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A CONCRETE REALIZATION OF THE DUAL SPACE OF L^1-SPACES OF CERTAIN VECTOR AND OPERATOR-VALUED MEASURES

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Abstract

For a given vector measure n, an important problem, but in practice a difficult one, is to give a concrete description of the dual space of $L^1(n)$. In this note such a description is presented for an important class of measures n, namely the spectral measures (in the sense of N. Dunford) and certain other vector and operator-valued measures that they naturally induce. The basic idea is to represent the L^1-spaces of such measures as a more familiar space whose dual space is known.

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Introduction and statement of results

Let X be a locally convex Hausdorff space, always assumed to be quasicomplete, and let X' denote its continuous dual space. An X-valued measure is a σ-additive map $n\colon \Sigma \to X$ whose domain is a σ-algebra of subsets of a set Ω. The locally convex Hausdorff space of (equivalence classes of) n-integrable functions is denoted by $L^1(n)$; see Section 1 for the definition.

An important problem, but in practice a difficult one, is to give a satisfactory description of the dual space, $L^1(n)'$. Various identifications of this space, such as a subspace of the space of all finite, countably additive measures on Σ, for example, are well known, However, such descriptions are often abstract and give little insight into the nature of the individual elements of $L^1(n)'$. It is therefore desirable to have available a more concrete description of the elements of $L^1(n)'$, whenever possible.

A first and perhaps obvious step in this direction is to note that there is a very natural way to generate a class of continuous linear functionals on $L^1(n)$ which have a particularly simple form. For, with each element x' of X' there is associated the complex-measure

$$\langle n, x' \rangle : E \to \langle n(E), x' \rangle, \qquad E \in \Sigma,$$

which has the property that each function $f \in L^1(n)$ is $\langle n, x' \rangle$-integrable and the linear functional

$$(1) \qquad f \to \left\langle \int_\Omega f dn, x' \right\rangle = \int_\Omega f d\langle n, x' \rangle, \qquad f \in L^1(n),$$

is continuous. Unfortunately, this somewhat appealing approach is too simple in general. The space of functionals of the form (1), for each $x' \in X'$, may turn out to be only a very small part of the dual space $L^1(n)'$. This happens already in the simplest of cases when X is the space of complex numbers. For, if $\Omega = [0, 1]$ and $n: \Sigma \to X$ is Lebesgue measure on the Borel σ-algebra Σ of Ω, then the space of all functionals of the form (1), for each $x' \in X'$, can be identified with the 1-dimensional subspace of constant functions of the infinite dimensional dual space $L^1(n)' = L^\infty(n)$.

Yet, as we shall see, for a very important class of measures, namely the spectral measures (in the sense of N. Dunford, [4]) which play a fundamental role in the spectral theory of linear operators, and certain other vector and operator-valued measures that they naturally induce, this naive approach is entirely satisfactory (cf. Theorems 1–4 below). For such measures the space of functionals of the form (1) actually turns out to be *all* of $L^1(n)'$; and hence we are in the desirable situation of having a concrete description of the space dual to $L^1(n)$. In fact, for the operator-valued measures it turns out that it is not necessary to use all of the space X'; a much smaller *subset* which generates it already suffices. Of course, it is the multiplicativity of spectral measures which plays a crucial role and distinguishes their situation from the case of arbitrary vector measures.

The purpose of this note, therefore, is to prove Theorems 1–4 below. A key element in their proofs is a representation theorem, recently proved in [2], which characterizes the L^1-space of certain spectral measures as a suitable algebra of operators (generated by the range of the given measure) whose dual space is well known and has a particularly simple description (cf. Lemma 1.2). We remark that this representation theorem is new even in the setting of Banach spaces. Before the results can be properly formulated, we need some further notation and definitions.

A vector measure $n: \Sigma \to X$ is said to be *closed* if the space $L^1(n)$ is complete. This agrees with the original definition given in [7]; see [9, IV, Theorem 4.1] and [11, Proposition 1].

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Let X be a locally convex Hausdorff space. Then $\mathbf{L}(X)$ denotes the space of continuous linear operators on X equipped with the topology of pointwise convergence in X. The space $\mathbf{L}(X)$ is always assumed to be sequentially complete. The adjoint of an operator $T \in \mathbf{L}(X)$ is denoted by T'. The correspondence $\sum_i x_i \otimes x_i' \to \xi \in (\mathbf{L}(X))'$, defined by

$$\xi: T \to \sum_i \langle Tx_i, x_i' \rangle, \qquad T \in \mathbf{L}(X), \tag{2}$$

is an (algebraic) isomorphism of the tensor product $X \otimes X'$ onto the dual of $\mathbf{L}(X)$. Call an element of $(\mathbf{L}(X))'$ *elementary* if it is of the form

$$T \to \langle Tx, x' \rangle, \qquad T \in \mathbf{L}(X),$$

for some $x \in X$ and $x' \in X'$, in which case we will denote it by $x \otimes x'$. Then $(\mathbf{L}(X))'$ is precisely the linear span of its elementary functionals.

