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ALTERNATING COLOURINGS OF THE VERTICES OF A REGULAR POLYGON

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Abstract

Let $n, r, k \in \mathbb{N}$. An r-colouring of the vertices of a regular n-gon is any mapping $\chi : \mathbb{Z}_n \to \{1, 2, \ldots, r\}$. Two colourings are equivalent if one of them can be obtained from another by a rotation of the polygon. An r-ary necklace of length n is an equivalence class of r-colourings of $\mathbb{Z}_n$. We say that a colouring is k-alternating if all k consecutive vertices have pairwise distinct colours. We compute the smallest number r for which there exists a k-alternating r-colouring of $\mathbb{Z}_n$ and we count, for any r, 2-alternating r-colourings of $\mathbb{Z}_n$ and 2-alternating r-ary necklaces of length n.

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1. Introduction

Let $n, r, k \in \mathbb{N}$. An *r-colouring* of the vertices of a regular n-gon is any mapping $\chi : \mathbb{Z}_n \to \{1, 2, \ldots, r\}$. The group $\mathbb{Z}_n$ naturally acts on its colourings by

$$(\chi + a)(x) = \chi(x - a).$$

Colourings χ and ψ are *equivalent* if there is $a \in \mathbb{Z}_n$ such that $\chi + a = \psi$, that is, if one of them can be obtained from another by a rotation of the polygon. An r-ary *necklace* of length n is an equivalence class of r-colourings of $\mathbb{Z}_n$. It is well-known that there are

$$\frac{1}{n} \sum_{d|n} \varphi(d) r^{n/d}$$

r-ary necklaces of length n, where φ is the Euler totient function (see [2]).

In [5] and [4] symmetric colourings of $\mathbb{Z}_n$ and symmetric necklaces were counted. A colouring χ of $\mathbb{Z}_n$ is *symmetric* if there is $a \in \mathbb{Z}_n$ such that $\chi(a - x) = \chi(x)$ for all

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$x \in \mathbb{Z}_n$, that is, if it is invariant under some reflection of the polygon. There are

$$S_r(n) = \begin{cases} \sum_{d|n} d \prod_{p|(n/d)} (1-p) r^{(d+1)/2} & n \text{ odd}, \\ \sum_{d|(n/2)} d \prod_{p|(n/2d)} (1-p)(r^{d+1} + r^d) - \sum_{d|m} d \prod_{p|(m/d)} (1-p) r^{(d+1)/2} & n \text{ even}, \end{cases}$$

symmetric r-colourings of $\mathbb{Z}_n$ (m is the greatest odd divisor of n) and

$$s_r(n) = \begin{cases} \frac{1}{2}(r+1) r^{n/2} & \text{if } n \text{ is even}, \\ r^{(n+1)/2} & \text{if } n \text{ is odd}, \end{cases}$$

symmetric r-ary necklaces of length n.

In this paper we study alternating colourings of $\mathbb{Z}_n$. We say that a colouring χ of $\mathbb{Z}_n$ is *k-alternating* if all k consecutive vertices have pairwise distinct colours, that is, if for every $x \in \mathbb{Z}_n$, the restriction of χ to $\{x, x+1, \dots, x+k-1\}$ is injective. Clearly, a colouring equivalent to a k-alternating one is also k-alternating. We compute the smallest number r for which there exists a k-alternating r-colouring of $\mathbb{Z}_n$ and we count, for any r, 2-alternating r-colourings of $\mathbb{Z}_n$ and 2-alternating r-ary necklaces of length n.

2. Computing the smallest number of colours

Given $n, k \in \mathbb{N}$ with $k \le n$, let $\rho(k,n)$ denote the smallest number r for which there exists a k-alternating r-colouring of the vertices of a regular n-gon. It is clear that $k \le \rho(k,n) \le n$.

