J. Austral. Math. Soc. Ser. B 31(1990), 425–433

THE ISOTROPY MAPPINGS OF MINKOWSKI SPACE-TIME GENERATE THE ORTHOCHRONOUS POINCARÉ GROUP

JOHN W. SCHUTZ[1]

(Received 17 November 1988; revised 22 June 1989)

Abstract

Minkowski space-time is specified with respect to a single coordinate frame by the set of timelike lines. Isotropy mappings are defined as automorphisms which leave the events of one timelike line invariant. We demonstrate the existence of two special types of isotropy mappings. The first type of isotropy mapping induce orthogonal transformations in position space. Mappings of the second type can be composed to generate Lorentz boosts. It is shown that isotropy mappings generate the orthochronous Poincaré group of motions. The set of isotropy mappings then maps the single assumed coordinate frame onto a set of coordinate frames related by transformations of the orthochronous Poincaré group.

1. Introduction

In this paper we describe the orthochronous Poincaré group of motions of Minkowski space-time in terms of isotropy mappings, where an isotropy mapping is a bijection of the set of timelike lines such that the events of one timelike line are invariant and one given "spacelike direction" is mapped onto a second given "spacelike direction". Thus the property of isotropy is stated without reference to metric properties and therefore differs from the usual metric definitions of isotropy for geometries and spacetimes. In physical terms, the property of isotropy corresponds to the isotropy of "kinematic

[1]Bendigo College of Advanced Eduction, Bendigo, Victoria 3550, Australia.

observations" which could be made by a freely-moving observer and which could be stated solely in terms of properties of incidence restricted to the set of timelike lines.

Minkowski space-time is characterised as an affine $\mathbf{R}^4$ in which timelike lines have velocities with magnitude less than unity. Thus we are assuming the existence of a single coordinate frame and the constancy of the speed of light with respect to direction, as observed from this coordinate frame. We do not assume the existence of other coordinate frames and so we do not need to assume the constancy of the speed of light with respect to transformations between coordinate frames, nor do we assume the "Principle of Relativity" of Einstein [4].

In Section 2 we make several definitions including that of an "isotropy mapping" which is a concept central to the subsequent discussion. Section 3 contains Theorem 1 in which isotropy mappings are characterised by necessary and sufficient conditions which apply to the matrices of the corresponding linear transformations. Two special types of isotropy mappings are described. Compositions of these mappings are used to generate Lorentz boosts and then the full orthochronous Lorentz group. In the penultimate Section 4 we compose isotropy mappings to generate translations through any interval. Finally we characterise the orthochronous Poincaré group as the largest group of transformations which can be generated by isotropy mappings.

Alexandrov [1, 2] and Zeeman [11] have considered transformations which preserve the set of future light cones and have thereby obtained the orthochronous Poincaré group augmented by dilatations. However this symmetry group is too large to admit an invariant inner product or squared interval. Additionally Alexandrov [1, 2], Borchers and Hegerfeldt [3] and Lester and McKiernan [7] have considered the larger group of transformations which leave the set of light cones invariant and have obtained the full Poincaré group augmented by dilatations. Lenard [6] has considered transformations which leave the set of timelike lines invariant and has obtained a similar result. By contrast, in the present paper we consider bijections of the set of timelike lines and obtain the orthochronous Poincaré group as the group of symmetries generated by the isotropy mappings about timelike lines.

The advantages of the present approach are:

(i) we assume the existence of only one coordinate frame,
(ii) the "Principle of Relativity" is not assumed since its role is replaced by the property of isotropy which is assumed with respect to only one timelike line. This property is subsequently shown to apply with respect to all timelike lines.

