Canad. Math. Bull. Vol. **53** (2), 2010 pp. 206–217
doi:10.4153/CMB-2010-003-0

Semi-Slant Submanifolds of an Almost Paracontact Metric Manifold

Mehmet Atçeken

Abstract. In this paper, we define and study the geometry of semi-slant submanifolds of an almost paracontact metric manifold. We give some characterizations for a submanifold to be semi-slant submanifold to be semi-slant product and obtain integrability conditions for the distributions involved in the definition of a semi-slant submanifold.

1 Introduction

Slant submanifolds have been studied by many geometers in the last two decades. They arise naturally and play important roles in the study of the geometry of submanifolds [2, 3, 6]. Slant immersions in complex geometry were first introduced by B.-Y. Chen as a natural generalization of both holomorphic immersions and totally real immersions [4, 5].

Later, A. Lotta introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold and he proved some properties of slant immersions [7].

Recently, Papaghuic introduced a class of submanifolds in an almost Hermitian manifold, called the semi-slant submanifold such that the class of proper CR-submanifolds and the class of slant submanifolds appear as particular cases in the class of semi-slant submanifolds [1, 9].

The purpose of the present paper is to define and study a paracontact version of semi-slant submanifolds so that both semi-invariant and paracontact slant submanifolds appear as particular cases of the introduced notion. Furthermore, we also give sufficient and necessary conditions for a distribution to be slant.

2 Preliminaries

In this section, we review basic formulas and definitions for almost paracontact metric manifolds and their submanifolds, which we shall use later.

Let M be an $(m+1)$-dimensional differentiable manifold. If there exist on M a $(1,1)$ type tensor field F, a vector field ξ and 1-form η satisfying

$$F^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

Received by the editors April 5, 2007.
Published electronically December 4, 2009.
AMS subject classification: **53C15**, 53C25, 53C4.
Keywords: paracontact metric manifold, slant distribution, semi-slant submanifold, semi-slant product.

then M is said to be an almost paracontact manifold. In the almost paracontact manifold, the following relations hold:

$$F\xi = 0, \quad \eta \circ F = 0, \quad \operatorname{rank}(F) = m. \tag{2.2}$$

An almost paracontact manifold M is said to be an almost paracontact metric manifold if Riemannian metric g satisfies

$$g(FX, FY) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2.3}$$

for all $X, Y \in \Gamma(TM)$. From (2.2) and (2.3), we can easily derive the relation

$$g(FX, Y) = g(X, FY). \tag{2.4}$$

Now let $\bar{M}$ be an isometrically immersed submanifold in an almost paracontact metric manifold M. We denote by $\bar{\nabla}$ and ∇ the Levi–Civita connections on $\bar{M}$ and M, respectively. Then the Gauss and Weingarten formulas are defined by

$$\nabla_X Y = \bar{\nabla}_X Y + h(X, Y) \quad \text{and} \quad \nabla_X V = -A_V X + \nabla^{\perp}_X V$$

for any $X, Y \in \Gamma(T\bar{M})$, $V \in \Gamma(T\bar{M}^{\perp})$, where $\nabla^{\perp}$ is the connection in the normal bundle $T\bar{M}^{\perp}$, h is the second fundamental form of $\bar{M}$, and A_V is the shape operator. The second fundamental form h and the shape operator A are related by

$$g(A_V X, Y) = g(h(X, Y), V). \tag{2.5}$$

An almost paracontact metric manifold is said to be an almost paracontact manifold with (F, η, ξ, g)-connection if $\nabla F = 0$ and $\nabla \eta = 0$, where ∇ denotes a connection on M. Since $F^2 = I - \eta \oplus \xi$, the vector field ξ is also parallel with respect to ∇ [8].

In the rest of this paper, we assume that M is an almost paracontact metric manifold with a structure (F, η, ξ, g).

Now let $\bar{M}$ be an n-dimensional differentiable manifold and suppose that $\bar{M}$ is an isometrically immersed submanifold in almost paracontact metric manifold M. We denote by g the induced Riemannian metric for $\bar{M}$ as well as M. For any vector field X tangent to $\bar{M}$, we put

$$FX = fX + \omega X, \tag{2.6}$$

where fX and ωX denote the tangential and normal components of FX, respectively. For any vector field N normal to $\bar{M}$, we also put

$$FN = BN + CN, \tag{2.7}$$

where BN and CN denote the tangential and normal components of FN, respectively. The submanifold $\bar{M}$ is said to be invariant if ω is identically zero, *i.e.,* $FX = fX \in$

https://doi.org/10.4153/CMB-2010-003-0 Published online by Cambridge University Press

$\Gamma(T\bar{M})$ for any $X \in \Gamma(T\bar{M})$. On the other hand, $\bar{M}$ is said to be an anti-invariant submanifold if f is identically zero, *i.e.*, $FX = \omega X \in \Gamma(T\bar{M}^{\perp})$ for any $X \in \Gamma(T\bar{M})$.