Theorem 2.1. *Given n and $k \le n$, write $n = mk + l$ and $l = k_0 m + m_0$, where $0 \le l < k$ and $0 \le m_0 < m$. Then*

$$\rho(k,n) = \left\lceil \frac{n}{m} \right\rceil = k + \left\lceil \frac{l}{m} \right\rceil = \begin{cases} k + k_0 & \text{if } m_0 = 0, \\ k + k_0 + 1 & \text{otherwise.} \end{cases}$$

Proof. Let $\chi : \mathbb{Z}_n \to \{1, 2, \dots, r\}$ be a k-alternating r-colouring. Then for each $i \in \{1, 2, \dots, r\}$, one has $|\chi^{-1}(i)| \le m$. Indeed, otherwise there is an increasing sequence $(a_t)_{t=1}^{m+1}$ in $\{0, 1, \dots, n-1\}$ such that $a_{t+1} - a_t \ge k$ for each $t \le m$ and $a_1 + n - a_{m+1} \ge k$, which implies that $(m+1)k \le n$. It follows that $r \ge \lceil n/m \rceil$.

Conversely, let $r = k + \lceil l/m \rceil$. Partition $\mathbb{Z}_n$ into m consecutive blocks, the first m_0 of which have $(k + k_0 + 1)$ elements and the next $m - m_0$ have $(k + k_0)$ elements. We can do this because $m_0(k + k_0 + 1) + (m - m_0)(k + k_0) = m(k + k_0) + m_0 = n$. Define $\chi : \mathbb{Z}_n \to \{1, 2, \dots, r\}$ on each block $\{a+1, a+2, \dots, a+i, \dots\}$ by $\chi(a+i) = i$. Figure 1 illustrates the colouring for $n = 17$ and $k = 3$. It is easy to see that the colouring χ so defined is k-alternating. □

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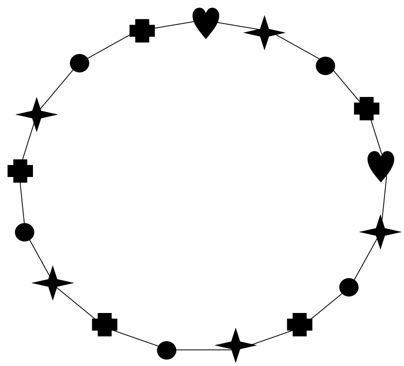

FIGURE 1. Example of a colouring for Theorem 2.1.

COROLLARY 2.2.

(1) $\rho(k, n) = k$ *if and only if* $k \mid n$.
(2) $\rho(k, n) = n$ *if and only if* $2k > n$.
(3) *If* $2k \leq n$, *then* $\rho(k, n) \leq \lceil n/2 \rceil$.

PROOF. (1) $\rho(k, n) = k$ if and only if $m_0 = 0$ and $k_0 = 0$, that is, $l = 0$.

(2) $2k > n$ if and only if $m = 1$. If $m = 1$, then $\rho(k, n) = \lceil n/1 \rceil = n$. If $m \geq 2$, then $\rho(k, n) = \lceil n/m \rceil \leq \lceil n/2 \rceil$ and $\lceil n/2 \rceil < n$ (since $n \geq m \geq 2$).

(3) If $2k \leq n$, then $m \geq 2$ and so $\rho(k, n) \leq \lceil n/2 \rceil$. □

COROLLARY 2.3. *If* $n \geq k^2$, *then*

$$\rho(k, n) = \begin{cases} k & \text{if } k \mid n, \\ k + 1 & \text{otherwise.} \end{cases}$$

PROOF. Since $n \geq k^2$, one has $m \geq k$, so $k_0 = 0$ and $m_0 = l$. □

Using Theorem 2.1, we can compute $\rho(k, n)$ for small k and $n < k^2$. Since $\rho(2, 2) = 2$ and $\rho(2, 3) = 3$, we can extend Corollary 2.3 in the case $k = 2$.

COROLLARY 2.4. *We have*

$$\rho(2, n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{otherwise.} \end{cases}$$

Notice that $\rho(3, 3) = 3$, $\rho(3, 4) = 4$, $\rho(3, 5)=5$, $\rho(3, 6)=3$, $\rho(3, 7) = 4$ and $\rho(3, 8)=4$.

3. Counting alternating colourings and necklaces

Given $n, k, r \in \mathbb{N}$, let $A_r(k, n)$ denote the number of k-alternating r-colourings of the vertices of a regular n-gon and $a_r(k, n)$ the number of k-alternating r-ary necklaces of length n, and let $A_r(n) = A_r(2, n)$ and $a_r(n) = a_r(2, n)$.

