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TRAJECTORY FITTING ESTIMATION FOR REFLECTED STOCHASTIC LINEAR DIFFERENTIAL EQUATIONS OF A LARGE SIGNAL

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Abstract

In this paper we study the drift parameter estimation for reflected stochastic linear differential equations of a large signal. We discuss the consistency and asymptotic distributions of trajectory fitting estimator (TFE).

Keywords: trajectory fitting estimator; reflected stochastic linear differential equation; large signal; consistency; asymptotic distributions

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1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be equipped with a right continuous and increasing family of σ-algebras $\{\mathcal{F}_t, t \geq 0\}$, and let $\{W_t\}_{t\geq 0}$ be a given standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this paper we consider a reflected stochastic linear differential equation of a large signal,

$$\begin{cases} \mathrm{d}X_t = \dfrac{\theta}{\varepsilon} X_t \, \mathrm{d}t + \mathrm{d}W_t + \mathrm{d}L_t, \\ X_t \geq 0, \quad 0 \leq t \leq T, \end{cases} \tag{1}$$

where the initial value $X_0 = x_0 > 0$, $\varepsilon \in (0, 1]$, $\theta \in \mathbb{R}$ is unknown, and $L = \{L_t, \ t \geq 0\}$ is the minimal increasing non-negative process which makes the reflected stochastic process (1) satisfy $X_t \geq 0$ for all $t \geq 0$. The process L increases only when X hits the boundary zero, so that

$$\int_0^\infty I_{\{X_t > 0\}} \, \mathrm{d}L_t = 0.$$

It can be easily proved (see e.g. [9] and [22]) that the process L has the following explicit expression:

$$L_t = \max\left\{0, \sup_{u\in[0,t]} \left(-x_0 - \frac{\theta}{\varepsilon} \int_0^u X_v \, \mathrm{d}v - W_u \right) \right\}. \tag{2}$$

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Usually in applications to financial engineering, queueing systems, storage models, etc., the reflecting barrier is assumed to be zero. This is principally because of the physical restriction of the state processes. For example, inventory levels, stock prices, and interest rates should take non-negative values. We refer to [2], [3], [8], [9], [14], [17], [18], [19], [20], [21], and [22] for more details on reflected stochastic differential equations (RSDEs) and their wide applications.

But in reality, the drift parameter in RSDEs is seldom known. Parametric inference is one of the effective methods for solving this type of problem. In the case of statistical inference for RSDEs driven by Brownian motion, a popular approach is the maximum likelihood estimation method, based on the Girsanov density (see e.g. [23], [24], and [26]). For example, Bo *et al.* [4] established the maximum likelihood estimator (MLE) for the stationary reflected Ornstein–Uhlenbeck processes (OUPs) and studied the strong consistency and asymptotic normality of the MLE. Jiang and Yang [12] considered asymptotic properties of the MLE of the parameter occurring in ergodic reflected Ornstein–Uhlenbeck processes (ROUPs) with a one-sided barrier. Zang and Zhu [26] investigated the strong consistency and limiting distribution of the MLE in both the stationary and non-stationary cases for reflected OUPs. It is well known that the TFE was introduced by Kutoyants [15] as a numerically attractive alternative to the well-investigated MLE. Recently, Zang and Zhang [25] used the trajectory fitting estimation to investigate the asymptotic behaviour of the estimator for non-stationary reflected OUPs, including strong consistency and asymptotic distribution. Further, they have shown that the TFE for ergodic reflected OUPs is not strongly consistent.

On the other hand, trajectory fitting estimation for stochastic process without reflection have drawn increasing attention (see e.g. [5], [6], [7], [15], and [16]). For instance, Abi-ayad and Mourid [1] discussed the strong consistency and Gaussian limit distribution of the TFE for non-recurrent diffusion processes. Jiang and Xie [11] studied the asymptotic behaviours for the TFE in stationary OUPs with linear drift.

Motivated by the aforementioned works, in this paper we extend the work of Zang and Zhang [25] and study the consistency and asymptotic distributions of the TFE for RSDE (1) based on continuous observation of $X = \{X_t, 0 \le t \le T\}$. In order to obtain our estimators, we divide RSDE (1) by $\varepsilon^{1/2}$ and change the variable $t_\varepsilon = t\varepsilon^{-1}$. So $t_\varepsilon \in [0, T_\varepsilon]$ with $T_\varepsilon = T\varepsilon^{-1}$. From the scaling properties of Brownian motion, we find that there exists another standard Brownian motion $\{\widetilde{W}_t\}_{t\ge 0}$ on the enlarged probability space such that $\widetilde{W}_t \overset{d}{=} \varepsilon^{-1/2} W_{\varepsilon t}$. Denote $Y_{t_\varepsilon} = X_{t_\varepsilon \varepsilon}\varepsilon^{-1/2}$. Then, for reflected stochastic process (1), we have

$$\begin{cases} \mathrm{d}Y_{t_\varepsilon} = \theta Y_{t_\varepsilon}\,\mathrm{d}t_\varepsilon + \mathrm{d}\widetilde{W}_{t_\varepsilon} + \mathrm{d}\widetilde{L}_{t_\varepsilon}, \\ Y_{t_\varepsilon} \ge 0, \quad 0 \le t_\varepsilon \le T_\varepsilon, \\ Y_0 = x_0\varepsilon^{-1/2}, \end{cases} \tag{3}$$

where the realizations of $\widetilde{L}_{t_\varepsilon} = \varepsilon^{-1/2} L_{\varepsilon t_\varepsilon}$. It follows from (2) that

$$\widetilde{L}_{t_\varepsilon} = \max\left\{0, \sup_{s\in[0,t_\varepsilon]}\left(-x_0\varepsilon^{-1/2} - \theta\int_0^s Y_u\,\mathrm{d}u - \widetilde{W}_s\right)\right\}. \tag{4}$$

Let

$$A_{t_\varepsilon} = \int_0^{t_\varepsilon} Y_s\,\mathrm{d}s.$$

RSDE (3) can be written as

$$Y_{t_\varepsilon} = Y_0 + \theta A_{t_\varepsilon} + \widetilde{W}_{t_\varepsilon} + \widetilde{L}_{t_\varepsilon}, \quad 0 \le t_\varepsilon \le T_\varepsilon.$$

The TFE of θ should minimize

$$\int_0^{T_\varepsilon} |Y_{t_\varepsilon} - (Y_0 + \theta A_{t_\varepsilon})|^2 \, \mathrm{d}t_\varepsilon.$$

It can easily be seen that the minimum is attained when θ is given by

$$\widehat{\theta}_\varepsilon = \frac{\int_0^{T_\varepsilon} A_{t_\varepsilon}(Y_{t_\varepsilon} - Y_0)\, \mathrm{d}t_\varepsilon}{\int_0^{T_\varepsilon} A_{t_\varepsilon}^2 \, \mathrm{d}t_\varepsilon}.$$