We note that for an invariant submanifold $\bar{M}$ of an almost paracontact metric manifold M, if ξ is normal to $\bar{M}$, then the induced almost paracontact structure on $\bar{M}$ is an almost product Riemannian structure. But if ξ is tangent to $\bar{M}$, then the induced almost paracontact metric structure on $\bar{M}$ is an almost paracontact metric structure.

Furthermore, we say that $\bar{M}$ is a semi-invariant submanifold if there exist two orthogonal distributions D_1 and D_2 such that

(i) $T\bar{M}$ has the orthogonal direct sum $T\bar{M} = D_1 \oplus D_2$,
(ii) the distribution D_1 is invariant, *i.e.*, $F(D_1) = D_1$,
(iii) the distribution D_2 is anti-invariant, *i.e.*, $F(D_2) \subset T\bar{M}^{\perp}$.

Given any submanifold $\bar{M}$ of M, from (2.4) and (2.6) we have

$$g(fX, Y) = g(X, fY) \tag{2.8}$$

for any $X, Y \in \Gamma(T\bar{M})$.

Henceforth we suppose that the vector field ξ is tangent to $\bar{M}$. If we denote by D the orthogonal distribution to ξ in $T\bar{M}$, then we can consider the orthogonal direct sum $T\bar{M} = D \oplus \xi$.

For each nonzero vector X tangent to $\bar{M}$ at x such that X is not proportional to ξ_x, we denote by $\theta(X)$ the angle between FX and $T_x\bar{M}$. In fact, since $F\xi = 0$, θ agrees with the angle between FX and D_x. Then $\bar{M}$ is said to be slant if the angle $\theta(X)$ is constant, which is independent of the choice of $x \in \bar{M}$ and $X \in T_x\bar{M} - sp\{\xi_x\}$. The angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion that is neither invariant nor anti-invariant is called a proper slant immersion.

3 Slant Submanifolds in Almost Paracontact Metric Manifolds

Next we will give an example of a slant submanifold in an almost paracontact metric manifold to illustrate our results.

Example 3.1 Let $\mathbb{R}^7$ be the Euclidean space endowed with the usual Euclidean metric and with coordinates $(x_1, x_2, y_1, y_2, y_3, y_4, t)$. We define an almost paracontact metric structure on $\mathbb{R}^7$ by

$$F\Big(\frac{\partial}{\partial x_i}\Big) = \frac{\partial}{\partial x_i}, \quad F\Big(\frac{\partial}{\partial y_j}\Big) = -\frac{\partial}{\partial y_j}, \ i = 1,2,\ j = 1,2,3,4, \quad F\Big(\frac{\partial}{\partial t}\Big) = 0,$$

$$\xi = \frac{\partial}{\partial t}, \eta = dt.$$

https://doi.org/10.4153/CMB-2010-003-0 Published online by Cambridge University Press

For any $Z = \lambda_i \frac{\partial}{\partial x_i} + \mu_j \frac{\partial}{\partial y_j} + \nu \frac{\partial}{\partial t} \in T\mathbb{R}^7$, we have

$$g(Z, Z) = \lambda_i^2 + \mu_j^2 + \nu^2 \quad \text{and} \quad g(FZ, FZ) = \lambda_i^2 + \mu_j^2$$

$$F^2 Z = \lambda_i \frac{\partial}{\partial x_i} + \mu_j \frac{\partial}{\partial y_j} = Z - \eta(Z)\xi \quad \text{and} \quad \eta(\xi) = 1.$$

for $i = 1, 2,\ j = 1, 2, 3, 4$. It follows that $g(FZ, FZ) = g(Z, Z) - \eta(Z)\eta(Z)$. Now, consider, for any $u, v \in (0, \frac{\pi}{2})$ and constant $k \neq 0$,

$$\varphi(u, v) = (u, v, -k \sin u, -k \sin v, k \cos u, k \cos v)$$

defines a slant submanifold in $\mathbb{R}^7$ with slant angle $\theta = \cos^{-1}(\frac{1-k^2}{1+k^2})$.

The following theorem is a useful characterization of slant submanifolds in an almost paracontact manifold.