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THEOREM 3.1. *We have*

$$A_r(n) = (r-1)^n + (-1)^n(r-1).$$

Theorem 3.1 is a known fact. But we give its proof for the convenience of the reader. Our proof is direct and differs a bit from that in [3].

PROOF. There are $r(r-1)^{n-2}$ colourings $\chi : \{0,1,\ldots,n-2\} \to \{1,\ldots,r\}$ such that $\chi(i) \neq \chi(i+1)$ for each $i < n-2$, and the number of such colourings with $\chi(0) = \chi(n-2)$ is $A_r(n-2)$. Consequently,

$$\begin{aligned} A_r(n) &= A_r(n-2)(r-1) + (r(r-1)^{n-2} - A_r(n-2))(r-2) \\ &= (r^2-2r)(r-1)^{n-2} + A_r(n-2). \end{aligned}$$

Since

$$A_r(2) = r(r-1) = (r^2-2r) + r \quad \text{and} \quad A_r(3) = r(r-1)(r-2) = (r^2-2r)(r-1),$$

it follows that if n is even, then

$$\begin{aligned} A_r(n) &= (r^2-2r)((r-1)^{n-2} + (r-1)^{n-4} + \cdots + 1) + r \\ &= (r^2-2r)\frac{(r-1)^n - 1}{(r-1)^2-1} + r \\ &= (r-1)^n - 1 + r \\ &= (r-1)^n + (r-1), \end{aligned}$$

and if n is odd, then

$$\begin{aligned} A_r(n) &= (r^2-2r)((r-1)^{n-2} + (r-1)^{n-4} + \cdots + (r-1)) \\ &= (r^2-2r)(r-1)\frac{(r-1)^{n-1} - 1}{(r-1)^2-1} \\ &= (r-1)((r-1)^{n-1} - 1) \\ &= (r-1)^n - (r-1). \end{aligned}$$

□

Now we turn to counting $a_r(k,n)$. For every $m \mid n$, let X_m denote the set of all k-alternating r-colourings of $\mathbb{Z}_m$.

LEMMA 3.2. *For every* $g \in \mathbb{Z}_n$,

$$|\{\chi \in X_n : \chi + g = \chi\}| = A_r(k, n/|\langle g\rangle|).$$

PROOF. Let $d = |\langle g\rangle|$. For every $\psi \in X_{n/d}$, define $\overline{\psi} \in X_n$ by

$$\overline{\psi}(i + (n/d)j) = \psi(i),$$

where $i \in \{0, 1\ldots, n/d-1\}$ and $j \in \{0,1,\ldots,d-1\}$. Then $\overline{\psi} + g = \overline{\psi}$ and the mapping

$$X_{n/d} \ni \psi \mapsto \overline{\psi} \in \{\chi \in X_n : \chi + g = \chi\}$$

is a bijection.

To see that it is a surjection, let $\chi \in X_n$ and $\chi + g = \chi$. Define $\psi \in X_{n/d}$ to be the restriction of χ to $\{0,1,\ldots,n/d-1\}$. Then $\overline{\psi} = \chi$. □

Theorem 3.3.

$$a_r(k,n) = \frac{1}{n}\sum_{d|n}\varphi(d)A_r(k,n/d).$$

Proof. Applying Burnside's lemma [1, I, Section 3] gives

$$a_r(k,n) = \frac{1}{n}\sum_{g\in\mathbb{Z}_n}|\{\chi\in X_n : \chi+g=\chi\}|,$$

and by Lemma 3.2,

$$|\{\chi\in X_n : \chi+g=\chi\}| = A_r(k,n/|\langle g\rangle|).$$

For every $d \mid n$, there is exactly one subgroup of $\mathbb{Z}_n$ of order d and the number of its generators is $\varphi(d)$. Hence,

$$a_r(k,n) = \frac{1}{n}\sum_{d|n}\varphi(d)A_r(k,n/d). \qquad \square$$

From Theorems 3.3 and 3.1 we obtain the following corollary.

Corollary 3.4. *We have*

$$a_r(n) = \frac{1}{n}\sum_{d|n}\varphi(d)[(r-1)^{n/d}+(-1)^{n/d}(r-1)].$$

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