By simple calculations, we have

$$\widehat{\theta}_\varepsilon - \theta = \frac{\int_0^{T_\varepsilon} A_{t_\varepsilon} \widetilde{W}_{t_\varepsilon} \, \mathrm{d}t_\varepsilon}{\int_0^{T_\varepsilon} A_{t_\varepsilon}^2 \, \mathrm{d}t_\varepsilon} + \frac{\int_0^{T_\varepsilon} A_{t_\varepsilon} \widetilde{L}_{t_\varepsilon} \, \mathrm{d}t_\varepsilon}{\int_0^{T_\varepsilon} A_{t_\varepsilon}^2 \, \mathrm{d}t_\varepsilon}. \tag{5}$$

2. Consistency of the TFE $\widehat{\theta}_\varepsilon$

In this section we discuss the consistency of the TFE $\widehat{\theta}_\varepsilon$ in both the non-ergodic and ergodic cases, respectively. We shall use the notation '$\rightarrow_p$' to denote 'convergence in probability' and the notation '$\Rightarrow$' to denote 'convergence in distribution'. We write '$\overset{d}{=}$' for equality in distribution.

We introduce two important lemmas as follows.

Lemma 2.1. (Dietz and Kutoyants [6].) *If φ_T is a probability measure defined on $[0, \infty)$ such that $\varphi_T([0, T]) = 1$ and $\varphi_T([0, K]) \rightarrow 0$ as $T \rightarrow \infty$ for each $K > 0$, then*

$$\lim_{T\rightarrow\infty} \int_0^T f_t \varphi_T(\mathrm{d}t) = f_\infty$$

for every bounded and measure function $f\colon [0, \infty) \rightarrow \mathbb{R}$ for which the limit $f_\infty := \lim_{t\rightarrow\infty} f_t$ exists.

Lemma 2.2. (Karatzas and Shreve [13].) *Let $z \geq 0$ be a given number and let $y(\cdot) = \{y(t); 0 \leq t < \infty\}$ be a continuous function with $y(0) = 0$. There exists a unique continuous function $k(\cdot) = \{k(t); 0 \leq t < \infty\}$ such that*

(i) $x(t) := z + y(t) + k(t) \geq 0,\ 0 \leq t < \infty$,

(ii) $k(0) = 0$, *$k(\cdot)$ is non-decreasing,*

(iii) *$k(\cdot)$ is flat off $\{t \geq 0; x(t) = 0\}$, that is,*

$$\int_0^\infty I_{\{x(s)>0\}} \, \mathrm{d}k(s) = 0.$$

Then the function $k(\cdot)$ is given by

$$k(t) = \max\Big[0, \max_{0\leq s\leq t} \{-(z + y(s))\}\Big], \quad 0 \leq t < \infty.$$

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Theorem 2.1.

(a) *Under $\theta > 0$, we have*

$$\lim_{\varepsilon\to 0} (\widehat{\theta}_\varepsilon - \theta) = 0 \quad a.s. \tag{6}$$

(b) *Under $\theta = 0$, we have*

$$\widehat{\theta}_\varepsilon - \theta \to_p 0 \quad as\ \varepsilon \to 0. \tag{7}$$

(c) *Under $\theta < 0$, we have*

$$\lim_{\varepsilon\to 0} \widehat{\theta}_\varepsilon = 0 \quad a.s., \tag{8}$$

that is, the TFE $\widehat{\theta}_\varepsilon$ is not strongly consistent.

Proof. (a) (i) If $x_0 > 0$, it is easy to see that

$$\mathrm{e}^{-\theta t_\varepsilon} Y_{t_\varepsilon} = x_0 \varepsilon^{-1/2} + \int_0^{t_\varepsilon} \mathrm{e}^{-\theta s}\, \mathrm{d}\widetilde{W}_s + \int_0^{t_\varepsilon} \mathrm{e}^{-\theta s}\, \mathrm{d}\widetilde{L}_s. \tag{9}$$

Because the process $\widetilde{L} = \{\widetilde{L}_{t_\varepsilon}\}_{t_\varepsilon \ge 0}$ increases only when $Y = \{Y_{t_\varepsilon}\}_{t_\varepsilon \ge 0}$ hits the boundary zero, $\int_0^{t_\varepsilon} \mathrm{e}^{-\theta s}\, \mathrm{d}\widetilde{L}_s$ is a continuous non-decreasing process for which

$$x_0 \varepsilon^{-1/2} + \int_0^{t_\varepsilon} \mathrm{e}^{-\theta s}\, \mathrm{d}\widetilde{W}_s + \int_0^{t_\varepsilon} \mathrm{e}^{-\theta s}\, \mathrm{d}\widetilde{L}_s \ge 0,$$

and increases only when

$$x_0 \varepsilon^{-1/2} + \int_0^{t_\varepsilon} \mathrm{e}^{-\theta s}\, \mathrm{d}\widetilde{W}_s + \int_0^{t_\varepsilon} \mathrm{e}^{-\theta s}\, \mathrm{d}\widetilde{L}_s = 0.$$

It follows from Lemma 2.2 that

$$\int_0^{t_\varepsilon} \mathrm{e}^{-\theta s}\, \mathrm{d}\widetilde{L}_s = \max\left[0, \max_{0\le s\le t_\varepsilon} \left\{-x_0 \varepsilon^{-1/2} - \int_0^s \mathrm{e}^{-\theta u}\, \mathrm{d}\widetilde{W}_u\right\}\right]. \tag{10}$$

For

$$M_t := -\int_0^t \mathrm{e}^{-\theta s}\, \mathrm{d}\widetilde{W}_s,$$

by time change for a continuous martingale, there exists another standard Brownian motion $\{\widehat{W}_t\}_{t\ge 0}$ on the enlarged probability space such that $M_t \overset{d}{=} \widehat{W}_{\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}}$. It follows from (10) that

$$\int_0^{t_\varepsilon} \mathrm{e}^{-\theta s}\, \mathrm{d}\widetilde{L}_s = \max\left[0, -x_0 \varepsilon^{-1/2} + \max_{0\le s\le \frac{1-\mathrm{e}^{-2\theta t_\varepsilon}}{2\theta}} \widehat{W}_s\right]. \tag{11}$$

Then, under $\theta > 0$ and $x_0 > 0$, by the fact that $\widehat{W}_s$, $s \in [0, 1/(2\theta)]$ is almost surely finite, we have

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2} \int_0^{t_\varepsilon} \mathrm{e}^{-\theta s}\, \mathrm{d}\widetilde{L}_s = 0 \quad \text{a.s.} \tag{12}$$