Theorem 3.2 *Let $\bar{M}$ be an immersed submanifold of an almost paracontact metric manifold M.*

(i) *Let ξ be tangent to $\bar{M}$. In this case, $\bar{M}$ is slant if and only if there exist a constant $\lambda \in [0, 1]$ such that $f^2 = \lambda(I - \eta \otimes \xi)$.*

(ii) *Let ξ be normal to $\bar{M}$. In this case, $\bar{M}$ is slant if and only if there exist a constant $\lambda \in [0, 1]$ such that $f^2 = \lambda I$.*

Furthermore, if θ is the slant angle of $\bar{M}$, it satisfies $\lambda = \cos^2 \theta$.

Proof (i) We suppose that ξ is tangent to $\bar{M}$ and $\bar{M}$ is a slant submanifold. Also, we assume $\cos\theta(X) = \frac{\|fX\|}{\|FX\|}$, where $\theta(X)$ is the slant angle. From (2.4) and (2.6) we have

$$\begin{aligned} g(f^2 X, X) &= g(fX, fX) = \cos^2 \theta(X) g(FX, FX) \\ &= \cos^2 \theta(X) g(F^2 X, X) = \cos^2 \theta(X) g(X - \eta(X)\xi, X) \end{aligned}$$

for all $X \in \Gamma(T\bar{M})$. Since g is a Riemannian metric, we induce

$$f^2 X = \cos^2 \theta (X - \eta(X)\xi).$$

Let $\lambda = \cos^2 \theta$. Then $\lambda \in [0, 1]$ and $f^2 = \lambda(I - \eta \otimes \xi)$.

Conversely, we suppose that there exists a constant $\lambda \in [0, 1]$ such that $f^2 = \lambda(I - \eta \otimes \xi)$. Then by using (2.3) and (2.4) we have

$$\begin{aligned} \cos\theta(X) &= \frac{g(FX, fX)}{\|FX\|\|fX\|} = \frac{g(X, f^2 X)}{\|FX\|\|fX\|} = \lambda \frac{g(X, X - \eta(X)\xi)}{\|FX\|\|fX\|} \\ &= \lambda \frac{g(X, F^2 X)}{FX\|\|fX\|} = \lambda \frac{g(FX, FX)}{\|FX\|\|fX\|} = \lambda \frac{\|FX\|}{\|fX\|}, \end{aligned}$$

for any $X \in \Gamma(T\bar{M})$. On the other hand, since $\cos\theta(X) = \frac{\|fX\|}{\|FX\|}$, we conclude that $\cos^2 \theta(X) = \lambda$, that is, $\theta(X)$ is a constant and so $\bar{M}$ is slant.

(ii) If ξ is a normal vector field to $\bar{M}$, then we conclude that $\eta(X) = 0$. Thus from Theorem 3.2(i), we mean that $\bar{M}$ is a slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that $f^2 = \lambda I$. Moreover, if θ is the slant angle of $\bar{M}$, it satisfies $\lambda = \cos^2\theta$. ■

Corollary 3.3 *Let $\bar{M}$ be a slant submanifold of an almost paracontact metric manifold M with slant angle θ such that ξ is tangent to $\bar{M}$. Then we have*

$$g(fX, fY) = \cos^2\theta\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{3.1}$$

$$g(\omega X, \omega Y) = \sin^2\theta\{g(X,Y) - \eta(X)\eta(Y)\} \tag{3.2}$$

for any $X, Y \in \Gamma(T\bar{M})$.

Proof From (2.8) and Theorem 3.2(i), a direct expansion gives (3.1). To prove (3.2), it is enough to take into account (2.3) and (2.6). ■

Let $\bar{M}$ be an immersed submanifold of an almost paracontact metric manifold M. Then from (2.1), (2.6), and (2.7) we have

$$X - \eta(X\xi = f^2X + \omega fX + B\omega X + C\omega X \tag{3.3}$$

for any $X \in \Gamma(T\bar{M})$. If the vector field ξ is tangent to $\bar{M}$, then from the tangential and normal components of (3.3), we have

$$f^2 + B\omega = I - \eta \otimes \xi, \tag{3.4}$$

$$\omega f + C\omega = 0. \tag{3.5}$$

On the other hand, if the vector field ξ is normal to $\bar{M}$, then, (3.4) and (3.5) become

$$I = f^2 + B\omega, \tag{3.6}$$

$$-\eta \otimes \xi = \omega f + C\omega. \tag{3.7}$$

Thus we have the following results.

Corollary 3.4 *Let $\bar{M}$ be an immersed submanifold of an almost paracontact metric manifold M.*

(i) *Let ξ be tangent to $\bar{M}$. In this case, $\bar{M}$ is a slant submanifold of M if and only if there exists a constant $\mu \in [0, 1]$ such that $B\omega = \mu(I - \eta \otimes \xi)$.*
(ii) *Let ξ be normal to $\bar{M}$. In this case, $\bar{M}$ is a slant submanifold of M if and only if there exists a constant $\mu \in [0, 1]$ such that $B\omega = \mu I$.*

Furthermore, if θ is the slant angle of $\bar{M}$, it satisfies $\mu = \sin^2\theta$.

