THE GENERAL ECONOMIC PREMIUM PRINCIPLE*

BY HANS BÜHLMANN

Zürich

ABSTRACT

We give an extension of the Economic Premium Principle treated in Astin Bulletin, Volume 11 where only exponential utility functions were admitted. The case of arbitrary risk averse utility functions leads to similar quantitative results. The role of risk aversion in the treatment is essential. It also permits an easy proof for the existence of equilibrium.

KEYWORDS

Mathematical economics, equilibrium theory, premium principles.

1. THE PROBLEM

In BÜHLMANN (1980) it was argued that in many real situations premiums are not only depending on the risk to be covered but also on the surrounding market conditions. The standard actuarial techniques are not geared to produce such a dependency and one has to construct a model for the whole market, if one wants to study the interrelationships between market conditions and premiums.

Such models exist in mathematical economics. For the purpose of this paper we borrow the model of mathematical economics for a pure exchange economy and we use the usual Walrasian equilibrium concept.

The more practically oriented reader might consider the model as an idealization of e.g., a reinsurance market where premiums of the contracts are determined by the market. Of course, the Walrasian model is not the only way to describe a reinsurance market. In oligopolistic situations one would rather have to rely on the theoretical framework provided by game theory. On the other hand the model used in this paper extends far beyond reinsurance.

The more theoretically minded reader will note that the model of an exchange economy used in the following has infinitely many commodities. The classical result of existence of equilibrium [see e.g., DEBREU (1959, 1974)] therefore does not hold. The existence proof given here is the theoretically most important aspect of the present paper.

2. THE MODEL FOR THE MARKET

We have agents i, $i = 1, 2, \ldots, n$ (typically reinsurers, insurers, buyers of direct insurance etc.).

* presented at the Meeting on Risk Theory September 1982 in Oberwolfach

The commodities to be traded are quantities of money, conditional on the random outcome ω, where ω stands for an element of a probability space $(\Omega, \mathfrak{A}, \Pi)$.

Let $Y_i(\omega)$ stand for the function as *traded* by agent i assigning to each state ω the payment received by i from the participants in the market. In insurance terminology Y_i describes an insurance policy or a reinsurance contract (Think of the sum of all insurance policies and reinsurance contracts bought and sold by i as if it were exactly one contract).

On the other hand we have conditional payments caused to agent *i from outside the market.* These payments—conditional on ω—are described by $X_i(\omega)$. In insurance terms X_i represents the risk of the agent i before (re-)insurance.

Using the terminology of BÜHLMANN (1980) we call X_i the original risk of agent i, Y_i the exchange function (or exchange variable) of agent i. In addition we characterize each agent by his utility function $u_i(x)$ [as usual $u_i'(x)>0$, $u_i''(x)\leq 0$] and his initial wealth W_i.

Whereas the original risk X_i belongs to agent i from the start we imagine that Y_i can be freely bought by him at a price which is given by

$$\text{Price}\,[Y_i]=\int_\Omega Y_i(\omega)\phi(\omega)\,d\Pi(\omega). \tag{1}$$

The function $\phi:\Omega\to\mathbb{R}$ appearing in (1) is called the *price density* The random vector $(Y_1, Y_2, \ldots, Y_n)$ representing the exchange variables bought by all agents will be denoted by $\boldsymbol{Y}$ in the sequel.

3. EQUILIBRIUM

DEFINITION. $(\tilde{\phi}, \tilde{\boldsymbol{Y}})$ is called an equilibrium if

(a) for all i: $E[u_i(W_i-X_i+\tilde{Y}_i-\int \tilde{Y}_i(\omega')\tilde{\phi}(\omega')\,d\Pi(\omega'))]=\max$ for all possible choices of the exchange variable Y_i.

(b) $\sum_{i=1}^n \tilde{Y}_i(\omega)=0$ for all $\omega\in\Omega$.

TERMINOLOGY. If conditions (a) and (b) are satisfied we call $\tilde{\phi}$ equilibrium price density, $\tilde{\boldsymbol{Y}}$ equilibrium risk exchange.

Hint. It might be worthwhile to look up in BÜHLMANN (1980) the definition in the special case of a finite probability space. The special case coincides with the standard equilibrium definitions in mathematical economics.

In BÜHLMANN (1980) it was shown that for exponential utility functions $u_i(x)=1-e^{-\alpha_i x}$ the equilibrium price density has the following form

$$\tilde{\phi}(\omega)=\frac{e^{\alpha Z(\omega)}}{E[e^{\alpha Z}]}\quad\text{where}\ \frac{1}{\alpha}=\sum_{i=1}^n\frac{1}{\alpha_i} \tag{2}$$

where Z has the precise meaning

$$Z(\omega)=\sum_{i=1}^n X_i(\omega). \tag{3}$$

In this paper we show that equation (3) defines the "market conditions" also in the case of arbitrary utility functions. We shall see that locally (but not globally) even (2) carries over to the case of arbitrary utility functions.