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In view of the definition of the quadratic variation of a continuous local martingale, we find that

$$\lim_{\varepsilon\to 0}\langle M, M\rangle_{t_\varepsilon}=\frac{1}{2\theta}\quad \text{a.s.}$$

and

$$\breve{M}_{t_\varepsilon} := M_{t_\varepsilon}^2-\langle M, M\rangle_{t_\varepsilon}$$

is a continuous local martingale. Writing this as

$$M_{t_\varepsilon}^2=\breve{M}_{t_\varepsilon}+\langle M, M\rangle_{t_\varepsilon},$$

it follows from the convergence theorem of non-negative semi-martingales that

$$\lim_{\varepsilon\to 0} M_{t_\varepsilon}^2<\infty\quad \text{a.s.},$$

and hence

$$\lim_{\varepsilon\to 0} M_{t_\varepsilon}=-\int_0^\infty \mathrm{e}^{-\theta s}\,\mathrm{d}\widetilde{W}_s\quad \text{a.s.} \tag{13}$$

Applying (9), (12), and (13), we obtain

$$\lim_{\varepsilon\to 0}\varepsilon^{1/2}\,\mathrm{e}^{-\theta t_\varepsilon}Y_{t_\varepsilon}=x_0\quad \text{a.s.} \tag{14}$$

Combining Lemma 2.1 with (14) yields

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon^{1/2}\,\mathrm{e}^{-\theta T_\varepsilon}A_{T_\varepsilon}&=\lim_{\varepsilon\to 0}\int_0^{T_\varepsilon}\big(\varepsilon^{1/2}\,\mathrm{e}^{-\theta t_\varepsilon}Y_{t_\varepsilon}\big)\frac{\mathrm{e}^{\theta t_\varepsilon}}{\int_0^{T_\varepsilon}\mathrm{e}^{\theta t_\varepsilon}\,\mathrm{d}t_\varepsilon}\,\mathrm{d}t_\varepsilon\times\mathrm{e}^{-\theta T_\varepsilon}\int_0^{T_\varepsilon}\mathrm{e}^{\theta t_\varepsilon}\,\mathrm{d}t_\varepsilon\\&=\frac{x_0}{\theta}\quad \text{a.s.}\end{aligned} \tag{15}$$

By (4) and the self-similarity property for the Brownian motion, we get

$$\widetilde{L}_{t_\varepsilon}\le\max\left\{0,\sup_{0\le s\le t_\varepsilon}\big(-x_0\varepsilon^{-1/2}-\widetilde{W}_s\big)\right\}\overset{d}{=}t_\varepsilon^{1/2}\max\left\{0,\max_{0\le u\le 1}\big(-x_0t^{-1/2}-\breve{W}_u\big)\right\}, \tag{16}$$

where $\{\breve{W}_u,\ u\ge 0\}$ is a fixed Wiener process on the enlarged probability space. Using (16) and the fact that $\breve{W}_u$, $u\in[0, 1]$ is almost surely finite, we see that

$$\lim_{\varepsilon\to 0}\frac{\widetilde{L}_{t_\varepsilon}}{t_\varepsilon}=0\quad \text{a.s.}$$

Obviously,

$$\lim_{\varepsilon\to 0}\frac{\widetilde{L}_{t_\varepsilon}}{\varepsilon^{-1/2}\,\mathrm{e}^{\theta t_\varepsilon}}=0\quad \text{a.s.} \tag{17}$$

By the strong law of large numbers, we obtain

$$\lim_{\varepsilon\to 0}\frac{\widetilde{W}_{t_\varepsilon}}{t_\varepsilon}=0 \quad \text{a.s.}$$

This implies that

$$\lim_{\varepsilon\to 0}\frac{\widetilde{W}_{t_\varepsilon}}{\varepsilon^{-1/2}\,\mathrm{e}^{\theta t_\varepsilon}}=0 \quad \text{a.s.} \tag{18}$$

By (15), (17), (18), and Lemma 2.1, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}(\widehat{\theta}_\varepsilon-\theta) &= \lim_{\varepsilon\to 0}\frac{\int_0^{T_\varepsilon}\frac{A_{t_\varepsilon}}{\varepsilon^{-1/2}\,\mathrm{e}^{\theta t_\varepsilon}}\frac{\widetilde{W}_{t_\varepsilon}}{\varepsilon^{-1/2}\,\mathrm{e}^{\theta t_\varepsilon}}\frac{\varepsilon^{-1}\,\mathrm{e}^{2\theta t_\varepsilon}}{\int_0^{T_\varepsilon}\varepsilon^{-1}\,\mathrm{e}^{2\theta t_\varepsilon}\,\mathrm{d}t_\varepsilon}\,\mathrm{d}t_\varepsilon}{\int_0^{T_\varepsilon}\big(\frac{A_{t_\varepsilon}}{\varepsilon^{-1/2}\,\mathrm{e}^{\theta t_\varepsilon}}\big)^2\frac{\varepsilon^{-1}\,\mathrm{e}^{2\theta t_\varepsilon}}{\int_0^{T_\varepsilon}\varepsilon^{-1}\,\mathrm{e}^{2\theta t_\varepsilon}\,\mathrm{d}t_\varepsilon}\,\mathrm{d}t_\varepsilon} \\
&\quad + \lim_{\varepsilon\to 0}\frac{\int_0^{T_\varepsilon}\frac{A_{t_\varepsilon}}{\varepsilon^{-1/2}\,\mathrm{e}^{\theta t_\varepsilon}}\frac{\widetilde{L}_{t_\varepsilon}}{\varepsilon^{-1/2}\,\mathrm{e}^{\theta t_\varepsilon}}\frac{\varepsilon^{-1}\,\mathrm{e}^{2\theta t_\varepsilon}}{\int_0^{T_\varepsilon}\varepsilon^{-1}\,\mathrm{e}^{2\theta t_\varepsilon}\,\mathrm{d}t_\varepsilon}\,\mathrm{d}t_\varepsilon}{\int_0^{T_\varepsilon}\big(\frac{A_{t_\varepsilon}}{\varepsilon^{-1/2}\,\mathrm{e}^{\theta t_\varepsilon}}\big)^2\frac{\varepsilon^{-1}\,\mathrm{e}^{2\theta t_\varepsilon}}{\int_0^{T_\varepsilon}\varepsilon^{-1}\,\mathrm{e}^{2\theta t_\varepsilon}\,\mathrm{d}t_\varepsilon}\,\mathrm{d}t_\varepsilon} \\
&= 0 \quad \text{a.s.}
\end{aligned} \tag{19}$$

This completes the desired proof.

(ii) If $x_0=0$, by Theorem 2.1 of Zang and Zhang [23], it follows that (6) holds. This completes the proof of Theorem 2.1(a).