Proof If ξ is tangent to $\bar{M}$, then from Theorem 3.2(i) and (3.6) we get the proof of (i). On the other hand, if ξ is normal to $\bar{M}$, then Theorem 3.2(ii) and (3.6) give (ii), where $\mu = 1 - \lambda$, which satisfies our assertion. ■

4 Semi-Slant Submanifolds in Almost Paracontact Metric Manifolds

Let $\bar{M}$ be an immersed submanifold of an almost paracontact metric manifold M.

Definition 4.1 We call a differentiable distribution D on M a *slant distribution* if for each $x \in M$ and each nonzero $X \in D_x$, the angle θ_x between FX and D_x is a constant that is independent of the choice $x \in M$ and $X \in D_x$. In this case, the constant angle θ_x is called the *slant angle* of the distribution D_x.

Let $\bar{M}$ be an immersed submanifold of almost paracontact metric manifold M and D be a differentiable distribution on $\bar{M}$. We denote by $D^\perp$ the orthogonal distribution to D in $\bar{M}$. Also, P_1 and P_2 denote the orthogonal projections on D and $D^\perp$, respectively. Then for any $X \in \Gamma(T\bar{M})$, we can write

$$FX = P_1 fX + P_2 fX + \omega X. \tag{4.1}$$

Thus we have the following theorem.

Theorem 4.2 *Let D be a differentiable distribution on $\bar{M}$ such that ξ is tangent to D. Then D is a slant distribution if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$(P_1 f)^2 = \lambda(I - \eta \otimes \xi). \tag{4.2}$$

Furthermore, in such a case, if θ is the slant angle of D, then $\lambda = \cos^2\theta$.

Proof We suppose that there exists a constant $\lambda \in [0,1]$ such that

$$(P_1 f)^2 X = \lambda(X - \eta(X)\xi)$$

for any $X \in \Gamma(D)$. Then from (2.4) and (4.2) we have

$$\begin{aligned}\cos\theta(X) &= \frac{g(FX, P_1 fX)}{\|FX\| \cdot \|P_1 fX\|} = \frac{g(X, FP_1 fX)}{\|FX\| \cdot \|P_1 fX\|} = \frac{g(X, (P_1 f)^2 X)}{\|FX\| \cdot \|P_1 fX\|} \\ &= \lambda\frac{g(X, X - \eta(X)\xi)}{\|FX\| \cdot \|P_1 fX\|} = \lambda\frac{g(X, F^2X)}{\|FX\| \cdot \|P_1 fX\|} = \lambda\frac{\|FX\|}{\|P_1 fX\|}.\end{aligned}$$

Moreover, we know that $\cos\theta(X) = \frac{\|P_1 fX\|}{\|FX\|}$. Thus we can derive $\lambda = \cos^2\theta$, *i.e.*, θ is a constant and so D is slant.

Conversely, we assume that D is a slant distribution. Then from (4.1) and $\|P_1 fX\| = \cos\theta\|FX\|$ we have

$$g(X, (P_1 f)^2 X) = \cos^2\theta g(FX, FX) = \cos^2\theta g(X, F^2X) = \cos^2\theta g(X, X - \eta(X)\xi),$$

which implies $(P_1 f)^2 X = \cos^2\theta(X - \eta(X)\xi)$ for any $X \in \Gamma(D)$. Setting $\lambda = \cos^2\theta$, we get the desired result. Here we note that if ξ is normal to $\bar{M}$, then (4.2) becomes $(P_1 f)^2 = \lambda I$. ■

https://doi.org/10.4153/CMB-2010-003-0 Published online by Cambridge University Press

Lemma 4.3 *Let $\bar{M}$ be a submanifold of an almost paracontact metric manifold M and D be a distribution on $\bar{M}$. Then $\bar{M}$ is a slant submanifold if and only if D is a slant distribution with the same slant angle.*

Proof It is obvious that if $\bar{M}$ is a slant submanifold, then it is easy to see that D is a slant distribution with the same slant angle, because $\theta(X) = \theta_D(X)$ for any $X \in \Gamma(D)$. Conversely, given $X \in \Gamma(T\bar{M}) - \mathrm{sp}\{\xi\}$, we have

$$\cos\theta(X) = \frac{g(fX, FX)}{\|fX\|\|FX\|} = \frac{\|fX\|}{\sqrt{\|X\|^2 - \eta^2(X)}}. \tag{4.3}$$

On the other hand, taking into account $X - \eta(X)\xi \in \Gamma(D)$, we derive

$$\cos\theta_D = \frac{\|P(X - \eta(X)\xi)\|}{\|X - \eta(X)\xi\|}, \tag{4.4}$$

where P denotes the orthogonal projection of F on D. But in almost paracontact manifolds, by virtue of $\sqrt{\|X\|^2 - \eta^2(X)} = \|X - \eta(X)\|$ and $fX = P(X - \eta(X)\xi)$, (4.3) is equal to (4.4), which gives our assertion. ∎

Semi-slant submanifolds are generalizations of semi-invariant submanifolds.