Since $\mathbf{L}(X)$ is a locally convex Hausdorff space in its own right, we can speak of $\mathbf{L}(X)$-valued measures. Such measures are call *equicontinuous* if their range is an equicontinuous part of $\mathbf{L}(X)$. A measure $P: \Sigma \to \mathbf{L}(X)$ is said to be a *spectral measure* if it is multiplicative, and if $P(\Omega) = I$, the identity operator on X. Of course, the multiplicativity of P means that $P(E \cap F) = P(E)P(F)$ for each $E \in \Sigma$ and $F \in \Sigma$.

THEOREM 1. *Let $P: \Sigma \to \mathbf{L}(X)$ be a closed equicontinuous spectral measure. Then a linear functional ξ on $L^1(P)$ is continuous if, and only if, there is an elementary functional $x \otimes x'$ in $(\mathbf{L}(X))'$ such that*

$$\langle f, \xi \rangle = \int_\Omega f d\langle P, x \otimes x' \rangle = \int_\Omega f d\langle Px, x' \rangle, \qquad f \in L^1(P). \tag{3}$$

If $P: \Sigma \to \mathbf{L}(X)$ is a spectral measure and f a P-integrable function, then the indefinite integral of f with respect to P is the set function P_f given by

$$P_f: E \to \int_E f dP, \qquad E \in \Sigma.$$

It follows from the Orlicz-Pettis lemma that P_f is a σ-additive measure in $\mathbf{L}(X)$. We remark that although P_f is a closed measure whenever P is a closed measure [9, IV, Theorem 7.2], it is not necessarily multiplicative (unless f is the characteristic function, χ_E, of some set $E \in \Sigma$). Also, the equicontinuity of P may not imply the equicontinuity of P_f (unless f is P-essentially bounded or the underlying space X satisfies additional properties, such as barrelledness, for example).

THEOREM 2. *Let $P: \Sigma \to \mathbf{L}(X)$ be a closed equicontinuous spectral measure and let f be a P-integrable function such that $1/f$ is also P-integrable* (*cf*. §2). *Then a linear functional ξ on $L^1(P_f)$ is continuous if, and only if, there is an elementary*

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functional $x \otimes x'$ *in* $(\mathbf{L}(X))'$ *such that*

$$(4) \qquad \langle g, \xi \rangle = \int_\Omega g d\langle P_f, x \otimes x' \rangle = \int_\Omega g f d\langle Px, x' \rangle, \qquad g \in L^1(P_f).$$

There are results analogous to Theorem 1 and 2 for the case of certain X-valued vector measures, namely those induced by an $\mathbf{L}(X)$-valued spectral measure or the indefinite integral of a spectral measure via evaluation at points of the underlying space X. If $P\colon \Sigma \to L(X)$ is a measure, then for each $x \in X$, the X-valued measure $E \to P(E)x$, $E \in \Sigma$, is denoted by Px.

THEOREM 3. *Let* $P\colon \Sigma \to L(X)$ *be an equicontinuous spectral measure and let* x *be an element of* X *such that the* X*-valued measure* Px *is closed. Then a linear functional* ξ *on* $L^1(Px)$ *is continuous if, and only if, there is an element* $x' \in X'$ *such that*

$$(5) \qquad \langle f, \xi \rangle = \left\langle \int_\Omega f dPx, x' \right\rangle = \int_\Omega f d\langle Px, x' \rangle, \qquad f \in L^1(Px).$$

It is worth noting that there are large classes of locally convex spaces X, including all metrizable spaces and all Suslin spaces, for example, in which *any* X-valued measure is necessarily closed (cf. [9, 12], for example). Also, if the spectral measure P is itself closed, then necessarily Px is a closed measure for *every* $x \in X$, [2, Proposition 1.7(iii)].

THEOREM 4. *Let* $P\colon \Sigma \to \mathbf{L}(X)$ *be an equicontinuous spectral measure, let* f *be a* P*-integrable function such that* $1/f$ *is also* P*-integrable (cf. Section* 2), *and let* x *be an element of* X *such that* Px *is a closed measure in* X. *Then a linear functional* ξ *on* $L^1(P_f x)$ *is continuous if, and only if, there exists an element* $x' \in X'$ *such that*

$$\langle g, \xi \rangle = \left\langle \int_\Omega g dP_f x, x' \right\rangle = \int_\Omega g f d\langle Px, x' \rangle, \qquad g \in L^1(P_f x).$$

It is clear from the above results that it is valuable to have criteria available which guarantee the closedness of a given spectral measure. Some criteria of this type are given in §4.

1. Preliminaries

In this section we establish the notation to be used in the text and summarize those aspects of the theory of integration with respect to vector measures (see [9] for a more comprehensive treatment) and, in particular, spectral measures that are needed in the sequel.