(b) Under $\theta=0$, we have $Y_{T_\varepsilon}=x_0\varepsilon^{-1/2}+\widetilde{W}_{T_\varepsilon}+\widetilde{L}_{T_\varepsilon}$. Then

$$\int_0^{T_\varepsilon}A_{t_\varepsilon}^2\,\mathrm{d}t_\varepsilon=T_\varepsilon^4\int_0^1\bigg(\int_0^t\bigg(\frac{x_0}{\sqrt{T}}+\frac{\widetilde{W}_{sT_\varepsilon}}{\sqrt{T_\varepsilon}}+\frac{\widetilde{L}_{sT_\varepsilon}}{\sqrt{T_\varepsilon}}\bigg)\,\mathrm{d}s\bigg)^2\,\mathrm{d}t$$

and

$$\int_0^{T_\varepsilon}(\widetilde{W}_{t_\varepsilon}+\widetilde{L}_{t_\varepsilon})A_{t_\varepsilon}\,\mathrm{d}t_\varepsilon=T_\varepsilon^3\int_0^1\bigg(\frac{\widetilde{W}_{sT_\varepsilon}}{\sqrt{T_\varepsilon}}+\frac{\widetilde{L}_{sT_\varepsilon}}{\sqrt{T_\varepsilon}}\bigg)\int_0^s\bigg(\frac{x_0}{\sqrt{T}}+\frac{\widetilde{W}_{uT_\varepsilon}}{\sqrt{T_\varepsilon}}+\frac{\widetilde{L}_{uT_\varepsilon}}{\sqrt{T_\varepsilon}}\bigg)\,\mathrm{d}u\,\mathrm{d}s.$$

By the scaling properties of Brownian motion and Lemma 2.2, it follows that

$$\begin{aligned}
\big\{(\widetilde{W}_\nu,\widetilde{L}_\nu);\nu\ge 0\big\} &\overset{d}{=} \Big\{T_\varepsilon^{1/2}\Big(\widehat{W}_{\nu/T_\varepsilon},\max\Big\{0,\max_{0\le s\le\nu}\big(-x_0T^{-1/2}-\widehat{W}_{s/T_\varepsilon}\big)\Big\}\Big);\nu\ge 0\Big\} \\
&=: \big\{T_\varepsilon^{1/2}\big(\widehat{W}_{\nu/T_\varepsilon},\widehat{L}_{\nu/T_\varepsilon}\big);\nu\ge 0\big\},
\end{aligned}$$

where $\{\widehat{W}_\nu,\ \nu\ge 0\}$ is another standard Wiener process on the enlarged probability space. By the continuous mapping theorem, we have

$$\begin{aligned}
\frac{\int_0^{T_\varepsilon}(\widetilde{W}_{t_\varepsilon}+\widetilde{L}_{t_\varepsilon})A_{t_\varepsilon}\,\mathrm{d}t_\varepsilon}{\int_0^{T_\varepsilon}A_{t_\varepsilon}^2\,\mathrm{d}t_\varepsilon} &\overset{d}{=} \frac{1}{T_\varepsilon}\frac{\int_0^1(\widehat{W}_s+\widehat{L}_s)\int_0^s(x_0T^{-1/2}+\widehat{W}_u+\widehat{L}_u)\,\mathrm{d}u\,\mathrm{d}s}{\int_0^1\big(\int_0^t(x_0T^{-1/2}+\widehat{W}_s+\widehat{L}_s)\,\mathrm{d}s\big)^2\,\mathrm{d}t} \\
&\to_p 0 \quad \text{as } \varepsilon\to 0,
\end{aligned} \tag{20}$$

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where

$$\widehat{L}_s = \max\Big\{0, \max_{0\le u\le s}\big(-x_0 T^{-1/2} - \widehat{W}_u\big)\Big\}.$$

Combining (5) with (20) implies that (7) holds. This completes the desired proof.

(c) We consider the case $\theta < 0$. We first introduce the reflected OUP

$$d\bar{Y}_{t_\varepsilon} = \theta \bar{Y}_{t_\varepsilon}\, dt_\varepsilon + d\widetilde{W}_{t_\varepsilon} + d\bar{L}_{t_\varepsilon}, \quad \bar{Y}_0 = 0, \quad 0 \le t_\varepsilon \le T_\varepsilon. \tag{21}$$

It is easy to see that

$$e^{-\theta t_\varepsilon}\bar{Y}_{t_\varepsilon} = \int_0^{t_\varepsilon} e^{-\theta s}\, d\widetilde{W}_s + \int_0^{t_\varepsilon} e^{-\theta s}\, d\bar{L}_s. \tag{22}$$

Similarly to the discussion of (10), we see that

$$\int_0^{t_\varepsilon} e^{-\theta s}\, d\bar{L}_s = \max\bigg[0, \max_{0\le s\le t_\varepsilon}\bigg\{-\int_0^s e^{-\theta u}\, d\widetilde{W}_u\bigg\}\bigg]. \tag{23}$$

By (9), (10), (22), and (23), we have

$$\begin{aligned} |Y_{T_\varepsilon} - \bar{Y}_{T_\varepsilon}| &= e^{\theta T_\varepsilon}\bigg|\max\bigg[0, \max_{0\le s\le T_\varepsilon}\bigg\{-x_0\varepsilon^{-1/2} - \int_0^s e^{-\theta u}\, d\widetilde{W}_u\bigg\}\bigg] \\ &\quad - \max\bigg[0, \max_{0\le s\le T_\varepsilon}\bigg\{-\int_0^s e^{-\theta u}\, d\widetilde{W}_u\bigg\}\bigg]\bigg| + x_0\varepsilon^{-1/2} e^{\theta T_\varepsilon} \\ &\le 2x_0\varepsilon^{-1/2} e^{\theta T_\varepsilon}. \end{aligned} \tag{24}$$

For the linear system (21), according to the mean ergodic theorem (see Hu *et al.* [10]), we have

$$\lim_{\varepsilon\to 0}\frac{1}{T_\varepsilon}\int_0^{T_\varepsilon} \bar{Y}_s\, ds = \frac{1}{\sqrt{-\pi\theta}} \quad \text{a.s.} \tag{25}$$

Combining (24) with (25) yields

$$\lim_{\varepsilon\to 0}\frac{A_{T_\varepsilon}}{T_\varepsilon} = \lim_{\varepsilon\to 0}\frac{1}{T_\varepsilon}\int_0^{T_\varepsilon}(Y_s - \bar{Y}_s)\, ds + \lim_{\varepsilon\to 0}\frac{1}{T_\varepsilon}\int_0^{T_\varepsilon}\bar{Y}_s\, ds = \frac{1}{\sqrt{-\pi\theta}} \quad \text{a.s.} \tag{26}$$

It follows from Zang and Zhang [25] that

$$\lim_{\varepsilon\to 0}\frac{\bar{L}_{T_\varepsilon}}{T_\varepsilon} = \sqrt{\frac{-\theta}{\pi}} \quad \text{a.s.}$$