Definition 4.4 We define $\bar{M}$ to be a semi-slant submanifold of an almost paracontact metric manifold M if there exist two orthogonal distributions D_1 and D_2 on $\bar{M}$ such that

(i) $T\bar{M}$ admits the orthogonal direct sum $T\bar{M} = D_1 \oplus D_2 \oplus \mathrm{sp}\{\xi\}$,
(ii) the distribution D_1 is invariant, *i.e.*, $F(D_1) = D_1$,
(iii) the distribution D_2 is slant with slant angle $\theta \neq 0, \pi/2$.

In this case, we call θ the slant angle of submanifold $\bar{M}$.

It is easily seen that the invariant and anti-invariant distributions of a semi-slant submanifold are slant distributions with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. Thus it is obvious that semi-invariant submanifolds are particular cases of semi-slant submanifolds. Furthermore, if we denote the dimension of D_i by d_i for $i = 1, 2$, then we have the following cases.

(i) If $d_2 = 0$, then $\bar{M}$ becomes an invariant submanifold.
(ii) If $d_1 = 0$ and $\theta = \pi/2$, then $\bar{M}$ becomes an anti-invariant submanifold.
(iii) If $d_1 = 0$ and $\theta \neq 0, \pi/2$, then $\bar{M}$ becomes a proper slant submanifold with slant angle θ.
(iv) If $d_1 \cdot d_2 \neq 0$ and $\theta \neq 0, \pi/2$, then $\bar{M}$ becomes a proper semi-slant submanifold.

Next, given a semi-slant submanifold $\bar{M}$ in an almost paracontact metric manifold M, we denote P_i the projections on the distributions D_i for $i = 1, 2$. Then we have

$$X = P_1X + P_2X \quad \text{and} \quad FX = fP_1X + fP_2X + \omega P_2X \tag{4.5}$$

and

$$g(fX, fP_2Y) = \cos^2\theta g(X, P_2Y) \quad \text{and} \quad g(\omega X, \omega P_2Y) = \sin^2\theta g(X, P_2Y), \tag{4.6}$$

for any $X, Y \in \Gamma(T\bar{M})$.

Now let $\bar{M}$ be an immersed submanifold of an almost paracontact metric manifold M. From the Gauss–Weingarten formulas and (2.6) and (2.7) we have

$$(\bar{\nabla}_X fY) = A_{\omega Y}X + Bh(X,Y), \tag{4.7}$$

$$(\nabla_X \omega)Y = Ch(X,Y) - h(X, fY), \tag{4.8}$$

for any $X, Y \in \Gamma(T\bar{M})$, where the covariant derivatives of f and ω are defined by

$$\bar{\nabla}_X fY = \bar{\nabla}_X fY - f(\bar{\nabla}_X Y) \quad \text{and} \quad (\nabla_X \omega)Y = \nabla^{\perp}_X \omega Y - \omega(\bar{\nabla}_X Y).$$

Next we shall characterize semi-slant submanifolds in almost paracontact metric manifolds by the following theorems.

Theorem 4.5 *Let $\bar{M}$ be an immersed submanifold of an almost paracontact metric manifold M. Then $\bar{M}$ is a semi-slant submanifold if and only if there exists a constant $\lambda \in [0,1)$ such that*

(i) *$D^{'} = \{X \mid f^2X = \lambda X\}$ is a distribution on $\bar{M}$.*
(ii) *For any $X \in \Gamma(T\bar{M})$ orthogonal to D', $\omega X = 0$. Furthermore, if θ is the slant angle of $\bar{M}$, in this case it satisfies $\lambda = \cos^2\theta$.*

Proof Let $\bar{M}$ be a semi-slant submanifold and $T\bar{M} = D_1 \oplus D_2 \oplus \mathrm{sp}\{\xi\}$, where D_1 is invariant and D_2 is slant. We put $\lambda = \cos^2\theta$. For any $X \in D^{'}$, if $X \in D_1$, then

$$X = F^2X - \eta(X)\xi = F^2X = (fP_1)^2X = \lambda X.$$

It follows that $\lambda = 1$, but this is a contradicton to $\lambda \in [0,1)$, that is, $D' \subseteq D_2$. On the other hand, since D_2 is a slant distribution, we have $f^2X = (fP_2)^2X = \lambda X$. It follows that $D_2 \subseteq D'$. Thus we conclude that $D_2 = D'$.

