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A family of inequalities for convex sets

P.R. Scott

Let K be a bounded, closed convex set in the euclidean plane. We denote the diameter, width, perimeter, area, inradius, and circumradius of K by d, w, p, A, r, and R respectively. We establish a number of best possible upper bounds for $(w-2r)d$, $(w-2r)R$, $(w-2r)p$, $(w-2r)A$ in terms of w and r. Examples are:

$$(w-2r)d < w^2/2 ,$$

$$(w-2r)d \leq 2wr/\sqrt{3} .$$

1. Introduction

Let K be a bounded, closed convex set in the euclidean plane. We denote the diameter, width, perimeter, area, inradius, and circumradius of K by d, w, p, A, r, and R respectively.

The inequalities we shall establish will be shown to be best possible; we either obtain equality when K is an equilateral triangle (denoted E), or the upper bound is the limit as K approaches an "infinite isosceles triangle" of fixed base and unbounded altitude (denoted I). The inequalities, together with the critical figures, are given below.

THEOREM 1. $(w-2r)d < w^2/2 \quad (I)$.

THEOREM 2. $(w-2r)d \leq 2wr/\sqrt{3} \quad (E)$.

THEOREM 3. $(w-2r)R < w^2/4 \quad (I)$.

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THEOREM 4. $(w-2r)R \le 2wr/3$ (E) .

THEOREM 5. $(w-2r)p \le 2w^2/\sqrt{3}$ (E) .

THEOREM 6. $(w-2r)A < w^3/4$ (I) .

THEOREM 7. $(w-2r)A \le w^2r/\sqrt{3}$ (E) .

By Blaschke's Theorem [1], every bounded convex figure of width w contains a circle of radius $w/3$. It follows that $w \le 3r$; equality holds here when and only when the figure is an equilateral triangle. We thus obtain immediately the following corollaries.

COROLLARY 1. $(w-2r)d \le 2\sqrt{3}\, r^2$ (E) .

COROLLARY 2. $(w-2r)R \le 2r^2$ (E) .

COROLLARY 3. $(w-2r)p \le 2\sqrt{3}\, wr \le 6\sqrt{3}\, r^2$ (E) .

COROLLARY 4. $(w-2r)A \le \sqrt{3}\, wr^2 \le 3\sqrt{3}\, r^3$ (E) .

None of these inequalities appears in [2], [3]; the first corollary is proved independently in [4].

2. Some preliminaries

We shall require the following result.

LEMMA 1. *Let two triangles have the same vertex angle and*

(a) the same perimeter, or

(b) the same area.

Then in either case, the triangle for which the difference in base angles is smaller has the smaller circumradius and the larger inradius.

Proof. Let the triangles be ΔBCD , $\Delta B'CD'$, with $\angle B' \le \angle B$, $\angle D \le \angle D'$ (Figure 1). Let R, R' and r, r' be the circumradii and inradii of ΔBCD, $\Delta B'CD'$, respectively.

We note that $\angle B + \angle D = \angle B' + \angle D'$ $(= \pi - \angle C)$, and $\angle D - \angle B \le \angle D' - \angle B'$ by assumption.

(a) Let the triangles have common perimeter p . Then in ΔBCD ,

$$p = 2R(\sin B + \sin C + \sin D) .$$

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Hence

$$p/2R = \sin C + 2\sin \tfrac{1}{2}(B+D)\cos \tfrac{1}{2}(B-D)$$
$$\geq \sin C + 2\sin \tfrac{1}{2}(B'+D')\cos \tfrac{1}{2}(B'-D')$$
$$= p/2R \ .$$

Hence $R \leq R'$.

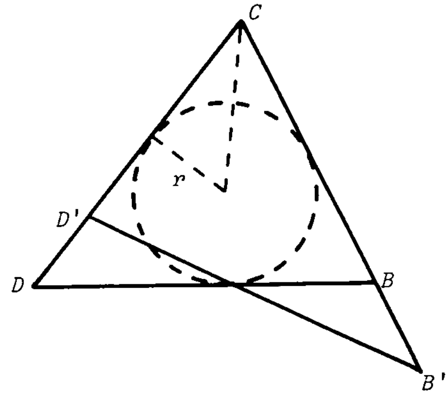

FIGURE 1

(b) Let the triangles have the same area A . Then

$$A = \tfrac{1}{2}BC.DC.\sin C$$
$$= 2R^2 \sin B \sin C \sin D \ .$$

Hence

$$A/R^2 = \sin C\{\cos(B-D) - \cos(B+D)\}$$
$$\geq \sin C\{\cos(B'-D') - \cos(B'+D')\}$$
$$= A/R'^2 \ .$$

Hence again $R \leq R'$.

In either case we now deduce that

$$BD = 2R \sin C \leq 2R' \sin C = B'D' \ ,$$

and so

$$r = (p-2BD)\tan(C/2) \geq (p-2B'D')\tan(C/2) = r' \ ,$$

as required.

This completes the proof of the lemma.