REMARK. In the case of an arbitrary probability space existence of an equilibrium as defined is usually not discussed in the economic literature. Exceptions are BEWLEY (1972) and TOUSSAINT (1981) who treat the problem of existence for economies with infinitely many commodities by imposing some topological structure on the space of random variables Y_i. In this paper we shall prove that equilibrium exists making only risk theoretical assumptions. This is, however, postponed to section 8. Up to this section we therefore *assume* existence of an equilibrium.

4. PRICE EQUILIBRIUM AND PARETO OPTIMUM

It is shown in BÜHLMANN (1980) that condition (a) is equivalent to condition

(c) for all i: $u_i'[W_i - X_i(\omega) + \tilde{Y}_i(\omega) - \int \tilde{Y}_i(\omega')\tilde{\phi}(\omega')\, d\Pi(\omega')]$

$$= \tilde{\phi}(\omega) \underbrace{\int u_i'[W_i - X_i(\omega) + \tilde{Y}_i(\omega) - \int \tilde{Y}_i(\omega')\tilde{\phi}(\omega')\, d\Pi(\omega')]\, d\Pi(\omega)}_{C_i}$$

for almost all ω.

COROLLARY. *From* (c) *we see that* $\int \tilde{\phi}(\omega)\, d\Pi(\omega) = 1$.

As $\tilde{Y}_i$ is only determined up to an additive constant there is no loss of generality in assuming

(d) $$\int \tilde{Y}_i(\omega')\tilde{\phi}(\omega')\, d\Pi(\omega') = 0 \quad \text{for all } i.$$

For convenience we write $X_i - Y_i = Z_i$ (and quite naturally $X_i - \tilde{Y}_i = \tilde{Z}_i$) and use either the Y-variables or the Z-variables to describe the exchange. In the Z-language conditions (c) and (d) yield

(4) $$\text{for all } i: \quad u_i'[W_i - \tilde{Z}_i(\omega)] = C_i\tilde{\phi}(\omega) \qquad (C_i > 0)$$

which—according to Borch's theorem [see BORCH (1960)]—shows that an equilibrium risk exchange (conditions (b), (c), (d)) is automatically a Pareto optimum (condition (b)) plus (4)).

Conversely if we start with a Pareto optimum (condition (b) plus (4) because of Borch's theorem) all we need to render $(\tilde{\phi}, \tilde{Y})$ an equilibrium is a change of the initial wealth W_i by the "free amounts" $A_i = E[\tilde{\phi}\tilde{Y}_i]$ where $\tilde{Y}_i = X_i - \tilde{Z}_i$. (Observe that $\sum_{i=1}^n A_i = 0$).

Before we continue our analysis it is important to note that the random variables $\tilde{Z}_i$ $(i=1,2,\ldots,n)$ and $\tilde{\phi}$ can be and very often must be chosen to *depend on ω only through $Z(\omega)=\sum_{i=1}^n X_i(\omega)$*. This result by BORCH (1962) can also be obtained from the following argument: Assume a Pareto optimal risk exchange $\tilde{\boldsymbol{Z}}$ with

$$\text{(I)} \qquad E[u_i(W_i-\tilde{Z}_i)]$$

and

$$\text{(II)} \qquad \sum_{i=1}^n \tilde{Z}_i(\omega)=\sum_{i=1}^n X_i(\omega)=Z(\omega) \quad \text{for all } \omega.$$

Define $\tilde{\tilde{Z}}_i=E[\tilde{Z}_i|Z]$ for each i.

$\tilde{\tilde{\boldsymbol{Z}}}$ is again a balancing risk exchange (i.e., satisfies (II)). From Jensen's inequality for the conditional expectation given Z we conclude that $\tilde{\tilde{\boldsymbol{Z}}}$ is at least as good as $\tilde{\boldsymbol{Z}}$ for all i. Namely

$$E[u_i(W_i-\tilde{Z}_i)|Z]\leq E[u_i(W_i-\tilde{\tilde{Z}}_i)|Z] \quad \text{for all } i$$

and hence

$$\text{(I)} \qquad E[u_i(W_i-\tilde{Z}_i)]\leq E[u_i(W_i-\tilde{\tilde{Z}}_i)] \quad \text{for all } i.$$

The inequality is strict unless either $\tilde{Z}_i=\tilde{\tilde{Z}}_i$ and/or $u_i(x)$ is linear on the probabilistic support of $\tilde{Z}_i$. Excluding linearity of u_i for all but one agent, $\tilde{Z}_i$ *must* depend on ω through Z for all i. In the case of linearity of u_i for several agents there is indifference of splitting the risk among them. Also in this case we *may* therefore assume that $\tilde{Z}_i$ depends on ω through Z for all i.