Conversely, we consider the orthogonal direct sum $T\bar{M} = D \oplus D^{\perp} \oplus \mathrm{sp}\{\xi\}$. It is obvious that $fD \subseteq D$. For any $X \in D^{\perp}$ and $Y \in D$, from (2.4) we have $g(FX,Y) = g(X,FY) = g(X,fY) = 0$, that is, $D^{\perp}$ is an invariant submanifold. The last statement of Theorem 4.2 implies that D is a slant distribution with slant angle θ satisfying $\lambda = \cos^2\theta$. ■

Theorem 4.6 *Let $\bar{M}$ be a semi-slant submanifold of almost paracontact metric manifold M. Then we have*

(i) *The distribution D_1 is integrable if and only if*

$$h(X, fY) = h(fX, Y) \tag{4.9}$$

for any $X, Y \in \Gamma(D_1)$.

(ii) *The distribution D_2 is integrable if and only if*

$$P_1(\nabla_X fY - \nabla_Y fX) = P_1(A_{\omega P_2 Y}X - A_{\omega P_2 X}Y)$$

for any $X, Y \in \Gamma(D_2)$.

https://doi.org/10.4153/CMB-2010-003-0 Published online by Cambridge University Press

Proof (i) From the Gauss–Weingarten formulas and making use of (4.5), we have

$$\nabla_X FY = F\nabla_X Y,$$

$$\begin{aligned}\bar{\nabla}_X fY + h(X, fY) &= F(\bar{\nabla}_X Y) + Fh(X,Y)\\ &= fP_1(\bar{\nabla}_X Y) + fP_2(\bar{\nabla}_X Y) + \omega(\bar{\nabla}_X Y) + Bh(X,Y)\\ &\quad + Ch(X,Y),\end{aligned} \tag{4.10}$$

for any $X, Y \in \Gamma(D_1)$. From the normal components of (4.10) we have

$$h(X, fY) = \omega P_2(\nabla_X Y) + Ch(X,Y).$$

Taking account of h being symmetric, we arrive at

$$\omega P_2[X,Y] = h(X, fY) - h(fX, Y). \tag{4.11}$$

Hence if D_1 is integrable, then (4.11) holds directly form (4.9).

Conversely, making use of (4.9) and (4.11), it follow that $\omega P_2[X,Y] = 0$. So we can easily deduce that $P_2[X,Y]$ must vanish.

(ii) Since D_2 is a slant distribution, we have

$$\begin{aligned}&\bar{\nabla}_X fP_2Y + h(X, fP_2Y) - A_{\omega P_2 Y}X + \nabla^{\perp}_X \omega P_2 Y\\ &\qquad = f(\bar{\nabla}_X Y) + \omega\bar{\nabla}_X Y + Bh(X,Y) + Ch(X,Y).\end{aligned}$$

Since h is symmetric, it follows that

$$f[X,Y] = \bar{\nabla}_X fP_2Y - \bar{\nabla}_Y fP_2X + A_{\omega P_2 X}Y - A_{\omega P_2 Y}X \tag{4.12}$$

for any $X, Y \in \Gamma(D_2)$. Applying P_1 to (4.12), we conclude that

$$P_1 f[X,Y] = P_1\{\bar{\nabla}_X fP_2Y - \bar{\nabla}_Y fP_2X\} - P_1\{A_{\omega P_2 Y}X - A_{\omega P_2 X}Y\}.$$

Hence D_2 is integrable if and only if $P_1 f[X,Y] = 0$. ∎

Lemma 4.7 *Let $\bar{M}$ be a mixed-geodesic semi-slant submanifold of an almost paracontact metric manifold M. Then the distribution D_1 is integrable if and only if the shape operator of $\bar{M}$ satisfies $FA_N X = A_N FX$ for any $N \in \Gamma(T\bar{M}^{\perp}), X \in \Gamma(D_1)$.*

Proof Since $\bar{M}$ is mixed-geodesic, from (2.5) we find that $A_N X$ has no component on D_2. Thus we conclude $g(FA_N X - A_N FX, Y) = g(h(X, FY) - h(FX, Y), N)$ for any $X, Y \in \Gamma(D_1)$. Also considering Theorem 4.6(i), it is easy to verify that D_1 is integrable if and only if $FA_N X = A_N FX$. ∎

