpha$, where f^α is an arbitrary map of extended comodules. Finally, given arbitrary comodules M^α, we have exact sequences of comodules

$$0 \to M^\alpha \xrightarrow{\psi} \Gamma \otimes_A M^\alpha \xrightarrow{f^\alpha} \Gamma \otimes_A N^\alpha,$$

where N^α is the cokernel of ψ, and f^α is the composite

$$\Gamma \otimes_A M^\alpha \to N^\alpha \xrightarrow{\psi} \Gamma \otimes_A N^\alpha.$$

It follows that $\prod_\Gamma M^\alpha \cong \ker \prod_\Gamma f^\alpha$. Details can be found in [**6**].

This construction shows that the product of comodules is more complicated than one would want; in particular, it is not always exact (see the example before Proposition 1.2.3 of [**6**]). As a right adjoint, of course, the product is left exact. Since there are enough injective Γ-comodules, the product will have right derived functors $\prod^s_\Gamma M^\alpha$ for $s \geq 0$. Almost nothing is known about these right derived functors, but they are what will appear as the E_2-term in our spectral sequence.

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For the construction of our spectral sequence, we need the following proposition.

PROPOSITION 2.1. *Suppose E is a topologically flat commutative ring spectrum, and $\{I_\alpha\}$ is a family of injective E_*E-comodules. Then there is a natural isomorphism*

$$D\left(\prod_{E_*E} I_\alpha\right) \to \prod D(I_\alpha).$$

Here the notation $\prod_{E_*E}$ denotes the product in the category of E_*E-comodules.

Proof. Note that $\prod_{E_*E} I_\alpha$ is again an injective comodule. The functoriality of D guarantees the existence of this map. Now, if X is an arbitrary spectrum, we have a chain of isomorphisms

$$\left[X, D\left(\prod_{E_*E} I_\alpha\right)\right] \cong \mathrm{Hom}_{E_*E}\left(E_*X, \prod_{E_*E} I_\alpha\right) \cong \prod \mathrm{Hom}_{E_*E}(E_*X, I_\alpha)$$
$$\cong \prod [X, D(I_\alpha)] \cong \left[X, \prod D(I_\alpha)\right].$$

This gives us the desired isomorphism. □

3. Construction of the spectral sequence. We can now use the modified Adams towers of Lemma 1.4 to construct our spectral sequence.

THEOREM 3.1. *Let E be a topologically flat commutative ring spectrum, and let $\{X^\alpha\}$ be a family of spectra. There is a natural spectral sequence $E_*^{**}(\{X^\alpha\})$ with $d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ and E_2-term*

$$E_2^{s,t} \cong \left(\textstyle\prod^s_{E_*E} E_*X^\alpha\right)_t.$$

This is a spectral sequence of E_*E-comodules, in the sense that each $d_r^{s,*}$ is a map of E_*E comodules of degree $r-1$. Furthermore, every element in $E_2^{0,t}$ in the image of the natural map

$$\bigoplus E_*X^\alpha \to \textstyle\prod_{E_*E} E_*X^\alpha$$

is a permanent cycle.

Proof. We have modified Adams towers X_s^α for each X^α. Taking the product gives us the tower below.

$$\begin{array}{ccccccc}
\prod X^\alpha & \xleftarrow{\prod g_0^\alpha} & \prod X_1^\alpha & \xleftarrow{\prod g_1^\alpha} & \prod X_2^\alpha & \xleftarrow{\prod g_2^\alpha} & \cdots \\
& & \downarrow{\scriptstyle \prod f_0^\alpha} & & \downarrow{\scriptstyle \prod f_1^\alpha} & & \\
& & \prod K_0^\alpha & & \prod K_1^\alpha & &
\end{array}$$

By applying E_*-homology, we get an associated exact couple and spectral sequence. That is, we let $D_1^{s,t} = E_{t-s}(\prod X_s^\alpha)$ and $E_1^{s,t} = E_{t-s}(\prod K_s^\alpha)$. We define $i_1\colon D \to D$ of bidegree $(-1,-1)$ by $i_1^{s,t} = E_{t-s}(\prod g_s^\alpha)$, we define $j_1\colon D \to E$ of bidegree $(0,0)$ by $j_1^{s,t} = E_{t-s}(\prod f_s^\alpha)$, and we define $k_1\colon E \to D$ of bidegree $(1,0)$ in bidegree (s,t) to be E_{t-s} of the boundary map $\prod K_s^\alpha \to \Sigma \prod X_{s+1}^\alpha$. All of these maps are maps of comodules,

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and therefore the resulting spectral sequence will be a spectral sequence of comodules, as claimed.

