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A NEW PROOF OF LAGUERRE'S THEOREM ABOUT THE ZEROS OF POLYNOMIALS

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An elementary new, simple and purely analytic proof of Laguerre's theorem about the zeros of the polar derivative of a polynomial $P(z)$ with respect to the point α, is given. The proof is based on a lemma which is also of independent interest.

Let $P(z)$ be a polynomial of degree n and α be a real or a complex number. The polar derivative $D_\alpha P(z)$ of $P(z)$ with respect to α is defined by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z) .$$

Clearly the polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$. According to the Gauss-Lucas theorem, any circle C which encloses all the zeros of $P(z)$ also encloses all the zeros of its derivative $P'(z)$. Now concerning the zeros of $D_\alpha P(z)$, we have the following famous result given by Laguerre in 1880 and which is a natural generalization of the Gauss-Lucas theorem.

THEOREM A. (Laguerr's theorem). *If all the zeros* $z_1, z_2, \ldots, z_n$ *of a polynomial* $P(z)$, *of degree* n *lie in a circular region* C *and if* w

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is any zero of $D_\alpha P(z)$, *the polar derivative of* $P(z)$, *then not both points* w *and* α *may lie outside of* C. *Furthermore, if* $P(w) \neq 0$, *any circle* K *through* w *and* α *either passes through all the zeros of* $P(z)$ *or separates these zeros.*

Here by a circular region we mean the closure of not merely the interior of a circle but also the exterior of a circle or a half-plane.

Several proofs of Laguerre's theorem can be found in [1], [2] and [3]. But these are based mainly on considerations from mechanics (spherical and plane fields of forces, points of equilibrium, centres of mass, etc.). In the present paper, we shall give a new, simple and purely analytic proof of Theorem A which in essence involves no considerations from mechanics. The proof is based on the following lemma which is also of independent interest.

LEMMA. *If* $z_1, z_2, \ldots, z_n$ *are the zeros of a polynomial* $P(z)$ *of degree* n *and if* w *is any zero of* $D_\alpha P(z)$ *such that* $P(w) \neq 0$, *then for every complex number* c

$$(1)\qquad (w-c)\left\{\left(\sum_{j=1}^{n} \frac{1}{|w-z_j|^2}\right) - \frac{n}{|w-\alpha|^2}\right\} = \left(\sum_{j=1}^{n} \frac{z_j - c}{|w-z_j|^2}\right) - \frac{n(\alpha-c)}{|w-\alpha|^2},$$

and

$$(2)\qquad |w-c|^2\left\{\left(\sum_{j=1}^{n} \frac{1}{|w-z_j|^2}\right) - \frac{n}{|w-\alpha|^2}\right\} = \left(\sum_{j=1}^{n} \frac{|z_j-c|^2}{|w-z_j|^2}\right) - \frac{n|\alpha-c|^2}{|w-\alpha|^2}.$$

Proof of the Lemma. Let w be a zero of $D_\alpha P(z)$, then we have

$$(3)\qquad \{D_\alpha P(z)\}_{z=w} = nP(w) + (\alpha-w)P'(w) = 0 .$$

Since $P(w) \neq 0$, therefore, $w \neq \alpha$ and from (3) we get

$$(4)\qquad \frac{P'(w)}{P(w)} = \frac{n}{w-\alpha} .$$

If $z_1, z_2, \ldots, z_n$ are the zeros of $P(z)$, then $w \neq z_j$, $j = 1, 2, \ldots, n$ and (4) implies