Finally if $\tilde{Z}_i$ is a function of Z for all i so must be $\tilde{\phi}$ as seen from (4).

Because of this we use also the notation $\tilde{Z}_i(\zeta)$, $\tilde{\phi}(\zeta)$, where ζ is the generic element of the probability space obtained by the mapping $Z:\Omega\to\mathbb{R}$.

5. RISK AVERSION

We rewrite (4) as

$$\text{(5)} \qquad \text{for all } i: \quad u_i'(W_i-\tilde{Z}_i(\zeta))=C_i\tilde{\phi}(\zeta) \quad \text{with} \quad \sum_{i=1}^n Z_i(\zeta)=\zeta.$$

Taking the logarithmic derivative on both sides we obtain

$$-\frac{u_i''(W_i-\tilde{Z}_i(\zeta))}{u_i'(W_i-\tilde{Z}_i(\zeta))}\tilde{Z}_i'(\zeta)=\frac{\tilde{\phi}'(\zeta)}{\tilde{\phi}(\zeta)}.$$

We introduce the *individual risk aversion* $\rho_i(x)=u_i''(x)/u_i'(x)$ and obtain

$$\text{(6)} \qquad \rho_i(W_i-\tilde{Z}_i(\zeta))\tilde{Z}_i'(\zeta)=\frac{\tilde{\phi}'(\zeta)}{\tilde{\phi}(\zeta)}$$

and because $\sum_{i=1}^{n} \tilde{Z}_i'(\zeta) = 1$ also

$$1 = \frac{\tilde{\phi}'(\zeta)}{\tilde{\phi}(\zeta)} \sum_i \frac{1}{\rho_i(W_i - \tilde{Z}_i(\zeta))}. \tag{7}$$

The sum on the right-hand side adds up the *individual risk tolerance units* and hence can be understood as the *total risk tolerance unit.* We express this by the abbreviated notation

$$\sum_i \frac{1}{\rho_i(W_i - \tilde{Z}_i(\zeta))} = \frac{1}{\rho(\zeta)}. \tag{8}$$

This notation suggests to call $\rho(\zeta)$ the *total risk aversion*. Observe, however, that this concept does not only depend on ζ but also on the functions $\tilde{Z}_i(\zeta)$ representing a particular fixed Pareto optimal splitting of the total risk.

With this understanding we also obtain from (6) and (7)

$$\tilde{Z}_i'(\zeta) = \frac{\rho(\zeta)}{\rho_i(W_i - \tilde{Z}_i(\zeta))} = \frac{1/[\rho_i(W_i - \tilde{Z}_i(\zeta))]}{1/\rho(\zeta)} \quad \text{Quotient of risk tolerance units} \tag{9}$$

This formula—as far as the author believes—not appearing elsewhere in the literature, is quite remarkable in two respects.

(a) Borch's condition (our (5) above) characterizes the Pareto optima by a system of differential equations with $n-1$ free parameters. In (9) these parameters have disappeared and we have a *unique system* of differential equations.

This means that one can now characterize the set of all Pareto optimal exchanges by the initial values $\tilde{Z}_i(0)$.

(b) The notion of risk aversion has been derived for the study of one single agent and the relationship between his certainty-equivalent and the risk variance [PRATT (1964)]. The appearance in the characterization of Pareto optimal risk exchanges is a surprise and gives the risk aversion a new additional meaning.

6. A NEW INTERPRETATION OF PARETO OPTIMAL RISK EXCHANGES

As just indicated, formula (9) allows us to characterize the set of all Pareto optima from their initial values. This shall now be done explicitly. Before we start we might, however, ask how these initial values $\tilde{Z}_i(0)$ should be interpreted.

Using the definitions as introduced in section 4

$$\tilde{Z}_i(0) = (X_i - \tilde{Y}_i)(0)$$

we see that $\tilde{Z}_i(0)$ stands for the total balance of payments to be made by i in the case when the total claims to the market $Z = \sum_{i=1}^{n} X_i$ are zero. This justifies the following

TERMINOLOGY. $T_i = -\tilde{Z}_i(0)$ is called *initial receipt* by agent i (before any positive or negative claims come in).