The incircle of K meets the boundary of K either in two diametrically opposite points, or in three points forming the vertices of an acute angled triangle. In the first case, $w = 2r$, and each theorem is trivially true. In the second case, K is contained in a triangle T formed by three lines of support common to K and the circle.

Our procedure in the proof of each theorem will be to show that K must be a triangle T_i satisfying certain conditions; we shall then use the following lemma to show that T_i is isosceles.

LEMMA 2. *Let* T_i *be a triangle,* ΔXYZ *, with* $\underline{/X} \le \underline{/Y} \le \underline{/Z}$ *. Choose a point* Z' *so that* $ZZ' \| XY$ *and* $T_i^* = \Delta XYZ'$ *satisfies* $\underline{/X} \le \underline{/Y} = \underline{/Z}$ *(Figure 2). Then*

$$w(T_i) = w(T_i^*) , \quad d(T_i) = d(T_i^*) , \quad A(T_i) = A(T_i^*) ,$$

$$p(T_i) \le p(T_i^*) , \quad r(T_i) \ge r(T_i^*) , \quad R(T_i) \le R(T_i^*) .$$

Proof. The assertions about the width, diameter, and area follow easily from the choice of Z' , and the constraints on the angles. The inequality on perimeter results from a well known shortest path problem. Since $2A = rp$ for each triangle, we deduce that $r(T_i^*) \le r(T_i)$. Finally,

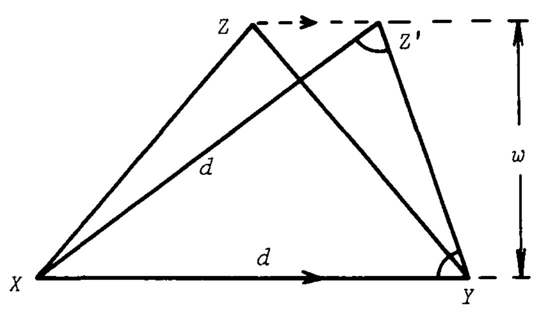

FIGURE 2

$$XY/2R(T_i) = \sin Z \le \sin Z' = XY/2R(T_i^*) ,$$

and hence

$$R(T_i) \le R(T_i^*) .$$

3. Proof of Theorems 1 and 2

Let K be contained in the triangle $T = \Delta BCD$, where $\underline{/B} \le \underline{/C} \le \underline{/D}$. Now $d(T) = BC$, and

$$w(K) \le w(T) , \quad r(K) = r(T) , \quad d(K) \le d(T) .$$

In proving Theorem 1, we seek to maximise $(w-2r)d$ for fixed w . We choose a point D' on DC distant w from BC , and let T_1 denote $\Delta D'BC$. Then

$$w(K) = w(T_1) , \quad d(K) \le d(T_1) ,$$

and

$$r(K) = r(T) \ge r(T_1) ,$$

since T_1 is a subset of T. Hence we may assume that K is the triangle T_1. From Lemma 2, we see that w is left invariant and $(w-2r)d$ is not decreased by taking T_1 isosceles.

We notice that the statement of Theorem 2 is equivalent to

$$(1/w) + \left(1/(\sqrt{3}\, d)\right) \geq (1/2r) .$$

Taking $K = T$ fixes r and does not decrease w, d; now Lemma 2 shows that we can assume T to be an isosceles triangle.

Let K be the isosceles triangle in Figure 3. We have

$w = d \sin B = 2d \sin D \cos D$.

Also

$2A = wd = pr$

$= r(2d+2d \cos D)$.

Hence

$w = 2r(1 + \cos D)$,

and

$(w-2r)d = 2rd \cos D$.

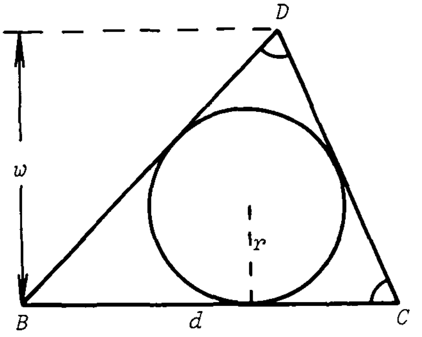

FIGURE 3

It follows that

$$\left(w^2\right)/\left((w-2r)d\right) = \left(2d \sin D \cos D . 2r(1 + \cos D)\right)/(2rd \cos D)$$
$$= 2 \sin D + \sin 2D$$
$$> 2 \sin (\pi/2) + \sin \pi$$
$$= 2 ,$$

since $2 \sin D + \sin 2D$ is a decreasing function of D over the allowable range $\pi/3 \leq D < \pi/2$.