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(5) $$\sum_{j=1}^{n} \frac{1}{w - z_j} = \frac{n}{w - \alpha}$$

which can be written as

$$\sum_{j=1}^{n} \frac{1}{(w-c) - (z_j-c)} = \frac{n}{(w-c) - (\alpha-c)}$$

where c is any given real or complex number. That is,

(6) $$\sum_{j=1}^{n} \frac{1}{W - Z_j} = \frac{n}{W - \beta}$$

where $W = w-c$, $Z_j = z_j-c$, $j = 1,2,\ldots,n$, $\beta = \alpha-c$ and $W \neq \beta$. This gives

$$\sum_{j=1}^{n} \frac{W - Z_j}{|W - Z_j|^2} = \frac{n(W - \beta)}{|W - \beta|^2},$$

or equivalently

(7) $$W\left\{\left(\sum_{j=1}^{n} \frac{1}{|W-Z_j|^2}\right) - \frac{n}{|W-\beta|^2}\right\} = \left(\sum_{j=1}^{n} \frac{Z_j}{|W-Z_j|^2}\right) - \frac{n\beta}{|W-\beta|^2}.$$

Now replacing W by $w-c$, Z_j by z_j-c and β by $\alpha-c$ in (7), we obtain (1) and this proves the first part of the lemma.

To prove the second part of the lemma, we write (6) in the form

$$\sum_{j=1}^{n} \left(\frac{1}{W - Z_j} - \frac{1}{W - \beta}\right) = 0.$$

This gives

$$\sum_{j=1}^{n} \frac{Z_j - \beta}{W - Z_j} = 0 \,, \quad \text{since} \quad W \neq \beta \,,$$

which implies by (6)

(8) $$\sum_{j=1}^{n} \frac{Z_j(\overline{W} - \overline{Z}_j)}{|W - Z_j|^2} = \sum_{j=1}^{n} \frac{\beta}{W - Z_j} = \frac{n\beta}{W - \beta}.$$

Now (8) can be written as

$$\left(\sum_{j=1}^{n} \frac{\overline{W}Z_j}{|W-Z_j|^2}\right) - \frac{n\beta(\overline{W} - \overline{\beta})}{|W - \beta|^2} = \sum_{j=1}^{n} \frac{|Z_j|^2}{|W - Z_j|^2}.$$

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This gives

$$(9)\qquad \overline{W}\left\{\left(\sum_{j=1}^{n}\frac{Z_j}{|W-Z_j|^2}\right)-\frac{n\beta}{|W-\beta|^2}\right\}=\left(\sum_{j=1}^{n}\frac{|Z_j|^2}{|W-Z_j|^2}\right)-\frac{n|\beta|^2}{|W-\beta|^2}$$

Multiplying the two sides of (7) by $\overline{W}$ and using the result in (9), we obtain

$$(10)\qquad |W|^2\left\{\left(\sum_{j=1}^{n}\frac{1}{|W-Z_j|^2}\right)-\frac{n}{|W-\beta|^2}\right\}=\left(\sum_{j=1}^{n}\frac{|Z_j|^2}{|W-Z_j|^2}\right)-\frac{n|\beta|^2}{|W-\beta|^2}\,.$$

Replacing W by $w-c$, Z_j by z_j-c and β by $\alpha-c$ in (10, we obtain (2) and the lemma is completely proved.

Proof of Laguerre's Theorem. We suppose $P(z)\neq(z-\alpha)^n$, so that $D_\alpha P(z)\not\equiv 0$. Assume that all the zeros $z_1, z_2, \ldots, z_n$ of $P(z)$ lie in a circular region C and let w be a zero of $D_\alpha P(z)$. If w is also a zero of $P(z)$, then w lies in C and the result of the first part of the theorem follows. Henceforth we assume that $P(w)\neq 0$. Now from (5) we have

$$\left|\frac{n}{w-\alpha}\right| = \left|\sum_{j=1}^{n}\frac{1}{w-z_j}\right| \leq \sum_{j=1}^{n}\frac{1}{|w-z_j|}\,.$$

with equality sign holding only if

$$\frac{1}{w-z_j} = \left|\frac{1}{w-z_j}\right| e^{i\theta} = r_j e^{i\theta} \quad \text{(say)},\quad j=1,2,\ldots,n,\quad \theta \text{ real.}$$