Let n: $\Sigma \to X$ be a vector measure. For each $x' \in X'$, the total variation of $\langle n, x' \rangle$ is denoted by $|\langle n, x' \rangle|$. If q is a continuous seminorm on X, let U_q^0 denote the polar of the closed unit ball of q. Then the *q-semivariation* of n is the set function $q(n)$ defined by

(6) $$q(n)(E) = \sup\{|\langle n, x' \rangle|(E) \colon x' \in U_q^0\}, \qquad E \in \Sigma.$$

For each $E \in \Sigma$, the inequalities

(7) $$\sup\{q(n(F)) \colon F \in \Sigma, F \subseteq E\} \leqslant q(n)(E) \leqslant 4\sup\{q(n(F)) \colon F \in \Sigma, F \subseteq E\}$$

hold [9, II, Lemma 1.2].

A complex-valued, Σ-measurable function f on Ω is said to be *n-integrable* if it is integrable with respect to each measure $\langle n, x' \rangle$, $x' \in X'$, and if, for every $E \in \Sigma$, there exists an element $\int_E f dn$ of X such that

$$\left\langle \int_E f dn, x' \right\rangle = \int_E f d\langle n, x' \rangle,$$

for each $x' \in X'$. The vector measure n_f: $\Sigma \to X$ defined by

$$n_f(E) = \int_E f dn, \qquad E \in \Sigma,$$

is called the *indefinite integral* of f with respect to n. The element $n_f(\Omega) = \int_\Omega f dn$ is denoted simply by $n(f)$.

The set of all n-integrable functions is denoted by $L(n)$. Members of Σ are freely identified with their characteristic function. An n-integrable function is said to be *n-null* if its indefinite integral is the zero vector measure. Two n-integable functions f and g are *n-equivalent* or *equal n-almost everywhere* (*n-a.e.*) if $|f - g|$ is n-null. A set $E \in \Sigma$ is *n-null* if $\chi_E = 0$, n-a.e.

If f is an n-integrable function, then for each continuous seminorm q on X we define the *q-upper integral*, $q(n)(f)$, by $q(n)(f) = q(n_f)(\Omega)$. The function

(8) $$f \to q(n)(f), \qquad f \in L(n),$$

is then a seminorm on $L(n)$.

Denote by $\tau(n)$ the topology on $L(n)$ which is defined by the family of seminorms (8), for each continuous seminorm q on X. This resulting locally convex space is not necessarily Hausdorff. The quotient space of $L(n)$ with respect to the subspace of all n-null functions is denoted by $L^1(n)$. The resulting Hausdorff topology on $L^1(n)$ is again denoted by $\tau(n)$. It is clear from (6) and (7) that $\tau(n)$ is the topology of uniform convergence on Σ of indefinite integrals.

LEMMA 1.1. *Let n: $\Sigma \to X$ be a vector measure and let f be an n-integrable function. Then a Σ-measurable function g is n_f-integrable if, and only if, fg is n-integrable. In this case*

(9) $$\int_E g dn_f = \int_E fg dn, \qquad E \in \Sigma.$$

PROOF. It follows from the identities

$$\langle n_f, x'\rangle(E) = \langle n_f(E), x'\rangle = \left\langle \int_E f dn, x'\right\rangle = \int_E f d\langle n, x'\rangle, \qquad E \in \Sigma,$$

valid for each $x' \in X'$, that

$$\text{(10)} \qquad \int_E s d\langle n_f, x'\rangle = \int_E f s d\langle n, x'\rangle, \qquad E \in \Sigma,$$

for each $x' \in X'$, whenever s is a Σ-simple function. Accordingly, if g is n_f-integrable, then approximating g pointwise by Σ-simple functions, we have from (10) and from the Dominated Convergence Theorem that gf is $\langle n, x'\rangle$-integrable, and that

$$\text{(11)} \qquad \int_E g f d\langle n_f, x'\rangle = \int_E g d\langle n_f, x'\rangle, \qquad E \in \Sigma,$$

for each $x' \in X'$ [10, Lemma 2.3]. From this it is clear that gf is n-integrable and has indefinite integral given by (9).

Conversely, assume that fg is n-integrable. Again approximating g pointwise by Σ-simple functions, say $\{s_m\}$, we have from (10) and from the Dominated Convergence Theorem (applied to $\{fs_m\}$, which converges pointwise to fg) that (11) is valid for each $x' \in X'$. From this it is clear that g is $\langle n_f, x'\rangle$-integrable for each $x' \in X'$ [10, Lemma 2.3], and hence that g is in fact n_f-integrable with indefinite integral given by (9).

For the remainder of the section we specialize our attention to spectral measures.