Note that

$$\begin{aligned} \big|\widetilde{L}_{T_\varepsilon} - \bar{L}_{T_\varepsilon}\big| &= \bigg|\max\bigg\{0, \sup_{s\in[0,T_\varepsilon]}\bigg(-x_0\varepsilon^{-1/2} - \theta\int_0^s Y_u\, du - \widetilde{W}_s\bigg)\bigg\} \\ &\quad - \max\bigg\{0, \sup_{s\in[0,T_\varepsilon]}\bigg(-\theta\int_0^s \bar{Y}_u\, du - \widetilde{W}_s\bigg)\bigg\}\bigg| \\ &\le x_0\varepsilon^{-1/2} + |\theta|\int_0^{T_\varepsilon} 2x_0\varepsilon^{-1/2} e^{\theta u}\, du \\ &\le 3x_0\varepsilon^{-1/2}. \end{aligned}$$

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Then we have

$$\lim_{\varepsilon\to 0}\frac{\widetilde{L}_{T_\varepsilon}}{T_\varepsilon}=\lim_{\varepsilon\to 0}\frac{\widetilde{L}_{T_\varepsilon}-\bar{L}_{T_\varepsilon}}{T_\varepsilon}+\lim_{\varepsilon\to 0}\frac{\bar{L}_{T_\varepsilon}}{T_\varepsilon}=\sqrt{\frac{-\theta}{\pi}}\quad \text{a.s.} \tag{27}$$

By (26), (27), Lemma 2.1, and the strong law of large numbers, we have

$$\lim_{\varepsilon\to 0}\frac{\int_0^{T_\varepsilon}\widetilde{W}_{t_\varepsilon}A_{t_\varepsilon}\,\mathrm{d}t_\varepsilon}{\int_0^{T_\varepsilon}A_{t_\varepsilon}^2\,\mathrm{d}t_\varepsilon}=\lim_{\varepsilon\to 0}\frac{\int_0^{T_\varepsilon}\frac{\widetilde{W}_{t_\varepsilon}}{t_\varepsilon}\frac{A_{t_\varepsilon}}{t_\varepsilon}\frac{t_\varepsilon^2}{\int_0^{T_\varepsilon}t_\varepsilon^2\,\mathrm{d}t_\varepsilon}\,\mathrm{d}t_\varepsilon}{\int_0^{T_\varepsilon}\big(\frac{A_{t_\varepsilon}}{t_\varepsilon}\big)^2\frac{t_\varepsilon^2}{\int_0^{T_\varepsilon}t_\varepsilon^2\,\mathrm{d}t_\varepsilon}\,\mathrm{d}t_\varepsilon}=0\quad \text{a.s.} \tag{28}$$

and

$$\lim_{\varepsilon\to 0}\frac{\int_0^{T_\varepsilon}\widetilde{L}_{t_\varepsilon}A_{t_\varepsilon}\,\mathrm{d}t_\varepsilon}{\int_0^{T_\varepsilon}A_{t_\varepsilon}^2\,\mathrm{d}t_\varepsilon}=\lim_{\varepsilon\to 0}\frac{\int_0^{T_\varepsilon}\frac{\widetilde{L}_{t_\varepsilon}}{t_\varepsilon}\frac{A_{t_\varepsilon}}{t_\varepsilon}\frac{t_\varepsilon^2}{\int_0^{T_\varepsilon}t_\varepsilon^2\,\mathrm{d}t_\varepsilon}\,\mathrm{d}t_\varepsilon}{\int_0^{T_\varepsilon}\big(\frac{A_{t_\varepsilon}}{t_\varepsilon}\big)^2\frac{t_\varepsilon^2}{\int_0^{T_\varepsilon}t_\varepsilon^2\,\mathrm{d}t_\varepsilon}\,\mathrm{d}t_\varepsilon}=-\theta\quad \text{a.s.} \tag{29}$$

By (5), (28), and (29), we can conclude that (8) holds. This completes the proof. □

3. Asymptotic distribution of the TFE $\widehat{\theta}_\varepsilon$

In this section we investigate the asymptotic distribution of the TFE $\widehat{\theta}_\varepsilon$.

Theorem 3.1. *Let $\{\widehat{W}_u,\ u\geq 0\}$ be another standard Wiener process on the enlarged probability space.*

(a) *Assume $\theta>0$.*

(i) *If $x_0>0$, then*

$$(\widehat{\theta}_\varepsilon-\theta)\,\mathrm{e}^{\theta T_\varepsilon}\Rightarrow\frac{2\theta}{x_0}T^{1/2}N\quad \textit{as } \varepsilon\to 0, \tag{30}$$

where N is a random variable with the standard normal distribution.

(ii) *If $x_0=0$, then*

$$\frac{\mathrm{e}^{\theta T_\varepsilon}}{\sqrt{T_\varepsilon}}(\widehat{\theta}_\varepsilon-\theta)\Rightarrow\frac{2\theta N}{\big|\widehat{W}_{1/(2\theta)}+\widehat{L}_{1/(2\theta)}\big|}\quad \textit{as } \varepsilon\to 0, \tag{31}$$

where

$$\widehat{L}_{1/(2\theta)}=\max\Big[0,\ \max_{0\leq u\leq 1/(2\theta)}(-\widehat{W}_u)\Big],$$

and N is a standard normal random variable which is independent of $\widehat{W}_{1/(2\theta)}$ and $\widehat{L}_{1/(2\theta)}$.

(b) *If $\theta=0$, then*

$$\varepsilon^{-1}\widehat{\theta}_\varepsilon\Rightarrow\frac{1}{T}\frac{\int_0^1\big(\widehat{W}_s+\widehat{L}_s\big)\int_0^s\big(x_0T^{-1/2}+\widehat{W}_u+\widehat{L}_u\big)\,\mathrm{d}u\,\mathrm{d}s}{\int_0^1\Big(\int_0^t\big(x_0T^{-1/2}+\widehat{W}_s+\widehat{L}_s\big)\,\mathrm{d}s\Big)^2\,\mathrm{d}t}\quad \textit{as } \varepsilon\to 0, \tag{32}$$

where

$$\widehat{L}_s=\max\Big\{0,\ \max_{0\leq u\leq s}\big(-x_0T^{-1/2}-\widehat{W}_u\big)\Big\}$$