2. Definitions

Minkowski space-time $\mathscr{M}$ is a four-dimensional affine space coordinatised as $\mathbf{R}^4$ such that each line has a parametric equation of the form

$$x_i = x_{i(initial)} + \lambda w_i \qquad (i = 0, 1, 2, 3) \tag{1}$$

where the four-component direction vector w_i is called a 4-*velocity vector*. We will use the index conventions that italic letters $i, j, k, \ldots$ range over the integers $0, 1, 2, 3$ while Greek letters $\alpha, \beta, \gamma, \ldots$ range over the integers $1, 2, 3$ and repeated subscripts imply a sum in accordance with the Einstein summation convention. Lines which satisfy the inequality

$$w_0^2 - w_1^2 - w_2^2 - w_3^2 > 0 \tag{2a}$$

are called *timelike lines* while lines which satisfy the corresponding equality

$$w_0^2 - w_1^2 - w_2^2 - w_3^2 = 0 \tag{2b}$$

are called *lightlike lines* and lines which satisfy the corresponding reversed inequality are called *spacelike lines*. Individual timelike lines are denoted by the symbols $\mathbf{Q}, \mathbf{R}, \mathbf{S}, \ldots$. Events which belong to a timelike line are denoted by the line symbol together with a subscript; thus, for example the events Q_1, Q_a, Q_x belong to the timelike line $\mathbf{Q}$. A timelike line $\mathbf{Q}$ and an event $e \notin \mathbf{Q}$ specify a *plane* $pl[\mathbf{Q}, e]$ which contains a subset of timelike lines "in one-dimensional rectilinear motion".

2.1. Space-time coordinates

In the subsequent discussion, let $\mathbf{Q}$ denote the timelike line along the the X_0-axis. Then the set of timelike lines parallel to $\mathbf{Q}$ is called the *position space* (*relative to* $\mathbf{Q}$). Each timelike line of the position space has 4-velocity $(1, 0, 0, 0)$ and is specified by a coordinate triple (x_1, x_2, x_3) called its *position-space coordinates*. Each event of $\mathscr{M}$ is specified by four coordinates (x_0, x_1, x_2, x_3): the 0-coordinate is called the *time coordinate* and the remaining coordinate triple is called its (*position*) *space coordinates*; together the four coordinates are called *space-time coordinates*. Equations (2) imply the existence of the usual euclidean metric on position space.

2.2. Measures of velocity, speed and rapidity

Besides the 4-velocity vector w_i, there are three (non-vectorial) measures of velocity relative to $\mathbf{Q}$: the 3-*velocity* v_α, more commonly referred to simply as the *velocity*, is defined to be

$$(v_1, v_2, v_3) := (w_1/w_0, w_2/w_0, w_3/w_0).$$

The *relative speed* is defined to be

$$|v| := (v_1^2 + v_2^2 + v_3^2)^{1/2}.$$

Finally there is the *relative rapidity*

$$|r| := \tanh^{-1}|v|.$$

Relative speeds for timelike lines have magnitude less than 1, while the speed of any lightlike line is equal to 1. Relative rapidities for timelike lines are unbounded. The concept of rapidity is due to Robb [8]: for the special case of rectilinear motion, directed rapidities are a "natural measure of speed" in the sense that they are arithmetically additive.

2.3. Isotropy mappings

Given a timelike line S and two half-planes with edge S, we say that a mapping of $\mathcal{M}$ which maps the set of timelike lines onto itself, leaves S and all its events invariant and maps the first half-plane onto the second, is an *isotropy mapping with invariant timelike line* S. If, for a given timelike line S and any two half-planes with edge S, there is an isotropy mapping with invariant timelike line S, we say that the space-time $\mathcal{M}$ is *isotropic with respect to* S. If $\mathcal{M}$ is isotropic with respect to every timelike line, we say that the *space-time is isotropic.*

2.4. The Poincaré group

The subgroup of affinities of $\mathbf{R}^4$ which leave the squared interval

$$\|\Delta x\|^2 := \Delta x_0^2 - \Delta x_1^2 - \Delta x_2^2 - \Delta x_3^2$$

invariant, is called the *Poincaré group.* The sub-group which leaves the origin of $\mathbf{R}^4$ invariant is called the *Lorentz group.* A Poincaré transformation

$$x_i \rightarrow x_i' = a_{ij}x_j + b_j$$

is said to be *orthochronous* if a_{00} is positive.