So, let $P\colon \Sigma \to L(X)$ be a spectral measure. If X is a Banach space, then a Σ-measurable function is P-integrable if, and only if, it is P-essentially bounded [4, XVIII, Theorem 2.11 (c)], and hence the product of two P-integrable functions is again P-integrable. For non-normable locally convex spaces X it need not be the case that P-integrable functions are P-essentially bounded. Nevertheless, it is still true, but now not so obvious, that the product of two P-integrable functions, f and g say, is again P-integrable [2, Lemma 1.3]. In fact, the indefinite integral of fg with respect to P is given by

$$\int_E fg dP = P(E)P(f)P(g) = P(E)P(g)P(f), \qquad E \in \Sigma.$$

If, in addition, P is closed and equicontinuous, then the integration map

$$\text{(12)} \qquad \Phi_P\colon f \to P(f) = \int_\Omega f dP, \qquad f \in L^1(P),$$

is a linear and multiplicative isomorphism of the (complete) locally convex algebra $L^1(P)$ onto the closed operator algebra $\langle \overline{P}\rangle$ in $L(X)$ generated by the range $P(\Sigma) = \{P(E)\colon E \in \Sigma\}$ of P [2, Proposition 1.5]. Noting that the closure in $L(X)$ of the range of an equicontinuous $L(X)$-valued spectral measure is an equicontinuous (Bade) complete Boolean algebra [15, Proposition 3.17], we obtain the following result immediately from [1, Proposition 3.2]; see also [5] for the case that the underlying space X is a Banach space.

LEMMA 1.2. *Let P: $\Sigma \to \mathbf{L}(X)$ be an equicontinuous spectral measure. If ν is any element of $(\mathbf{L}(X))'$, then there exists an elementary functional $x \otimes x'$ in $(\mathbf{L}(X))'$ such that ν and $x \otimes x'$ agree on $\langle \overline{P} \rangle$, that is,*

$$\langle T, \nu \rangle = \langle Tx, x' \rangle, \qquad T \in \langle \overline{P} \rangle. \tag{13}$$

Let P: $\Sigma \to \mathbf{L}(X)$ be a measure. For each $x \in X$, the *cyclic space* (in X) generated by x with respect to P is defined to be the closed subspace of X generated by the set $\{P(E)x\colon E \in \Sigma\}$; it is denoted by $P(\Sigma)[x]$. If $P(\Sigma)[x] = X$, then x is said to be *cyclic* for P.

LEMMA 1.3. *Let P: $\Sigma \to \mathbf{L}(X)$ be an equicontinuous spectral measure and let x be an element of X such that Px is a closed measure. Then the integration map*

$$\Phi_{Px}\colon f \to \int_\Omega f dPx, \qquad f \in L^1(Px),$$

is an isomorphism of $L^1(Px)$ onto the cyclic space $P(\Sigma)[x]$.

This is essentially [2, Proposition 2.1]. There the spectral measure P itself is assumed to be closed, but an examination of the proof shows that for a *particular* element $x \in X$, all that is needed is for the X-valued measure Px to be closed. Finally, we will need the following result [1, Proposition 1.1].

LEMMA 1.4. *Let P: $\Sigma \to \mathbf{L}(X)$ be an equicontinuous spectral measure and let f be a P-integrable function. Let $x \in X$. Then for each Px-integrable function g the function fg is also Px-integrable and has indefinite integral given by*

$$\int_E fg dPx = P(f) \int_E g dPx, \qquad E \in \Sigma.$$

2. Proofs of Theorems 1 and 2

To prove Theorem 1, let $x \otimes x'$ be an elementary functional in $(\mathbf{L}(X))'$. If q denotes the continuous seminorm on $\mathbf{L}(X)$ given by $q(T) = |\langle T, x \otimes x' \rangle|$, $T \in \mathbf{L}(X)$, in which case $x \otimes x' \in U_q^0$, then

$$\left| \int_\Omega f d\langle Px, x' \rangle \right| \leqslant \int_\Omega |f| d|\langle Px, x' \rangle| = \int_\Omega |f| d|\langle P, x \otimes x' \rangle|$$
$$\leqslant \sup\left\{ \int_\Omega |f| d|\langle P, \nu \rangle| \colon \nu \in U_q^0 \right\} = q(P)(f)$$

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for each $f \in L^1(P)$; see [9, II, Lemma 2.2] for the last identity. This shows that each functional of the form (3) is $\tau(P)$-continuous.

Conversely, let $\xi \in L^1(P)'$. Now every continuous linear functional on $\mathbf{L}(X)$ is of the form (2). However, when restricted to $\langle \overline{P} \rangle$, it can be specified more simply, as described in Lemma 1.2. So, if Φ_P is the linear isomorphism given by (12), then the adjoint map Φ'_P: $\langle \overline{P} \rangle' \to L^1(P)'$ is certainly a vector space isomorphism, and hence there exists $\nu \in \langle \overline{P} \rangle'$ such that $\xi = \Phi'_P(\nu)$. If $x \otimes x'$ is an elementary functional such that ν is given by (13), then (3) follows from (13) and from the identity $\langle f, \xi \rangle = \langle \Phi_P(f), \nu \rangle$, for each $f \in L^1(P)$. This completes the proof of Theorem 1.

The following result should be compared with Theorem 1.