Any Pareto optimal risk exchange $\tilde{\boldsymbol{Z}} = (\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_n)$ can then be described as follows:

(A) Define arbitrary initial receipts T_i ($\sum_{i=1}^n T_i = 0$). (This is equivalent to choosing the constants C_i in equation (4)).

(B) Solve the system of differential equations (9) ($i = 1, 2, \ldots, n$) with initial conditions $\tilde{Z}_i(0) = -T_i$ ($i = 1, 2, \ldots, n$).

This mathematical characterization allows the following interpretation: After having distributed the initial receipts, the increases (decreases) $d\zeta$ of total risk ζ are split in the proportion of the risk tolerance units

$$d\tilde{Z}_i(\zeta) = \frac{1/[\rho_i(W_i - \tilde{Z}_i(\zeta))]}{1/\rho(\zeta)}\, d\zeta. \tag{10}$$

It is clear how (10) would immediately allow for a numerical integration of the system of differential equations (9). In order to avoid any technical difficulties with the system of differential equations we make the hypothesis (from here on)

(H) The risk aversions $\rho_i(x)$ are positive continuous functions on $\mathbb{R}$, satisfying a Lipschitz condition $|\rho_i(x) - \rho_i(x')| \leq K|x - x'|$

Under (H) we have existence and uniqueness of the solution to (9) for arbitrary initial receipts $\boldsymbol{T} = (T_1, T_2, \ldots, T_n)$.

REMARKS

(1) The new interpretation of Pareto Optimum can also be used in the case of risk exchanges $\boldsymbol{Y}$ restricted by some bounds. In this case, however, not all the agents would always participate in the splitting of all increases (decreases) $d\zeta$.

(2) From our interpretation (10) it is clear that hypothesis (H) could be weakened to allow at most one function $\rho_i(x)$ to be zero for any specific argument x. We renounce this refinement.

7. THE GENERAL ECONOMIC PREMIUM PRINCIPLE DEPENDING ON THE INITIAL TRANSFER PAYMENTS

We start with an equilibrium $(\tilde{\boldsymbol{Z}}, \tilde{\phi})$ (remember $\tilde{Z}_i = X_i - \tilde{Y}_i$). As $\tilde{\boldsymbol{Z}}$ is Pareto optimal it can be constructed according to the description in section 6. The choice of the initial receipts T_i must be left open at the moment.

However, the equilibrium price density $\tilde{\phi}$ like the "after exchange" functions $\tilde{Z}_i$ ($i = 1, 2, \ldots, n$) can be determined from the basic equations in section 5 for any particular choice of $\boldsymbol{T} = (T_1, T_2, \ldots, T_n)$.

Combining (6) and (9) we obtain

$$\rho_T(\zeta) = \frac{\tilde{\phi}'_T(\zeta)}{\tilde{\phi}_T(\zeta)} \tag{11}$$

with

$$\frac{1}{\rho_T(\zeta)} = \sum_{i=1}^{n} \frac{1}{\rho_i(W_i - \tilde{Z}_i^T(\zeta))}. \tag{12}$$

Observe that from here on $\tilde{Z}_i^T(\zeta)$ (for all i) stand for those *unique* Pareto optimal risk exchange functions with $\tilde{Z}_i^T(0) = -T_i$.

From (11) and the norming condition $\int \tilde{\phi}(\omega)\, d\Pi(\omega) = 1$ we obtain

$$\boxed{\tilde{\phi}_T(Z(\omega)) = \frac{\exp \int_0^{Z(\omega)} \rho_T(\zeta)\, d\zeta}{E[\exp \int_0^{Z(\omega)} \rho_T(\zeta)\, d\zeta]}.} \tag{13}$$

We easily recognize (13) as the global generalization of (2). The local behaviour, described by (11), is even the same as for exponential utilities. The basic difference is, of course, that in general the total risk aversion is not constant but depends on the total risk ζ and the way this total risk is split up among the agents.

For the practically minded reader we might add that the price density $\tilde{\phi}_T$ can be understood as a distortion of the actuarially correct probabilities. Formula (13) explains how this distortion comes about.

8. EXISTENCE OF EQUILIBRIUM

We have now—in a very natural way—come back to the question of existence of equilibrium. With the tools at our disposal we can now pose it as follows:

Are there initial receipts $\tilde{\boldsymbol{T}} = (\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n)$ such that

$$E[\phi_{\tilde{T}} \tilde{y}_i^T] = E[\phi_{\tilde{T}}(X_i - \tilde{Z}_i^T)] = 0 \quad \text{for all } i = 1, 2, \ldots, n? \tag{14}$$

Observe that for arbitrary initial receipts the resulting $(\tilde{\phi}_T, \boldsymbol{Y}^T)$ satisfies (4) and (b). In order to be an equilibrium it must also satisfy (d) (which is the same as (14)). We could also say, in the spirit of section 3, that in equilibrium no change of initial wealth distribution by free amounts is needed.