By combining Proposition 2.1 with Theorem 1.3, we see that

$$E_1^{s,t} \cong E_{t-s}\Big(\prod \Sigma^{-s} D(I_s^{\alpha})\Big) \cong E_t D\left(\prod_{E_*E} I_s^{\alpha}\right) \cong \left(\prod_{E_*E} I_s^{\alpha}\right)_t .$$

One can easily check that the first differential d_1 is $\prod_{E_*E} \tau_s^{\alpha}$, and therefore that the E_2-term is as claimed.

Naturality now follows in the usual way; a collection of maps $X^{\alpha} \to Y^{\alpha}$ induces non-canonical maps of the injective resolutions in question, and hence the modified Adams towers. Taking products gives us a map of spectral sequences, which is canonical from E_2 onwards.

Finally, we can also construct a spectral sequence by taking the wedge of the modified Adams towers of the X^{α} and applying E_* homology. This gives a spectral sequence with $D_1^{s,t} = (\bigoplus C_s^{\alpha})_t$ and $E_1^{s,t} = (\bigoplus I_s^{\alpha})_t$. The d_1 differential is the obvious one, and so $E_2^{0,*} \cong \bigoplus E_*X^{\alpha}$ and $E_2^{s,t} = 0$ for $s > 0$. There is a map from the spectral sequence to the spectral sequence for the product of the X^{α}. Anything in the image of this map of spectral sequences must be a permanent cycle. □

4. Convergence of the spectral sequence. We now discuss the convergence of our spectral sequence. This is a delicate question, in general, as the example given at the beginning of the paper shows. However, the spectral sequence always converges when $E = E(n)$ and each X^{α} is $E(n)$-local.

THEOREM 4.1. *Suppose $E = E(n)$ and each X^{α} is L_n-local. Then the spectral sequence of Theorem* 3.1 *converges strongly to $E(n)_*(\prod X^{\alpha})$. Furthermore, it has a horizontal vanishing line at some E_r term.*

Proof. First note that each X_s^{α} is L_n-local, since $K_s^{\alpha} = \Sigma^{-s} D(I_s^{\alpha})$ is clearly L_n-local. Each map $g_s^{\alpha}\colon X_{s+1}^{\alpha} \to X_s^{\alpha}$ has $E(n)_*(g_s^{\alpha}) = 0$. It follows from [**10**, Theorem 5.10] that there is an N, depending on n but independent of α, such that each N-fold composite $X_{s+N}^{\alpha} \to X_s^{\alpha}$ is null. Hence each composite $\prod X_{s+N}^{\alpha} \to \prod X_s^{\alpha}$ is null, giving us our desired horizontal vanishing line. Hence

$$\lim_s E(n)_*\Big(\prod X_s^{\alpha}\Big) = \lim_s^1 E(n)_*\Big(\prod X_s^{\alpha}\Big) = 0$$

so the spectral sequence converges conditionally to $E(n)_*(\prod X^{\alpha})$ [**2**]. It is also clear that $\lim_r^1 E_r^{s,t} = 0$, and so the spectral sequence converges strongly as well [**2**, Theorem 7.3]. □

5. Relatively injective resolutions and an application. Although we cannot prove very much about the derived functors of products, we can at least show that one can use relatively injective comodules to compute them. This allows us to compute the $E(n)$-homology of products of $E(n)$-module spectra.

PROPOSITION 5.1. *Let (A, Γ) be a flat Hopf algebroid, and suppose M^{α} is a relatively injective Γ-comodule for all α. Then $\prod_{\Gamma}^{s} M^{\alpha} = 0$ for $s > 0$.*

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Proof. Since M^α is relatively injective, it is a retract of $\Gamma \otimes_A M^\alpha$. It therefore suffices to show that $\prod_\Gamma^s(\Gamma \otimes_A M^\alpha) = 0$ for all $s > 0$. To do so, choose an injective resolution I_*^α of M^α in the category of A-modules. Since Γ is flat over A, $\Gamma \otimes_A I_*^\alpha$ is a resolution of $\Gamma \otimes_A M^\alpha$ in the category of Γ-comodules. Furthermore, each $\Gamma \otimes_A I_s^\alpha$ is an injective Γ-comodule [**11**, Lemma A1.2.2]. Hence

$$\textstyle\prod_\Gamma^s(\Gamma \otimes_A M^\alpha) \cong H^s\big(\prod_\Gamma(\Gamma \otimes_A I_*^\alpha)\big) \cong H^s\big(\Gamma \otimes_A \big(\prod I_*^\alpha\big)\big).$$

Since products are exact on the category of A-modules, and since Γ is flat, these groups are 0 for $s > 0$. □

This yields an immediate topological corollary.