PROPOSITION 2.1. *Let P: $\Sigma \to \mathbf{L}(X)$ be an equicontinuous spectral measure. Suppose that x_0 is a cyclic vector for P and that Px_0 is a closed measure. Then a linear functional ξ on $L^1(P)$ is continuous if, and only if, there is an element $x' \in X'$ and a Px_0-integrable function h such that*

$$(14) \qquad \langle f, \xi \rangle = \int_\Omega fh d\langle Px_0, x' \rangle, \qquad f \in L^1(P).$$

PROOF. Let $x' \in X'$ and let h be a Px_0-integrable function. Then $x = \int_\Omega h dPx_0$ belongs to X. If $f \in L^1(P)$, then Lemma 1.4 implies that fh is Px_0-integrable and that $\int_\Omega fh dPx_0 = P(f) \int_\Omega h dPx_0 = P(f)x$, from which it follows that

$$\int_\Omega fh d\langle Px_0, x' \rangle = \left\langle \int_\Omega fh dPx_0, x' \right\rangle = \langle P(f)x, x' \rangle = \int_\Omega f d\langle Px, x' \rangle.$$

Since the linear functional $f \to \int_\Omega f d\langle Px, x' \rangle$, $f \in L^1(P)$, is $\tau(P)$-continuous, it follows that the right-hand side of (14) defines an element of $L^1(P)'$.

Conversely, if $\xi \in L^1(P)'$, then there exist elements $x \in X$ and $x' \in X'$ such that ξ is given by (3). Since $P(\Sigma)[x_0] = X$, Lemma 1.3 implies that there exists a Px_0-integrable function h such that $x = \int_\Omega h dPx_0$. Using Lemma 1.4 again and arguing as above, we see that ξ is given by (14).

The main idea of the proof of Theorem 1 is a simple one: we exploit the fact that $L^1(P)$ is isomorphic to some space whose dual space is known. This same principle will eventually be used to prove Theorem 2 but, unlike the case of spectral measures, the appropriate representation of $L^1(P_f)$ that is needed is not yet available. We now proceed to formulate it.

Let P: $\Sigma \to \mathbf{L}(X)$ be a spectral measure and let f be a Σ-measurable function. Let $\mathbf{Z}(f)$ denote the zero set $\{\omega: f(\omega) = 0\}$ of f, and let f^{-1} denote the function on Ω defined to be $1/f(\omega)$ if $\omega \notin \mathbf{Z}(f)$, and zero otherwise. We say that $1/f$ is

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P-integrable whenever $\mathbf{Z}(f)$ is a P-null set and f^{-1} is a P-integrable function. In this case $ff^{-1} = 1$ P-a.e. In fact, f^{-1} is the unique P-integrable function with this property in the sense that if h is another Σ-measurable function such that $fh = 1$ P-a.e., then $h = f^{-1}$ P-a.e. (and hence h is also P-integrable).

LEMMA 2.2. *Let P: $\Sigma \to \mathbf{L}(X)$ be an equicontinuous spectral measure and let f be a P-integrable function. The following statements are equivalent.*

(i) *The operator $P(f) = \int_\Omega f dP$ is invertible in the space $\mathbf{L}(X)$.*

(ii) *$1/f$ is P-integrable.*

(iii) *The set $\mathbf{Z}(f)$ is P-null and the spaces $L(P)$ and $L(P_f)$ are equal as vector spaces.*

PROOF. The equivalence of (i) and (ii) constitute Lemma 3 of [14].

(ii) $\Rightarrow$ (iii). Let g be any P_f-integrable function. Then gf is P-integrable (cf. Lemma 1.1), and hence so is the product $h = (gf)f^{-1}$. Since $h = g$ P-a.e., it follows that g is P-integrable. In fact, it is easily calculated that

$$\int_E g dP = P(f)^{-1}\left(\int_E g dP_f\right), \qquad E \in \Sigma.$$

So $L(P_f) \subseteq L(P)$. The reverse inclusion follows from Lemma 1.1 and from the remarks made just prior to (12). Finally, the fact that $\mathbf{Z}(f)$ is P-null forms part of the definition of $1/f$ being P-integrable.

(iii) $\Rightarrow$ (ii). Observe that the function f^{-1} is Σ-measurable and satisfies $ff^{-1} = \chi_{\Omega \setminus \mathbf{Z}(f)}$. Since $\mathbf{Z}(f)$ is P-null by hypothesis, we have $ff^{-1} = 1$, P-a.e., and hence ff^{-1} is P-integrable. By Lemma 1.1 then, the function f^{-1} is P_f-integrable, and so also P-integrable (by the hypotheses of (iii)).

REMARKS. (1) The assumption in (iii) that $\mathbf{Z}(f)$ is P-null cannot be removed. For, if $\Omega = \{v, w\}$ is a two point set and Q is any self-adjoint projection ($\neq 0$ or I) in a Hilbert space X, then the set function P: $2^\Omega \to \mathbf{L}(X)$ defined by $P(\varnothing) = 0$, $P(\{v\}) = Q$, $P(\{w\}) = I - Q$, and $P(\Omega) = I$ is an equicontinuous spectral measure, and $f = \chi_{\{v\}}$ is a P-integrable function for which the spaces of P-integrable and P_f-integrable functions coincide (but $\mathbf{Z}(f)$ is *not* a P-null set). It is clear in this example that Lemma 2.2(i) does not follow.