The condition $\nabla f = 0$ also plays an important role in almost paracontact manifolds as well as locally product manifolds. The following theorem characterizes it.

https://doi.org/10.4153/CMB-2010-003-0 Published online by Cambridge University Press

Theorem 4.8 *Let $\bar{M}$ be a semi-slant submanifold of an almost paracontact metric manifold M. If $\nabla f = 0$, then the distributions D_1 and D_2 are integrable and their leaves are totally geodesic in $\bar{M}$.*

Proof If $\nabla f = 0$, then from (4.7) we have $Bh(Y, X) = 0$, for any $Y \in \Gamma(D_1)$ and $X \in \Gamma(T\bar{M})$. Thus we get $g(h(X, Y), \omega P_2 Z) = 0$, and $g(Fh(X, Y), \omega P_2 Z) = 0$ for any $Y \in \Gamma(D_1)$ and $X, Z \in \Gamma(T\bar{M})$. Thus we arrive at

$$\begin{aligned} g(\omega P_2 \nabla_X Y, Fh(X, Y)) &= g(\omega P_2 \nabla_X Y, \bar{\nabla}_X FY) - g(\omega P_2 \nabla_X Y, F\nabla_X Y) \\ &= g(\omega P_2 \nabla_X Y, h(X, FY)) - g(\omega P_2 \nabla_X Y, \omega P_2 \nabla_X Y) \\ &= -sin^2\theta\{g(P_2 \nabla_X Y, P_2 \nabla_X Y) - \eta^2(P_2 \nabla_X Y)\} = 0, \end{aligned}$$

which is equivalent to $P_2 \nabla_X Y = 0$, that is $\nabla_X Y \in \Gamma(D_1)$. Since $\bar{M}$ is a Riemannian manifold, its metric is a Riemannian metric, and D_2 is orthogonal D_1, we conclude that D_2 is also integrable. ∎

Theorem 4.9 *Let $\bar{M}$ be a semi-slant submanifold of an almost paracontact metric manifold M. If $\nabla\omega = 0$, then M is a mixed geodesic submanifold. Furthermore, if $X, Y \in \Gamma(D_2)$, then either $\bar{M}$ is D_2-geodesic, or $h(X, Y)$ is an eigenvector of C^2 with eigenvalue $\cos^2\theta$. If $X, Y \in \Gamma(D_1)$, then either $\bar{M}$ is a D-geodesic submanifold or $h(X, Y)$ is an eigenvector of C^2 with eigenvalue 1.*

Proof If $\nabla\omega = 0$ for any $X, Y \in \Gamma(T\bar{M})$, then from (4.8) we have $Ch(X, Y) = h(X, fY)$. Since D_2 is a slant distribution with a slant angle θ and D_1 is an invariant distribution, we have

(4.13) $$C^2 h(X, Y) = Ch(X, fY) = h(X, f^2 Y) = \cos^2\theta h(X, Y),$$

(4.14) $$C^2 h(X, Y) = Ch(Y, fX) = h(Y, f^2 X) = h(Y, F^2 X) = h(Y, X)$$

for any $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. By virtue of (4.13) and (4.14), we have $\sin^2\theta h(X, Y) = 0$, which implies $h(X, Y) = 0$ because $\theta \neq 0, \pi/2$. Thus $\bar{M}$ is a mixed-geodesic semi-slant submanifold.

Similarly, we have

(4.15) $$C^2 h(X, Y) = Ch(X, fY) = h(X, f^2 Y) = h(X, Y)$$

for any $X, Y \in \Gamma(D_1)$ and by using (4.13) we arrive at

(4.16) $$C^2 h(X, Y) = cos^2\theta Ch(X, Y)$$

for any $X, Y \in \Gamma(D_2)$. Thus (4.15) and (4.16) give our assertion. ∎

Theorem 4.10 *Let $\bar{M}$ be a semi-slant submanifold of an almost paracontact metric manifold M. Then M is a semi-slant product if and only if its second fundamental form satisfies*

(4.17) $$Bh(Z, X) = 0 \quad \textit{and} \quad h(Z, fX) = Ch(Z, X)$$

for any $Z \in \Gamma(T\bar{M})$ and $X \in \Gamma(D_1)$.

https://doi.org/10.4153/CMB-2010-003-0 Published online by Cambridge University Press

Proof If $\bar{M}$ is a semi-slant product, then D_1 and D_2 are totally geodesic distributions in $\bar{M}$. From Theorem 4.8, (4.7), and (4.8) we have

$$(\bar{\nabla}_Z f)X = \bar{\nabla}_Z fX - f(\bar{\nabla}_Z X) = Bh(X, Z) = 0$$

and

$$(\nabla_Z \omega)X = \nabla_Z^{\perp} \omega X - \omega(\bar{\nabla}_Z X) = 0.$$

It follows that $Ch(Z, X) = h(Z, fX)$ for any $Z \in \Gamma(T\bar{M})$ and $X \in \Gamma(D_1)$.