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Proof. (a) (i) If $x_0 > 0$, we have

$$
\begin{aligned}
& \mathrm{e}^{\theta T_\varepsilon}(\widehat{\theta}_\varepsilon - \theta) \\
&= \frac{\varepsilon\, \mathrm{e}^{-\theta T_\varepsilon} \int_0^{T_\varepsilon} A_{t_\varepsilon} \widetilde{W}_{t_\varepsilon}\, \mathrm{d}t_\varepsilon + \varepsilon\, \mathrm{e}^{-\theta T_\varepsilon} \int_0^{T_\varepsilon} A_{t_\varepsilon} \widetilde{L}_{t_\varepsilon}\, \mathrm{d}t_\varepsilon}{\varepsilon\, \mathrm{e}^{-2\theta T_\varepsilon} \int_0^{T_\varepsilon} A_{t_\varepsilon}^2\, \mathrm{d}t_\varepsilon} \\
&= \frac{T^{1/2}}{\varepsilon\, \mathrm{e}^{-2\theta T_\varepsilon} \int_0^{T_\varepsilon} A_{t_\varepsilon}^2\, \mathrm{d}t_\varepsilon} \bigg(T_\varepsilon^{-1/2} \widetilde{W}_{T_\varepsilon} \varepsilon^{1/2}\, \mathrm{e}^{-\theta T_\varepsilon} \int_0^{T_\varepsilon} A_{t_\varepsilon}\, \mathrm{d}t_\varepsilon \\
&\quad + \varepsilon^{1/2}\, \mathrm{e}^{-\theta T_\varepsilon} T_\varepsilon^{-1/2} \int_0^{T_\varepsilon} A_{t_\varepsilon} \big(\widetilde{W}_{T_\varepsilon} - \widetilde{W}_{t_\varepsilon}\big)\, \mathrm{d}t_\varepsilon + \varepsilon^{1/2}\, \mathrm{e}^{-\theta T_\varepsilon} T_\varepsilon^{-1/2} \int_0^{T_\varepsilon} A_{t_\varepsilon} \widetilde{L}_{t_\varepsilon}\, \mathrm{d}t_\varepsilon \bigg) \\
&:= I_1(\varepsilon)(I_2(\varepsilon) + I_3(\varepsilon) + I_4(\varepsilon)).
\end{aligned} \tag{33}
$$

We shall study the asymptotic behaviour of $I_i(\varepsilon)$, $i = 1, \ldots, 4$. By (15) and Lemma 2.1, we can see that

$$
\begin{aligned}
& \lim_{\varepsilon \to 0} \varepsilon\, \mathrm{e}^{-2\theta T_\varepsilon} \int_0^{T_\varepsilon} A_{t_\varepsilon}^2\, \mathrm{d}t_\varepsilon \\
&\quad = \lim_{\varepsilon \to 0} \frac{\int_0^{T_\varepsilon} \varepsilon^{-1}\, \mathrm{e}^{2\theta t_\varepsilon}\, \mathrm{d}t_\varepsilon}{\varepsilon^{-1}\, \mathrm{e}^{2\theta T_\varepsilon}} \int_0^{T_\varepsilon} \bigg(\frac{A_{t_\varepsilon}}{\varepsilon^{-1/2}\, \mathrm{e}^{\theta t_\varepsilon}} \bigg)^2 \frac{\varepsilon^{-1}\, \mathrm{e}^{2\theta t_\varepsilon}}{\int_0^{T_\varepsilon} \varepsilon^{-1}\, \mathrm{e}^{2\theta t_\varepsilon}\, \mathrm{d}t_\varepsilon}\, \mathrm{d}t_\varepsilon \\
&\quad = \frac{x_0^2}{2\theta^3} \quad \text{a.s.}
\end{aligned}
$$

It follows that

$$
\lim_{\varepsilon \to 0} I_1(\varepsilon) = \frac{2\theta^3}{x_0^2} T^{1/2} \quad \text{a.s.} \tag{34}
$$

Now we consider $I_2(\varepsilon)$. Using (15) and Lemma 2.1 again, we have

$$
\begin{aligned}
& \lim_{\varepsilon \to 0} \varepsilon^{1/2}\, \mathrm{e}^{-\theta T_\varepsilon} \int_0^{T_\varepsilon} A_{t_\varepsilon}\, \mathrm{d}t_\varepsilon \\
&\quad = \lim_{\varepsilon \to 0} \frac{\int_0^{T_\varepsilon} \varepsilon^{-1/2}\, \mathrm{e}^{\theta t_\varepsilon}\, \mathrm{d}t_\varepsilon}{\varepsilon^{-1/2}\, \mathrm{e}^{\theta T_\varepsilon}} \int_0^{T_\varepsilon} \bigg(\frac{A_{t_\varepsilon}}{\varepsilon^{-1/2}\, \mathrm{e}^{\theta t_\varepsilon}} \bigg) \frac{\varepsilon^{-1/2}\, \mathrm{e}^{\theta t_\varepsilon}}{\int_0^{T_\varepsilon} \varepsilon^{-1/2}\, \mathrm{e}^{\theta t_\varepsilon}\, \mathrm{d}t_\varepsilon}\, \mathrm{d}t_\varepsilon \\
&\quad = \frac{x_0}{\theta^2} \quad \text{a.s.}
\end{aligned} \tag{35}
$$

For the second factor in $I_2(\varepsilon)$, we find that

$$
T_\varepsilon^{-1/2} W_{T_\varepsilon} = T_\varepsilon^{-1/2} \big(W_{T_\varepsilon} - W_{{T_\varepsilon}^{1/2}} \big) + T_\varepsilon^{-1/2} W_{T_\varepsilon^{1/2}}.
$$

It is easy to see that the random variable

$$
T_\varepsilon^{-1/2} \big(W_{T_\varepsilon} - W_{{T_\varepsilon}^{1/2}} \big)
$$

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has a normal distribution $N\big(0, 1-T_\varepsilon^{-1/2}\big)$, which converges weakly to a standard normal random variable N as $\varepsilon \to 0$. By the strong law of large numbers, we see that

$$\lim_{\varepsilon\to 0} \frac{W_{T_\varepsilon^{1/2}}}{T_\varepsilon^{1/2}} = 0 \quad \text{a.s.}$$

Hence we have

$$T_\varepsilon^{-1/2} W_{T_\varepsilon} \Rightarrow N \quad \text{as } \varepsilon \to 0. \tag{36}$$

Combining (35) with (36) gives

$$I_2(\varepsilon) \Rightarrow \frac{x_0}{\theta^2} N \quad \text{as } \varepsilon \to 0. \tag{37}$$