(2) If f is a P-integrable function for which any of the equivalent conditions of Lemma 2.2 is satisfied, then the measures P and P_f have the same null sets.

(3) If X is a Banach space, then it is well known that the conditions (i) and (ii) of Lemma 2.2 are equivalent to $1/f$ being P-essentially bounded [4, XVII, Corollary 2.11 (i)]. This may no longer be the case in non-normable spaces. For example, let $X = L^1_{\mathrm{loc}}(\mathbb{R})$ and let P be the (closed) equicontinuous spectral measure of pointwise multiplication by characteristic functions of Borel subsets of

$\Omega = \mathbb{R}$. If $f(\omega) = \exp(\omega)$, $\omega \in \mathbb{R}$, then both f and $1/f$ are P-integrable, but neither of them is P-essentially bounded.

We may now state the representation theorem for $L^1(P_f)$ that is needed to prove Theorem 2.

PROPOSITION 2.3. *Let $P\colon \Sigma \to \mathbf{L}(X)$ be an equicontinuous spectral measure and let f be a Σ-measurable function such that both f and $1/f$ are P-integrable. Then the map $\Phi\colon L^1(P_f) \to L^1(P)$ defined by*

$$\Phi\colon g \to fg, \qquad g \in L^1(P_f), \tag{15}$$

is a linear isomorphism.

PROOF. To show that Φ is well-defined, let g and h be two P_f-integrable functions such that $g = h$ P_f-a.e. By Lemma 1.1, both fg and fh are P-integrable, and in the notation of (12) it follows that

$$\Phi_P(fg - fh) = \int_\Omega f(g-h)\,dP = \int_\Omega (g-h)\,dP_f = 0$$

in $\mathbf{L}(X)$. Since Φ_P is injective, we have $fg = fh$ in the space $L^1(P)$. Hence Φ is well defined.

If g, $h \in L^1(P_f)$ and $\Phi(g) = \Phi(h)$, then $fg = fh$ P-a.e., and hence, for any $E \in \Sigma$, Lemma 1.1. implies that

$$\int_E (g-h)\,dP_f = \int_E (fg - fh)\,dP = 0.$$

So the indefinite integral of $(g - h)$ with respect to P_f is the zero measure or, equivalently, $g = h$ P_f-a.e. This shows that Φ is injective.

To verify that Φ is onto, let h be any P-integrable function. Since f^{-1} is P-integrable, the function $g = hf^{-1}$ is also P-integrable and hence is P_f-integrable (cf. Lemma 2.2(iii)). But then the identities $\Phi(g) = fg = f(f^{-1}h)$ and $ff^{-1} = 1$ P-a.e. imply that $h = \Phi(g)$.

Let q be any continuous seminorm on $\mathbf{L}(X)$. Then it follows from (7) and from Lemma 1.1 that

$$\begin{aligned} q(P)(\Phi(g)) = q(P)(fg) &\leqslant 4\sup\left\{q\left(\int_E fg\,dP\right): E \in \Sigma\right\} \\ &= 4\sup\left\{q\left(\int_E g\,dP_f\right): E \in \Sigma\right\} \leqslant 4q(P_f)(g) \end{aligned}$$

for each $g \in L^1(P_f)$. Accordingly, Φ is continuous.

The inverse map $\Phi^{-1}\colon L^1(P) \to L^1(P_f)$ is given by

$$\Phi^{-1}\colon h \to hf^{-1}, \qquad h \in L^1(P).$$

So, again by (7) and Lemma 1.1, it follows that if q is any continuous seminorm on $\mathbf{L}(X)$, then

$$q(P_f)(\Phi^{-1}(h)) = q(P_f)(hf^{-1}) \leqslant 4\sup\left\{q\left(\int_E hf^{-1}dP_f\right): E \in \Sigma\right\}$$
$$= 4\sup\left\{q\left(\int_E hdP\right): E \in \Sigma\right\} \leqslant 4q(P)(h)$$

for each $h \in L^1(P)$. This shows that Φ^{-1} is also continuous and so completes the proof.

The proof of Theorem 2 is now an easy application of Proposition 2.3 and Theorem 1.

REMARK. An examination of the previous proof shows that for *any* P-integrable function f (even if $1/f$ is not P-integrable), the map defined by (15) is a (continuous) linear injection of $L^1(P_f)$ into $L^1(P)$. However, if $1/f$ is not P-integrable, then the inclusion of $L^1(P_f)$ in $L^1(P)$ may be strict. For example, let $X = L^1([0,1])$ and let P be the spectral measure of pointwise multiplication by characteristic functions of Borel subsets of $\Omega = [0,1]$. Then the identity function f on Ω is P-integrable, and P_f is the $\mathbf{L}(X)$-valued measure defined by

$$P_f\colon E \to P(E)P(f) = P(f)P(E), \qquad E \in \Sigma,$$

where $P(f) \in \mathbf{L}(X)$ is the operator in X of pointwise multiplication by f. Then $\mathbf{Z}(f)$ is a P-null set, but $1/f$ is not P-integrable as f^{-1} is not P-essentially bounded. It can be shown that the function

$$g\colon \omega \to (1/\sqrt{\omega})\chi_{(0,1]}(\omega), \qquad \omega \in \Omega,$$

belongs to $L^1(P_f)$ but not to $L^1(P)$.