3. Isotropy mappings and Lorentz boosts

In the next theorem we obtain necessary and sufficient conditions for the existence of isotropy mappings. Particular types of isotropy mappings and compositions of isotropy mappings with the common invariant event $O(0,0,0,0)$ will be discussed after the proof. The set of timelike lines through the event $O(0,0,0,0)$ is denoted by $\mathscr{V}$, the set of lightlike lines through $O(0,0,0,0)$ forms the boundary of $\mathscr{V}$ and is denoted by $\partial\mathscr{V}$, while the closure of $\mathscr{V}$ (which is the union of $\mathscr{V}$ and $\partial\mathscr{V}$) is denoted by $\overline{\mathscr{V}}$. It is convenient to define a matrix η_{ij} whose non-zero components are $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$.

Theorem 1. (Isotropy Mappings)

(i) *Isotropy mappings which leave the event* $O(0,0,0,0)$ *invariant are Lorentz transformations.*

(ii) *An affinity* $x_i \to x_i' = a_{ij}x_j$ *is a Lorentz transformation if and only if:*

$$a_{0i}a_{0j} - a_{1i}a_{1j} - a_{2i}a_{2j} - a_{3i}a_{3j} = \eta_{ij}.$$

(iii) *Isotropy mappings satisfy the additional condition:*

$$a_{00} \geq 1.$$

(iv) *An affinity which has a fixed timelike line and satisfies the equations of* (ii) *and* (iii) *is an isotropy mapping.*

Proof. (a) *We first show that each isotropy mapping induces an affinity of* $\mathscr{M}$. In each plane which contains a timelike line there are two parallel classes of lightlike lines which can be used to specify the usual concept of "equality of intervals" on classes of parallel timelike lines. The same concept can then be extended to lightlike and spacelike lines in the plane. For any given lightlike line in the plane there is a distinct class of parallel lightlike lines which project equal intervals from any timelike line of the plane onto the given lightlike line. Similarly given any spacelike line **S** in the plane and any two distinct events of **S**, there are two pairs of lightlike lines which pass through the events and which meet at two distinct events which lie on some timelike line **T** in the plane. Properties of similarity imply that each event on the spacelike line **S** can be specified by a pair of events of **T** which are at equal affine intervals from the intersection **T** and **S**: furthermore, the interval measure on **T** can be projected by either one of the classes of lightlike lines to define an affine measure of intervals on **S**. Thus the affine structure of the plane and hence of $\mathscr{M}$ is specified by the set of timelike and lightlike lines which, in turn, are specified by the set of timelike lines. Consequently each isotropy mapping preserves the affine structure of $\mathscr{M}$ and induces an affinity.

(b) *We will next establish part* (i) *and then consider the sequence of parts* (ii), (iii) *and* (iv). By definition an isotropy mapping θ has an invariant timelike line $\mathbf{Q}^*$ and the affinity

$$x_i \to x_i' = a_{ij}x_j \tag{1}$$

induced by θ sends a lightlike line

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0 \tag{2}$$

onto a lightlike line

$$(x_0')^2 - (x_1')^2 - (x_2')^2 - (x_3')^2 = 0. \tag{3}$$

Now if equations (1) are substituted into the quadratic form of (3) and terms are grouped we obtain

$$(x_0')^2 - (x_1')^2 - (x_2')^2 - (x_3')^2 = (a_{0k}^2 - a_{1k}^2 - a_{2k}^2 - a_{3k}^2)x_k^2 \\ + 2(a_{0k}a_{0l} - a_{1k}a_{1l} - a_{2k}a_{2l} - a_{3k}a_{3l})x_k x_l \quad (k < l). \quad (4)$$