COROLLARY 5.2. *Suppose X^α is an $E(n)$-module spectrum for all α. Then*

$$E(n)_*\Big(\prod X^\alpha\Big) \cong \prod_{E(n)_*E(n)} E(n)_*(X^\alpha).$$

In particular,

$$E(n)_*\Big(\prod E(n) \wedge X^\alpha\Big) \cong E(n)_*E(n) \otimes_{E(n)_*} \Big(\prod E(n)_*X^\alpha\Big).$$

Proof. Since X^α is an $E(n)$-module spectrum, it is L_n-local. Furthermore, $E(n)_*X^\alpha$ is a retract of

$$E(n)_*(E(n) \wedge X^\alpha) \cong E(n)_*E(n) \otimes_{E(n)_*} E(n)_*X^\alpha,$$

so is relatively injective. Proposition 5.1 then implies that the E_2-term of our spectral sequence is 0 except in bidegree $(0, t)$. It therefore collapses, and we get the desired isomorphism. □

It also follows, using standard homological algebra, that we can use relatively injective resolutions to compute the derived functors of product. For example, we can use the cobar resolution $C^*(M)$ described in [**11**, Definition A1.2.10].

COROLLARY 5.3. *Suppose (A, Γ) is a flat Hopf algebroid, and $\{M^\alpha\}$ is a set of Γ-comodules. Let $C^*(M^\alpha)$ denote the cobar resolution on M^α. Then*

$$\textstyle\prod_\Gamma^s M^\alpha \cong H^s \prod_\Gamma C^* M^\alpha.$$

This corollary tells us, for example, that if $JM^\alpha = 0$ for some invariant ideal J and all α, then $J\prod_\Gamma^s M^\alpha = 0$ for all s.

6. BP_*BP-comodules and $E(n)_*E(n)$-comodules.

In this section, we exploit the close relationship between BP_*BP-comodules and $E(n)_*E(n)$-comodules studied in [**9**] to get some partial understanding of the product of comodules.

We begin with BP_*BP-comodules, which are easier to handle because BP_*BP is connective and projective over BP_*. As mentioned in Section 2, the product of a family $\{M^\alpha\}$ of BP_*BP-comodules is the submodule of $\prod M^\alpha$ consisting of those elements whose diagonal has finite length.

DEFINITION 6.1. A family of BP_*BP-comodules $\{M^\alpha\}$ is *uniformly bounded below* if there is a $d \in \mathbb{Z}$ such that $M_n^\alpha = 0$ for all $n < d$ and all α.

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The product and its derived functors are particularly simple for a uniformly bounded below family.

THEOREM 6.2. *Suppose $(A, \Gamma) = (BP_*, BP_*BP)$, and $\{M^\alpha\}$ is a family of Γ-comodules that is uniformly bounded below. Then*

$$\prod_\Gamma M^\alpha \cong \prod M^\alpha \text{ and } \prod_\Gamma^s M^\alpha = 0 \text{ for all } s > 0.$$

Proof. Since every element of $\prod M^\alpha$ must have finite diagonal, the first statement is clear. For the second statement, consider the cobar resolution C_*M^α of M^α by relatively injective comodules. We have $C_sM^\alpha = \Gamma \otimes_A \overline{\Gamma}^{\otimes s} \otimes_A M^\alpha$, so, since Γ is connective, the family $\{C_sM^\alpha\}$ is uniformly bounded below for each s. We therefore have

$$\prod_\Gamma^s M^\alpha \cong H^s \prod_\Gamma C_*M^\alpha \cong H^s \prod C_*M^\alpha = 0$$

for $s > 0$, using Corollary 5.3 and the fact that products of modules are exact. □

To relate this to $E(n)_*E(n)$-comodules, we recall from [**9**] and [**10**] the exact functor Φ_* from BP_*BP-comodules to $E(n)_*E(n)$-comodules defined by $\Phi_*M = E(n)_* \otimes_{BP_*} M$. The functor Φ_* has a fully faithful right adjoint Φ^*, the composite $\Phi_*\Phi^*$ is naturally isomorphic to the identity, and the composite $L_n = \Phi^*\Phi_*$ is the localization functor on the category of BP_*BP-comodules with respect to the hereditary torsion theory of v_n-torsion comodules. The functor L_n is left exact, but has right derived functors L_n^q for $0 \le q \le n$, studied in [**10**].