3. Proofs of Theorems 3 and 4

The proof of Theorem 3 follows easily from Lemma 1.3. For, if $\xi \in L^1(Px)'$, then in the notation of Lemma 1.3 there is an element $y' \in P(\Sigma)[x]'$ such that $\xi = \Phi'_{Px}(y')$. Let x' be any element of X' which agrees with y' on $P(\Sigma)[x]$. Then

$$\langle f, \xi\rangle = \langle \Phi_{Px}(f), y'\rangle = \langle \Phi_{Px}(f), x'\rangle = \int_\Omega fd\langle Px, x'\rangle$$

for each $f \in L^1(Px)$, and so ξ is of the form (5). Conversely, if $x' \in X'$, then its restriction to $P(\Sigma)[x]$ is continuous, and hence the right-hand side of (5), being equal to $x' \circ \Phi_{Px}$, defines an element of $L^1(Px)'$.

In order to prove Theorem 4 we first need some preliminary results.

LEMMA 3.1. *Let $P: \Sigma \to \mathbf{L}(X)$ be an equicontinuous spectral measure, let f be a P-integrable function such that $1/f$ is also P-integrable and let x be an element of X such that Px is a closed measure in X. Then $L(Px)$ and $L(P_f x)$ are equal as vector spaces.*

PROOF. If g is a Px-integrable function, then fg is also Px-integrable (cf. Lemma 1.4), and hence Lemma 1.1. implies that g is $P_f x$-integrable. Conversely, if g is a $P_f x$-intregrable function, then fg is Px-integrable (cf. Lemma 1.1). Since f^{-1} is a P-integrable function such that $ff^{-1} = 1$ P-a.e., it follows that

$$g = (ff^{-1})g = (fg)f^{-1} \tag{16}$$

P-a.e. and hence also Px-a.e. As f^{-1} is P-integrable and fg is Px-integrable, (16) implies that g is Px-integrable (cf. Lemma 1.4).

PROPOSITION 3.2. *Let P, f and x be as in the statement of Lemma 3.1. Then the map Φ: $L^1(P_f x) \to L^1(Px)$ defined by*

$$\Phi: g \to fg, \qquad g \in L^1(P_f x),$$

is a linear isomorphism.

PROOF. That Φ is well-defined and injective can be verified as in the proof of Proposition 2.3 (where now Φ_P is suitably replaced by Φ_{Px}; see Lemma 1.3 for the notation). To verify that Φ is onto, let h be any Px-integrable function. Since f^{-1} is P-integrable, the function $g = hf^{-1}$ is Px-integrable (cf. Lemma 1.4) and hence is $P_f x$-integrable by Lemma 3.1. But then the identities $\Phi(g) = fg = f(hf^{-1})$ and $ff^{-1} = 1$, which hold P-a.e. (hence also Px-a.e.), imply that $h = \Phi(g)$. Finally, the fact that Φ and Φ^{-1} are both continuous can be shown by calculations identical to those in the proof of Proposition 2.3, the only difference being that the $\mathbf{L}(X)$-continuous seminorm q used there is now specified to be a continuous seminorm on X, and the measures P and P_f are replaced by Px and $P_f x$, respectively.

The proof of Theorem 4 is now an easy application of Proposition 3.2 and Theorem 3.

4. Criteria for closedness of spectral measures

It is clear from earlier sections that it is important to be able to determine the closedness of spectral measures. For, as noted previously, the L^1-spaces of such measures are complete and have an easily describable dual space. The simplest known criterion states that in a separable Fréchet space every spectral measure is closed (cf. proof of Corollary 4.7 in [15], for example). Given a *particular* spectral measure, an improvement of this result is the following.

THEOREM 5. *Let $P\colon \Sigma \to L(X)$ be an equicontinuous spectral measure.*

(i) *Let X be a Fréchet space. If there exists a sequence $\{x_i\}$ in X such that the linear span of*

$$(17) \qquad \{P(E)x_i\colon E \in \Sigma,\ i = 1, 2, \ldots\}$$

is dense in X, then P is a closed measure.

(ii) *Let X be quasicomplete and separable. Suppose that for each $x \in X$ there is a metrizable locally convex Hausdorff topology on the cyclic space $P(\Sigma)[x]$ which is weaker than (or equal to) the relative X-topology. Then P is a closed measure.*

(iii) *Suppose that $x \in X$ is a cyclic vector for P. If Px is a closed measure in X, then P is also a closed measure.*

PROOF. (i) It will be established that every set of disjoint projections in $P(\Sigma)$ is at most countable. It will then follow from [3, IV, Lemma 11.5] that every set in $P(\Sigma)$ has a least upper bound which is also the least upper bound of a countable subset. Thus the Bade completeness of $P(\Sigma)$, which is equivalent to P being a closed measure [2], will follow from its Bade σ-completeness [15, Proposition 1.3].