Using the Cauchy-Schwarz inequality, it follows that

$$\frac{n^2}{|w-\alpha|^2} \leq \left\{\sum_{j=1}^{n}\frac{1}{|w-z_j|}\right\}^2 \leq n\sum_{j=1}^{n}\frac{1}{|w-z_j|^2}\,,$$

where now equality in the right hand inequality holds only if $r_1 = r_2 = \ldots = r_n = r$ (say). Thus equality holds in both inequalities only if

$$\frac{1}{w-z_1} = \frac{1}{w-z_2} = \ldots = \frac{1}{w-z_n} = re^{i\theta}\,.$$

This gives with the help of (5) that $z_1 = z_2 = \ldots z_n = \alpha$, so that

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$P(z) = (z-\alpha)^n$, which is not the case. Hence in fact, we have

$$B = \left\{\sum_{j=1}^{n} \frac{1}{|w-z_j|^2}\right\} - \frac{n}{|w-\alpha|^2} > 0. \tag{11}$$

To prove the result, we shall consider the three cases of C separately.

Case 1. Let $C : |z-c| \le R$, $R > 0$. By the second part of the above lemma

$$B|w-c|^2 = \left\{\sum_{j=1}^{n} \frac{|z_j - c|^2}{|w-z_j|^2}\right\} - \frac{n|\alpha-c|^2}{|w-\alpha|^2}. \tag{12}$$

Now let us assume that w lies exterior to C then $|w-c| > R$. Since $|z_j - c| \le R$, $j = 1,2, \ldots ,n$, it follows from (12) that

$$BR^2 < B|w - c|^2 \le \left\{\sum_{j=1}^{n} \frac{R^2}{|w - z_j|^2}\right\} - \frac{n|\alpha - c|^2}{|w - \alpha|^2},$$

which gives, with the help of (11),

$$\frac{n|\alpha - c|^2}{|w - \alpha|^2} < R^2 \left\{\left(\sum_{j=1}^{n} \frac{1}{|w - z_j|^2}\right) - B\right\} = \frac{nR^2}{|w-\alpha|^2},$$

or equivalently, $|\alpha - c| < R$.

Similarly if $|\alpha-c| > R$, that is if α lies exterior to C then, from (12), again we get

$$B|w - c|^2 < \left\{\sum_{j=1}^{n} \frac{R^2}{|w - z_j|^2}\right\} - \frac{nR^2}{|w - \alpha|^2} = R^2B,$$

which gives with the help of (11) that $|w - c| < R$. Thus in this case, not both points α and w may lie outside of $C : |z - c| \le R$.

Case 2. Let now $C : |z - c| \ge r$, $r > 0$.

Since all the zeros of $P(z)$ lie in C, therefore, $|z_j-c| \ge r, j = 1, 2, \ldots ,n$. If w lies exterior to C then $|w - c| < r$ and, from (12), we get with the help of (11), that

$$Br^2 > B|w - c|^2 \ge \left\{\sum_{j=1}^{n} \frac{r^2}{|w - z_j|^2}\right\} - \frac{n|\alpha - c|^2}{|w - \alpha|^2}$$

$$= Br^2 + \frac{nr^2}{|w - \alpha|^2} - \frac{n|\alpha - c|^2}{|w - \alpha|^2}.$$

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This gives $|\alpha-c| > r$. If now α lies exterior to C that is, if $|\alpha - c| < r$, Then from (11) and (12) we obtain as before that $|w - c| > r$. Thus in this case also not both points α and w may lie outside of $C : |z - c| \geq r$.

Case 3. Finally let C be a half-plane. That is, let $C : Re\ z \leq a$ or $Re\ z \geq b$ or $Im\ z \leq a'$ or $Im\ z \geq b'$, where a, b, a' and b' are real numbers. We prove the result for one of these four cases, say for $C : Re\ z \leq a$. The remaining three cases follow in a similar way. Since now we have $Re\ z \leq a$, therefore, $Re\ z_j \leq a$, $j = 1, 2, \ldots, n$. By the first part of the above lemma (with $c = 0$), we have

$$\text{(13)} \qquad wB = \left\{ \sum_{j=1}^{n} \frac{z_j}{|w - z_j|^2} \right\} - \frac{n\alpha}{|w - \alpha|^2},$$

where $B > 0$ is defined by (11).