As a left adjoint, we do not expect Φ_* to preserve products. We do, however, have the following result.

THEOREM 6.3. *Suppose $\{M^\alpha\}$ is a family of BP_*BP-comodules. Then there is a natural isomorphism*

$$\prod_{E(n)_*E(n)} \Phi_*M^\alpha \to \Phi_*\left(\prod_{BP_*BP} L_nM^\alpha\right).$$

In fact, there is a convergent first quadrant spectral sequence $E_r^{p,q}$ of $E(n)_*E(n)$-comodules with

$$E_2^{p,q} \cong \Phi_*\left(\prod^p_{BP_*BP} \left(L_n^qM^\alpha\right)\right) \Rightarrow \prod^{p+q}_{E(n)_*E(n)} \Phi_*M^\alpha.$$

Proof. Since Φ^* is a right adjoint, we have

$$\prod_{E(n)_*E(n)} \Phi_*M^\alpha \cong \Phi_*\Phi^*\left(\prod_{E(n)_*E(n)} \Phi_*M^\alpha\right) \cong \Phi_* \prod_{BP_*BP} (L_nM^\alpha),$$

as required. The spectral sequence is the Grothendieck spectral sequence for the derived functors of the composition, described in [**12**, Section 5.8]. Recall that this spectral sequence has $E_2^{p,q} = (R^pF)(R^qG)(-)$ and converges to $R^{p+q}(FG)(-)$, under the assumption that $(R^pF)(GI) = 0$ for all injectives I and $p > 0$. In the case at hand, the functor F is $\Phi_* \prod_{BP_*BP}(-)$ and the functor G is L_n (applied objectwise to the product category). Since L_n preserves injectives [**10**, Corollary 2.4], the Grothendieck spectral sequence exists. Since Φ_* is exact and products of injectives are injective, we can use

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another Grothendieck spectral sequence argument to see that $R^pF = \Phi_* \prod^p_{BP_*BP}(-)$. Similarly, since Φ_* is exact and preserves injectives [**10**, Corollary 2.5], another Grothendieck spectral sequence argument shows that

$$R^{p+q}(FG)(-) = R^{p+q}\big(\prod_{E(n)_*E(n)} \Phi_*\big)(-) = \prod^{p+q}_{E(n)_*E(n)} \Phi_*(-),$$

completing the proof. □

This proposition allows us to compute some products of $E(n)_*E(n)$-comodules. For example, we have

$$\prod_{E(n)_*E(n)} \Sigma^\alpha E(n)_* \cong E(n)_* \otimes_{BP_*} \prod \Sigma^\alpha BP_*,$$

as long as the α are bounded below. To see this, use the fact that BP_* is L_n-local [**10**], Theorem 6.2, and Theorem 6.3.

In fact, we have

$$\prod^s_{E(n)_*E(n)} \Sigma^\alpha E(n)_* = 0$$

for $0 < s < n$ and

$$\prod^s_{E(n)_*E(n)} \Sigma^\alpha E(n)_* \cong \Phi_* \prod^{s-n}_{BP_*BP} \Sigma^\alpha BP_* \big/ \big(p^\infty, v_1^\infty, \ldots, v_n^\infty\big)$$

for $s \geq n$, again under the hypothesis that the α are bounded below. This follows from the spectral sequence of Theorem 6.3 and the fact [**10**] that $L^q_n BP_* = 0$ except when $q = 0$ and $q = n$, where

$$L^0_n BP_* = BP_* \text{ and } L^n_n BP_* = BP_* \big/ \big(p^\infty, v_1^\infty, \ldots, v_n^\infty\big).$$

Note that we do not know whether $\prod_{BP_*BP} \Sigma^\alpha BP_*/(p^\infty, v_1^\infty, \ldots, v_n^\infty)$ is all v_n-torsion or not, and therefore we do not know whether $\prod^n_{E(n)_*E(n)} \Sigma^\alpha E(n)_*$ is zero or not.

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