Next, we show that $I_3(\varepsilon) \to 0$ in probability as $\varepsilon \to 0$. Note that

$$\begin{aligned}
|I_3(\varepsilon)| &\le \varepsilon^{1/2}\,\mathrm{e}^{-\theta T_\varepsilon} T_\varepsilon^{-1/2} \int_0^{T_\varepsilon} \bigg|\int_0^{t_\varepsilon} Y_s\,\mathrm{d}s\bigg| \big|\widetilde{W}_{T_\varepsilon} - \widetilde{W}_{t_\varepsilon}\big|\,\mathrm{d}t_\varepsilon \\
&\le \varepsilon^{1/2}\,\mathrm{e}^{-\theta T_\varepsilon} T_\varepsilon^{-1/2} \int_0^{T_\varepsilon} \bigg(\int_0^{t_\varepsilon} \big|\varepsilon^{1/2}\,\mathrm{e}^{-\theta s} Y_s\big|\,\mathrm{e}^{\theta s}\varepsilon^{-1/2}\,\mathrm{d}s\bigg) \big|\widetilde{W}_{T_\varepsilon} - \widetilde{W}_{t_\varepsilon}\big|\,\mathrm{d}t_\varepsilon \\
&\le \frac{1}{\theta} \sup_{\varepsilon\ge 0} \big|\varepsilon^{1/2}\,\mathrm{e}^{-\theta T_\varepsilon} Y_{T_\varepsilon}\big| T_\varepsilon^{-1/2}\,\mathrm{e}^{-\theta T_\varepsilon} \int_0^{T_\varepsilon} |W_{T_\varepsilon} - W_{t_\varepsilon}|\,\mathrm{e}^{\theta t_\varepsilon}\,\mathrm{d}t_\varepsilon,
\end{aligned} \tag{38}$$

which converges to zero in probability as $\varepsilon \to 0$. In fact, using Markov's inequality and Fubini's theorem, we find that for given $\delta > 0$,

$$\begin{aligned}
&\mathbb{P}\bigg(T_\varepsilon^{-1/2}\,\mathrm{e}^{-\theta T_\varepsilon} \int_0^{T_\varepsilon} |W_{T_\varepsilon} - W_{t_\varepsilon}|\,\mathrm{e}^{\theta t_\varepsilon}\,\mathrm{d}t_\varepsilon > \delta\bigg) \\
&\quad \le \delta^{-1}\mathbb{E}\bigg[T_\varepsilon^{-1/2}\,\mathrm{e}^{-\theta T_\varepsilon} \int_0^{T_\varepsilon} |W_{T_\varepsilon} - W_{t_\varepsilon}|\,\mathrm{e}^{\theta t_\varepsilon}\,\mathrm{d}t_\varepsilon\bigg] \\
&\quad = \delta^{-1} T_\varepsilon^{-1/2} \int_0^{T_\varepsilon} \mathbb{E}|W_{T_\varepsilon} - W_{t_\varepsilon}|\,\mathrm{e}^{-\theta(T_\varepsilon - t_\varepsilon)}\,\mathrm{d}t_\varepsilon \\
&\quad = \delta^{-1} T_\varepsilon^{-1/2} \int_0^{T_\varepsilon} \nu^{1/2}\,\mathrm{e}^{-\theta\nu}\,\mathrm{d}\nu \\
&\quad \le \delta^{-1} T_\varepsilon^{-1/2} \theta^{-3/2} \frac{1}{2}\sqrt{\pi},
\end{aligned}$$

which tends to zero as $\varepsilon \to 0$. Finally, we consider $I_4(\varepsilon)$. Combining the self-similarity property for the Brownian motion with (4) yields

$$\begin{aligned}
\frac{\widetilde{L}_{t_\varepsilon}}{t_\varepsilon^{1/2}} &= t_\varepsilon^{-1/2} \max\bigg\{0, \sup_{0\le ut_\varepsilon\le t_\varepsilon} \bigg(-x_0\varepsilon^{-1/2} - \theta\int_0^{ut_\varepsilon} Y_r\,\mathrm{d}r - \widetilde{W}_{ut_\varepsilon}\bigg)\bigg\} \\
&\overset{d}{=} \max\bigg\{0, \max_{0\le u\le 1} \bigg(-x_0 t^{-1/2} - \theta t_\varepsilon^{-1/2} \int_0^{ut_\varepsilon} Y_r\,\mathrm{d}r - \widehat{W}_u\bigg)\bigg\} \quad \text{a.s.}
\end{aligned} \tag{39}$$

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Note that for any given $u \in (0, 1]$ we can choose a positive number $\epsilon > 0$ such that $\epsilon < u$. Then we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \max_{u\in(0,1]} \left(-x_0 t^{-1/2} - \theta t_\varepsilon^{-1/2} \int_0^{ut_\varepsilon} Y_r \,\mathrm{d}r - \widehat{W}_u\right) \\
&\quad \le \lim_{\varepsilon\to 0} \max_{u\in(0,1]} \left(-\theta t_\varepsilon^{-1/2} \int_0^{ut_\varepsilon} Y_r \,\mathrm{d}r\right) + \max_{u\in(0,1]} (-\widehat{W}_u) \\
&\quad \le -\lim_{\varepsilon\to 0} \theta t_\varepsilon^{-1/2} \int_0^{\epsilon t_\varepsilon} Y_r \,\mathrm{d}r + \max_{u\in(0,1]} (-\widehat{W}_u). \qquad (40)
\end{aligned}$$

By (15), we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} \theta t_\varepsilon^{-1/2} \int_0^{\epsilon t_\varepsilon} Y_s \,\mathrm{d}s &= \theta \lim_{\varepsilon\to 0} \frac{\int_0^{\epsilon t_\varepsilon} Y_s \,\mathrm{d}s}{\varepsilon^{-1/2}\,\mathrm{e}^{\theta\epsilon t_\varepsilon}} \frac{\varepsilon^{-1/2}\,\mathrm{e}^{\theta\epsilon t_\varepsilon}}{t_\varepsilon^{1/2}} \\
&= \theta \lim_{\varepsilon\to 0} \frac{\int_0^{\epsilon t_\varepsilon} Y_s \,\mathrm{d}s}{\varepsilon^{-1/2}\,\mathrm{e}^{\theta\delta t_\varepsilon}} \frac{\mathrm{e}^{\theta\epsilon t_\varepsilon}}{t^{1/2}} \\
&= \infty \quad \text{a.s.} \qquad (41)
\end{aligned}$$

By (40), (41), and the fact that $\widehat{W}_u$, $u \in (0, 1]$ is almost surely finite, we have

$$\lim_{\varepsilon\to 0} \max_{u\in(0,1]} \left(-x_0 t^{-1/2} - \theta t_\varepsilon^{-1/2} \int_0^{ut_\varepsilon} Y_r \,\mathrm{d}r - \widehat{W}_u\right) = -\infty \quad \text{a.s.} \qquad (42)$$

If $u = 0$, we find that

$$-x_0 t^{-1/2} - \theta t_\varepsilon^{-1/2} \int_0^{ut_\varepsilon} Y_r \,\mathrm{d}r - \widehat{W}_u = -x_0 t^{-1/2}.$$