Assume now $Re\ w > a$, that is, w lies exterior to C then from (13) we obtain

$$aB < (Re\ w)B = \left\{ \sum_{j=1}^{n} \frac{Re\ z_j}{|w - z_j|^2} \right\} - \frac{n(Re\ \alpha)}{|w - \alpha|^2}$$

$$\leq \left\{ a \sum_{j=1}^{n} \frac{1}{|w - z_j|^2} \right\} - \frac{n(Re\ \alpha)}{|w - \alpha|^2}.$$

This gives with the help of (11) that

$$\frac{n(Re\ \alpha)}{|w - \alpha|^2} < a \left\{ \left(\sum_{j=1}^{n} \frac{1}{|w - z_j|^2} \right) - B \right\} = \frac{na}{|w - \alpha|^2},$$

so that $Re\ \alpha < a$. Similarly if we assume that $Re\ \alpha > a$, then from (13) we get $Re\ w < a$. This shows that not both points α and w lie outside of C. Hence the first part of the theorem is completely proved.

To prove the second part of the theorem, let us suppose first that a circle $K : |z-c| \leq r$, through w and α has at least one z_j in its interior, no z_j in its exterior and the remaining z_j on its circumference. Then we have, $|w-c| = r$, $|\alpha-c| = r$, $|z_j-c| < r$ for at least one j and $|z_j - c| = r$ for the remaining z_j. Using this in (12), we obtain

$$Br^2 < \left\{ \sum_{j=1}^{n} \frac{r^2}{|w-z_j|^2} \right\} - \frac{nr^2}{|w-\alpha|^2} = Br^2 ,$$

which is obviously a contradiction. Since we also get a contradiction if we assume that a circle K through w and α has at least one z_j in its exterior, no z_j in its interior and the remaining z_j on its circumference, we conclude that any circle K through w and α either passes through all the zeros of $P(z)$ or separates these zeros. This completes the proof of the theorem in full.

Many other interesting results follow easily from the above lemma. For example the following theorem is an immediate consequence of the lemma.

THEOREM 1. *If all the zeros of a polynomial $P(z)$ of degree n lie on $|z| = r$, then for every real θ, all the zeros of the polynomial*

$$nP(z) + (re^{i\theta} - z)P'(z)$$

also lie on $|z| = r$.

If $z_1, z_2, \ldots , z_n$ are the zeros of $P(z)$ and w is a zero of the polynomial $nP(z) + (\alpha-z)P'(z)$ such that $P(w) \neq 0$, then by (11)

$$\text{(14)} \qquad \frac{n}{|w - \alpha|^2} \leq \sum_{j=1}^{n} \frac{1}{|w - z_j|^2} \leq n \max_{1\leq j\leq n} \frac{1}{|w - z_j|^2} .$$

Again using (11) in the second part of the lemma, it follows that

$$\text{(15)} \qquad \frac{n|\alpha - c|^2}{|w - \alpha|^2} \leq \sum_{j=1}^{n} \frac{|z_j - c|^2}{|w - z_j|^2} \leq n \max_{1\leq j\leq n} \frac{|z_j - c|^2}{|w - z_j|^2} .$$

From (14) and (15), we obtain the following result.

THEOREM 2. *If $P(z)$ is a polynomial of degree n and w is a zero of $nP(z) + (\alpha-z)P'(z)$, then $P(z)$ has at least one zero in each of the regions*

$$|z - w| \leq |\alpha - w| \quad \textit{and} \quad |z - w||\alpha - c| \leq |\alpha - w||z - c| ,$$

where c is any given real or complex number.

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