Conversely, let us assume that (4.17) is satisfied. Then (4.8) implies that

$$(\nabla_Z \omega)X = -\omega(\bar{\nabla}_Z X) = 0,$$

that is, $\bar{\nabla}_Z X \in \Gamma(D_1)$. Since D_2 is orthogonal to D_1, we get $\bar{\nabla}_Z Y \in \Gamma(D_2)$ for any $X \in \Gamma(D_1)$, $Y \in \Gamma(D_2)$, and $Z \in \Gamma(T\bar{M})$. Hence the proof is complete. ■

Corollary 4.11 *Let $\bar{M}$ be a semi-slant submanifold of an almost paracontact metric manifold M. Then $\nabla\omega = 0$ if and only if the shape operator of $\bar{M}$ satisfies $A_{CN}Z = A_N fZ$ for any $N \in \Gamma(T\bar{M}^{\perp})$, $Z \in \Gamma(T\bar{M})$.*

Proof From (2.5), (2.7), and (4.8) we have

$$\begin{aligned} g((\nabla_X \omega)Y, N) &= g(Ch(X, Y), N) - g(h(X, fY), N) \\ &= g(h(X, Y), FN) - g(h(X, fY), N) = g(A_{CN}Y - A_N fY, X) \end{aligned}$$

for any $X, Y \in \Gamma(T\bar{M})$ and $N \in \Gamma(T\bar{M}^{\perp})$. It follows that $\nabla\omega = 0$ if and only if $A_{CN}Z = A_N fZ$. ■

Corollary 4.12 *Let $\bar{M}$ be a semi-slant submanifold of an almost paracontact metric manifold M. Then $\bar{\nabla} f = 0$ if and only if the shape operator of $\bar{M}$ satisfies $A_{\omega P_2 X}Y = -A_{\omega P_2 Y}X$ for any $X, Y \in \Gamma(T\bar{M})$.*

Proof Taking into account (2.5) and (4.7), we have

$$\begin{aligned} g((\bar{\nabla}_X f)Y, Z) &= g(A_{\omega P_2 Y}X, Z) + g(Bh(X, Y), Z) \\ &= g(h(X, Z), \omega P_2 Y) + g(h(X, Z), \omega P_2 Z) \\ &= g(A_{\omega P_2 Y}X, Z) + g(A_{\omega P_2 Z}X, Z) \end{aligned}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$. It is equivalent to our assertion. ■

Acknowledgement The author would like to thank the referee(s) for several comments and suggestions.

References

[1] Bejancu, A. and Papaghiuc, N.: Semi-Invariant Submanifolds of a Sasakian Manifold. An. Stiint. Al.I. Cuza Univ. Iasi.27(1981),163-170.

[2] Cabrerizo, J.L., Carriazo, A., Fernandez, L.M. and Fernandez.M.: Slant Submanifolds in Sasakian Manifolds. Glasgow Math. J. 42(2000), 125-138. doi:10.1017/S0017089500010156

[3] Cabrerizo, J.L., Carriazo, A., Fernandez, L.M. and Fernandez, M.: Semi-Slant Submanifolds of A Sasakian Manifold. Geometriae Dedicata 78:183-199,1999. doi:10.1023/A:1005241320631

[4] B.-Y. Chen, *Geometry of Slant Submanifolds.* Katholieke Universiteit Leuven, Louvain, 1990.

[5] B.-Y. Chen and Y. Tazawa, *Slant Submanifolds of Complex Projective and Complex Hyperbolic Spaces.* Glasgow. Math.J. **42**(2000), no. 3, 439–454. doi:10.1017/S0017089500030111

[6] H. Li and X Liu, *Semi-slant submanifolds of a locally Riemannian product manifold.* Georgian Math. J. **12**(2005), 273–282.

[7] Lotto, A.: Slant Submanifolds in Contact Geometry. Bull. Math. Soc. Roumanie 39(1996), 183-198.

[8] Nikic, J.: Conditions For Invariant Submanifold of A Manifold with the (φ, ξ, η, G)-Structure. Kragujevac J. Math. 25(2003), 147-154.

[9] Papaghiuc, N.: Semi-Slant Submanifolds of A Kaehlerian Manifold. An. Stiint. Al.I. Cuza Univ. Iasi.40(1994), 55-61.

GOP University, Faculty of Arts and Sciences, Department of Mathematics, 60200 Tokat, Turkey
e-mail: matceken@gop.edu.tr