Hence we see that

$$\lim_{\varepsilon\to 0} \max_{u\in[0,1]} \left(-x_0 t^{-1/2} - \theta t_\varepsilon^{-1/2} \int_0^{ut_\varepsilon} \widetilde{Y}_s \,\mathrm{d}s - \widehat{W}_u\right) = -x_0 t^{-1/2} \quad \text{a.s.} \qquad (43)$$

Applying (39) and (43), we obtain

$$\lim_{\varepsilon\to 0} \frac{\widetilde{L}_{t_\varepsilon}}{t_\varepsilon^{1/2}} = 0 \quad \text{a.s.} \qquad (44)$$

By (15) and (44) as well as Lemma 2.1, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} I_4(\varepsilon) &= \lim_{\varepsilon\to 0} \int_0^{T_\varepsilon} \frac{A_{t_\varepsilon}}{\varepsilon^{-1/2}\,\mathrm{e}^{\theta t_\varepsilon}} \frac{L_{t_\varepsilon}}{t_\varepsilon^{1/2}} \frac{\varepsilon^{-1/2} t_\varepsilon^{1/2}\,\mathrm{e}^{\theta t_\varepsilon}}{\int_0^{t_\varepsilon} \varepsilon^{-1/2} t_\varepsilon^{1/2}\,\mathrm{e}^{\theta t_\varepsilon}\,\mathrm{d}t_\varepsilon} \,\mathrm{d}t_\varepsilon \frac{\int_0^{T_\varepsilon} \varepsilon^{-1/2} t_\varepsilon^{1/2}\,\mathrm{e}^{\theta t_\varepsilon}\,\mathrm{d}t_\varepsilon}{\varepsilon^{-1/2} T_\varepsilon^{1/2}\,\mathrm{e}^{\theta T_\varepsilon}} \\
&\le \lim_{\varepsilon\to 0} \int_0^{T_\varepsilon} \frac{A_{t_\varepsilon}}{\varepsilon^{-1/2}\,\mathrm{e}^{\theta t_\varepsilon}} \frac{L_{t_\varepsilon}}{t_\varepsilon^{1/2}} \frac{\varepsilon^{-1/2} t_\varepsilon^{1/2}\,\mathrm{e}^{\theta t_\varepsilon}}{\int_0^{t_\varepsilon} \varepsilon^{-1/2} t_\varepsilon^{1/2}\,\mathrm{e}^{\theta t_\varepsilon}\,\mathrm{d}t_\varepsilon} \,\mathrm{d}t_\varepsilon \frac{\int_0^{T_\varepsilon} \mathrm{e}^{\theta t_\varepsilon}\,\mathrm{d}t_\varepsilon}{\mathrm{e}^{\theta T_\varepsilon}} \\
&= 0 \quad \text{a.s.} \qquad (45)
\end{aligned}$$

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Therefore, by (33), (34), (37), (38), and (45), we conclude that (30) holds. This completes the desired proof.

(ii) Under $x_0 = 0$, by Theorem 2.2 of Zang and Zhang [23], it follows that (31) holds. This completes the desired proof.

(b) Combining (5) with (20) implies that (32) holds. This completes the proof. □

4. Discussion

In this section we discuss the properties of the TFE $\widehat{\theta}_\varepsilon$ and MLE

$$\widehat{\theta}_\varepsilon^{\mathrm{MLE}} := \frac{\int_0^{T_\varepsilon} Y_s \, \mathrm{d}Y_s}{\int_0^{T_\varepsilon} Y_s^2 \, \mathrm{d}s}$$

separately in terms of the range of θ, i.e. $\theta > 0$, $\theta = 0$, and $\theta < 0$. By comparing the results of Zhang and Shu [27], we obtain the following claims.

(a) Under $\theta > 0$, both estimators are consistent. In addition, the following hold.

(i) If $x_0 > 0$, then

$$\varepsilon^{-1/2} \, \mathrm{e}^{\theta T_\varepsilon} (\widehat{\theta}_\varepsilon^{\mathrm{MLE}} - \theta) \Rightarrow N\left(0, \frac{2\theta}{x_0^2}\right) \quad \text{as } \varepsilon \to 0, \tag{46}$$

(ii) If $x_0 = 0$, then

$$\mathrm{e}^{\theta T_\varepsilon} (\widehat{\theta}_\varepsilon^{\mathrm{MLE}} - \theta) \Rightarrow \frac{\sqrt{2\theta} N}{\left|\widehat{W}_{1/(2\theta)} + \widehat{L}_{1/(2\theta)}\right|} \quad \text{as } \varepsilon \to 0, \tag{47}$$

where

$$\widehat{L}_{1/(2\theta)} = \max\left[0, \max_{0 \le u \le 1/(2\theta)} (-\widehat{W}_u)\right],$$

and N is a standard normal random variable which is independent of $\widehat{W}_{1/(2\theta)}$ and $\widehat{L}_{1/(2\theta)}$. For both estimators, the order of the convergence depends heavily on the true value of the parameter. It can also be seen that the MLE $\widehat{\theta}_\varepsilon^{\mathrm{MLE}}$ converges in distribution of higher order than the TFE $\widehat{\theta}_\varepsilon$.

(b) Under $\theta = 0$, it is easy to see that both estimators are consistent. The MLE $\widehat{\theta}_\varepsilon^{\mathrm{MLE}}$ has the limiting distribution

$$\varepsilon^{-1} \widehat{\theta}_\varepsilon^{\mathrm{MLE}} \Rightarrow \frac{1}{T} \frac{\int_0^1 \left(x_0 T^{-1/2} + \widehat{W}_u + \widehat{L}_u\right) \mathrm{d}\widehat{W}_u}{x_0 T^{-1} + \int_0^1 \left(\widehat{W}_u + \widehat{L}_u\right)^2 \mathrm{d}u + 2x_0 T^{-1/2} \int_0^1 \left(\widehat{W}_u + \widehat{L}_u\right) \mathrm{d}u} \quad \text{as } \varepsilon \to 0,$$

where

$$\widehat{L}_u = \max\left\{0, \max_{0 \le r \le u} \left(-x_0 T^{-1/2} - \widehat{W}_r\right)\right\}.$$

Both estimators are neither normal nor a mixture of normals. Further, both estimators have the same order of convergence in this case.

(c) Under $\theta < 0$, the MLE $\widehat{\theta}_\varepsilon^{\mathrm{MLE}}$ of θ is strongly consistent. But the TFE $\widehat{\theta}_\varepsilon$ is not strongly consistent. The MLE $\widehat{\theta}_\varepsilon^{\mathrm{MLE}}$ has the limiting distribution

$$\frac{\widehat{\theta}_\varepsilon^{\mathrm{MLE}} - \theta}{\varepsilon^{1/2}} \Rightarrow N\left(0, -\frac{2\theta}{x_0^2 + T}\right) \quad \text{as } \varepsilon \to 0.$$

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There were no competing interests to declare which arose during the preparation or publication process of this article.

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