So, let $\{P(E_\alpha)\colon \alpha \in \mathscr{A}\}$ be a disjoint family in $P(\Sigma)$. Then $E_\alpha \cap E_\beta$ is a P-null set whenever $\alpha \neq \beta$. Let $q_1, q_2, \ldots$ be a sequence of continuous seminorms which determines the topology of X. For each natural number i, let m_i denote the X-valued measure Px_i.

Fix a natural number i. If k is a positive integer, then there exists a finite positive measure λ^i_k on Σ such that

$$(18) \qquad \lambda^i_k(E) \leqslant q_k(m_i)(E), \qquad E \in \Sigma,$$

and such that $\lambda^i_k(E) = 0$ implies that $q_k(m_i)(E) = 0$; see II, Corollary 1.2, and II, Theorem 1.1, of [9]. Since $E_\alpha \cap E_\beta$ is P-null for $\alpha \neq \beta$, it follows that $E_\alpha \cap E_\beta$ is also m_i-null [2, Proposition 1.7(iii)], that is, $m_i(E) = 0$ whenever $E \in \Sigma$ and $E \subseteq E_\alpha \cap E_\beta$, or, equivalently, $q_k(m_i)(E_\alpha \cap E_\beta) = 0$ for each $k = 1, 2, \ldots$; see (7). It follows from (18) that for each $\alpha, \beta \in \mathscr{A}$ with $\alpha \neq \beta$ we have $\lambda^i_k(E_\alpha \cap E_\beta) = 0$, for each $k = 1, 2, \ldots$. Choose positive constants ε^i_k, $k = 1, 2, \ldots$, such that $\lambda_i = \sum_{k=1}^{\infty} \varepsilon^i_k \lambda^i_k$ is a finite positive measure on Σ. Then $m_i(E) = 0$ whenever $E \in \Sigma$ and $\lambda_i(E) = 0$. Since $\lambda_i(E_\alpha \cap E_\beta) = 0$ whenever $\alpha \neq \beta$, it follows that $\lambda_i(E_\alpha) = 0$ for all but countably many $\alpha \in \mathscr{A}$, and hence that $m_i(E_\alpha) = 0$ for all but countably many $\alpha \in \mathscr{A}$. Accordingly, there is a countable subset $\mathscr{B}$ of $\mathscr{A}$ such that $P(E_\alpha)x_i = 0$ for each $\alpha \notin \mathscr{B}$ and $i = 1, 2, \ldots$. The density of the linear span of (18) then implies that $P(E_\alpha) = 0$ whenever $\alpha \notin \mathscr{B}$, as required.

(ii) If $x \in X$, then Px can be interpreted as having values in the cyclic space $P(\Sigma)[x]$ equipped with the relative X-topology, and hence there is a finite positive measure λ_x on Σ such that Px is absolutely continuous with respect to λ_x [6, Proposition 4]. The conclusion then follows from an argument as in the proof of [13, Theorem 1].

(iii) It follows from [8, Corollary 13] that there is a localizable measure λ on Σ such that the vector measure Px is absolutely continuous with respect to λ. If $y \in X$ is any element of the form $\sum_{i=1}^{n} \alpha_i P(E_i)x$, where the α_i are complex numbers, and where $E_i \in \Sigma$, $1 \leqslant i \leqslant n$, then it is clear that $P(E)y = 0$ whenever $E \in \Sigma$ is λ-null. Since any element of X can be approximated by such vectors y, it follows that $P(E) = 0$ whenever $E \in \Sigma$ and $\lambda(E) = 0$. Then [9, Theorem 7.3, IV] implies that P is a closed measure.

REMARKS. (1) Part (i) of Theorem 5 is well known in the Banach space setting [4, XVII, Lemma 3.21]. It was noted earlier that any vector measure with values in a Fréchet space is necessarily a closed measure. Hence, for the case of metrizable spaces, (iii) is a special case of (i).

(2) The hypothesis of part (ii) is satisfied if, for each $x \in X$, there exists a countable subset of $P(\Sigma)[x]'$ which separates points in $P(\Sigma)[x]$.

(3) The separability of X in (ii) cannot be replaced by the separability of each cyclic space $P(\Sigma)[x]$, $x \in X$, even if the space X itself is metrizable. For example, let X denote the Hilbert space $l^2([0,1])$ and let P denote the $L(X)$-valued spectral measure of multiplication by characteristic functions of Borel subsets of $[0,1]$. Then it is easily verified that each cyclic space $P(\Sigma)[x]$, $x \in X$, is separable (since the support of x is a countable subset of $[0,1]$). However, since the range of P is clearly not a closed subset of $L(X)$, it follows that P is not a closed measure [11, Proposition 3].

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