THEOREM. *Under* (H) *and for bounded* X_i, $i = 1, 2, \ldots, n$ $\tilde{\boldsymbol{T}}$ *exists.*

PROOF.

(i) Consider the mapping $\mathbb{R}^n \to \mathbb{R}^n$ which sends $\boldsymbol{T} = (T_1, T_2, \ldots, T_n)$ into $\boldsymbol{S} = (S_1, S_2, \ldots, S_n)$ by the rule

$$E[\tilde{\phi}_T(\tilde{Z}_i^T + T_i - X_i)] = S_i.$$

(ii) Observe that $(\tilde{Z}_i^T + T_i)(0) = 0$ by definition. In view of (10) and hypothesis H we must have for all i

$$(Z_i^T + T_i) \leqslant \zeta \quad \text{for } \zeta \geqslant 0$$
$$(Z_i^T + T_i) \geqslant \zeta \quad \text{for } \zeta \leqslant 0$$

which can be written as

$$|Z_i^T + T_i| \leqslant \left| \sum_{i=1}^{n} X_i \right| \leqslant nM \qquad (|X_i| \leqslant M \text{ for all } i, \text{ by assumption})$$

hence

$$|S_i| = E[\tilde{\phi}_T \cdot (\tilde{Z}_i^T + T_i - X_i)] \leqslant (n+1)M \quad \text{for arbitrary } \boldsymbol{T}.$$

(iii) Consider now the compact rectangle $|T_i| \leqslant (n+1)M$ for all i. Call it $\boldsymbol{R}$. Consider the hyperplane $\sum_{i=1}^{n} T_i = 0$. Call it E.
The intersection $\boldsymbol{R} \cap E$ is non empty, compact and convex.
(iv) The mapping $\boldsymbol{T} \mapsto \boldsymbol{S}$ defined in (i) maps $\boldsymbol{R} \cap E$ into $\boldsymbol{R} \cap E$.

Check

$$\sum_{i=1}^{n} S_i = E\Big[\underbrace{\tilde{\phi}_T\Big(\sum_{i=1}^{n} Z_i^T + \sum_{i=1}^{n} T_i - \sum_{i=1}^{n} X_i\Big)}_{0}\Big] = 0$$

From (H) and boundedness of all X_i it follows by a standard theorem on differential equations that the solutions $\tilde{Z}_i^T$ $(i = 1, 2, \ldots, n)$ depend continuously on the initial conditions $\boldsymbol{T}$. Therefore the mapping $\boldsymbol{T} \mapsto \boldsymbol{S}$ is also continuous.

Applying Brouwer's Fixed-Point Theorem we have existence of $\tilde{\boldsymbol{T}}$ with

$$E[\tilde{\phi}_T \cdot (\tilde{Z}_i^{\tilde{T}} + \tilde{T}_i - X_i)] = \tilde{T}_i \quad \text{for all } i$$

and consequently

$$E[\tilde{\phi}_T(\tilde{Z}_i^{\tilde{T}} - X_i)] = 0 \quad \text{for all } i \qquad \text{q.e.d.}$$

Remark. Boundedness of X_i is a rather strong technical assumption which one might want to weaken. The general idea would be to approximate arbitrary random variables X_i by truncation and to perform a limit argument. For the correctness of this limit argument, however, one needs again some technical assumptions.

REFERENCES

Bewley, T. F. (1972) Existence of Equilibria in Economies with infinitely Many Commodities. *Journal of Economic Theory*. **4**, 514–540.

Borch, K. (1960) The Safety loading of Reinsurance Premiums. *Skandinavisk Aktuarietidskrift*. **43**, 163–184.

Borch, K. (1962) Equilibrium in a Reinsurance Market. *Econometrica*. **30**, 424–444.

BÜHLMANN, H. (1980) An Economic Premium Principle. *Astin Bulletin.* **11**, 52–60.

DEBREU, G. (1959) *Theory of Value. An Axiomatic Analysis of Economic Equilibrium.* Cowles Foundation. Wiley.

DEBREU, G. (1974) International Congress of Mathematicians Vancouver. pp. 65–77.

PRATT, J. W. (1964) Risk Aversion in the Small and in the Large. *Econometrica.* **32**, 122–136.

TOUSSAINT, S. (1981) On the Existence of Equilibria in Economies with Infinitely Many Commodities. Mimeograph Fakultät für Volkswirtschafslehre und Statistik, Universität Mannheim.