Hence

$$(w-2r)d < w^2/2 .$$

Similarly

$$(wr)/\big((w-2r)d\big) = (2d \sin D \cos D.r)/(2rd \cos D)$$
$$= \sin D$$
$$\geq \sin(\pi/3) \quad \text{for} \quad \pi/3 \leq D < \pi/2$$
$$= \sqrt{3}/2 \ .$$

Hence

$$(w-2r)d \leq 2wr/\sqrt{3} \ .$$

4. Proof of Theorems 3 and 4

To establish Theorem 3, we seek to maximise $(w-2r)R$ for given w. Let K be contained in the triangle $T = \Delta BCD$. Choose a point D' on DC distant w from BC, and choose a point B' on the ray $\overrightarrow{CB}$ so that the triangle $T_2 = \Delta B'CD'$ satisfies

$$p(T_2) = p(T) \ .$$

Then

$$w(K) = w(T_2) \ ,$$

and

$$r(K) = r(T) \geq r(T_2) \ , \quad R(K) \leq R(T) \leq R(T_2) \ ,$$

by Lemma 1.

Hence we may assume that K is the triangle T_2; also that T_2 is isosceles, by Lemma 2.

The statement of Theorem 4 is equivalent to

$$1/w + 1/3R \geq 1/2r \ .$$

Obviously we may here take K to be the triangle T; by Lemma 2, T may be assumed isosceles.

Now

$$\big(w^2\big)/\big((w-2r)R\big) = \Big(\big(w^2\big)/\big((w-2r)d\big)\Big).(d/R)$$
$$= (2 \sin D + \sin 2D).2 \sin D$$
$$> (2 \sin(\pi/2) + \sin \pi).2 \sin(\pi/2)$$
$$= 4 \ ,$$

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since $(2 \sin D + \sin 2D).2 \sin D$ assumes its minimum value of 4 at $D = \pi/2$ when D satisfies $\pi/3 \le D \le \pi/2$.

Hence

$$(w-2r)R < w^2/4 \ .$$

Also

$$\begin{aligned}(wr)/\big((w-2r)R\big) &= \big((wr)/\big((w-2r)d\big)\big).(d/R)\\ &= \sin D.2 \sin D\\ &\ge 2 \sin^2(\pi/3) \quad (\text{for } \pi/3 \le D \le \pi/2)\\ &= 3/2 \ .\end{aligned}$$

Thus

$$(w-2r)R \le 2wr/3 \ .$$

5. Proof of Theorem 5

We seek to maximise $(w-2r)p$ for given w . We may assume that K is the isosceles triangle T_2 defined in Section 4. We see that

$$\begin{aligned}\big(w^2\big)/\big((w-2r)p\big) &= (wr)/\big((w-2r)d\big)\\ &= \big((wr)/(2r \cos D)\big).\big((\sin 2D)/w\big)\\ &= \sin D\\ &\ge \sqrt{3}/2 \ ,\end{aligned}$$

since $\pi/3 \le D < \pi/2$. Hence

$$(w-2r)p \le 2w^2/\sqrt{3} \ .$$

6. Proof of Theorems 6 and 7

To prove Theorem 6, we maximise $(w-2r)A$ for given w . Let K be contained in triangle $T = \Delta BCD$. Choose point D' on DC distant w from BC , and choose a point B' on the ray CB so that triangle $T_3 = \Delta B'CD'$ satisfies

$$A\big(T_3\big) = A(T) \ .$$

Then

$$w(K) = w\big(T_3\big) \ , \quad A(K) \le A(T) = A\big(T_3\big) \ ,$$

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and

$$r(K) = r(T) \geq r(T_3) ,$$

by Lemma 1. By Lemma 2 we may take T_3 to be an isosceles triangle.

The inequality of Theorem 7 is equivalent to

$$(2/w^2) + (1/\sqrt{3}A) \geq (1/wr) .$$

Taking $K = T_3$ fixes w, does not decrease A and does not increase r; as usual, Lemma 2 gives T_3 isosceles.

Now

$$\begin{aligned}(w^3)/((w-2r)A) &= ((w^3)/(w-2r)).(2/wd) \\ &= (2w^2)/((w-2r)d) \\ &> 4 ,\end{aligned}$$

as in Section 3.

Hence

$$(w-2r)A < w^3/4 .$$

Finally,

$$\begin{aligned}(w^2r)/((w-2r)A) &= ((w^2r)/((w-2r))).(1/dw) \\ &\geq \sqrt{3} ,\end{aligned}$$

as in Section 3.

Hence $(w-2r)A \leq w^2r/\sqrt{3}$.

References

[1] Wilhelm Blaschke, *Kreis und Kugel*, 2te Aufl. (Walter de Gruyter, Berlin, 1956).

[2] O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović, P.M. Vasić, *Geometric inequalities* (Wolters-Noordhoff, Groningen, 1969).

[3] Marlow Sholander, "On certain minimum problems in the theory of convex curves", *Trans. Amer. Math. Soc.* 73 (1952), 139-173.

[4] P.R. Scott, "Two inequalities for convex sets in the plane", *Bull. Austral. Math. Soc.* **19** (1978), 131-133 (1979).

Department of Pure Mathematics,
University of Adelaide,
Adelaide,
South Australia.