We next show that

$$a_{0k}a_{0l} - a_{1k}a_{1l} - a_{2k}a_{2l} - a_{3k}a_{3l} = 0 \quad (k < l). \quad (5a)$$

First consider the two special cases of (2) with $(x_0, x_1, x_2, x_3) = (1, \pm 1, 0, 0)$. The resulting equations (3) and (4) then imply (5a) for the case $(k, l) = (0, 1)$ and the cases $(0, 2)$ and $(0, 3)$ are obtained similarly. Next the two special cases $(2, 1, \pm 1, 0)$ imply (5a) for the case $(k, l) = (1, 2)$ and the cases $(1, 3)$ and $(2, 3)$ are obtained similarly. Hence for a lightlike line (2), equations (3) and (4) become

$$(a_{0k}^2 - a_{1k}^2 - a_{2k}^2 - a_{3k}^2)x_k^2 = 0.$$

If we consider the special cases of (2) with vectors $(1, 1, 0, 0), (1, 0, 1, 0)$ and $(1, 0, 0, 1)$ we obtain (for $\alpha = 1, 2, 3$)

$$(a_{00}^2 - a_{10}^2 - a_{20}^2 - a_{30}^2) = -(a_{0\alpha}^2 - a_{1\alpha}^2 - a_{2\alpha}^2 - a_{3\alpha}^2) =: \lambda. \quad (5b)$$

Thus equations (4) and (5) imply that for any vector (x_0, x_1, x_2, x_3),

$$(x_0')^2 - (x_1')^2 - (x_2')^2 - (x_3')^2 = (a_{0k}a_{0l} - a_{1k}a_{1l} - a_{2k}a_{2l} - a_{3k}a_{3l})x_k x_l \\ = \lambda(x_0^2 - x_1^2 - x_2^2 - x_3^2). \quad (5c)$$

For an isotropy mapping the events on the timelike line $\mathbf{Q}^*$ are invariant, so for these events the two vectors are equal, whence

$$\lambda = 1 \quad (5d)$$

so the previous equation implies that the isotropy mapping θ is a Lorentz transformation which establishes (i).

(ii) Sufficiency is established by substituting the equations of (ii) into (4). To prove necessity we observe that, since Lorentz transformations leave the form $\|\cdot\|^2$ invariant, any Lorentz transformation maps a lightlike line (2) onto a lightlike line (3). A similar argument shows that equations (5a,b,c) apply, with $\lambda = 1$ in equation (5c) by the definition of a Lorentz transformation. Equations (5a,b) with $\lambda = 1$ imply the equations of (ii).

A Lorentz transformation $x_i \to x_i' = a_{ij}x_j$ has an inverse with coefficients $\overline{a}_{ij} = \eta_{il}a_{kl}\eta_{kj}$ as is easily verified using equations (ii). Thus we have a set of equations which correspond to those of (ii) but with all subscripts transposed; in particular we have

$$a_{00}^2 - a_{01}^2 - a_{02}^2 - a_{03}^2 = 1. \quad (6)$$

https://doi.org/10.1017/S0334270000006767 Published online by Cambridge University Press

(iii) Consider an invariant event (x_0, x_1, x_2, x_3) with $x_0 > 0$ on the fixed timelike line. For this invariant event $x_0 = a_{0j}x_j$ so that $x_0(1 - a_{00}) = a_{0\alpha}x_\alpha$. The Cauchy-Schwarz inequality implies that

$$x_0(1 - a_{00}) \leq (a_{01}^2 + a_{02}^2 + a_{03}^2)^{1/2}(x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

Then equation (6) and the inequality for timelike lines imply that

$$x_0(1 - a_{00}) < (a_{00}^2 - 1)^{1/2}(x_0^2)^{1/2}$$

whence $(1 - a_{00})^2 < a_{00}^2 - 1$ which implies (iii).

(iv) The equations of (ii) imply (as in the derivation of (5c)) that $\|\cdot\|^2$ is an invariant, so $\mathscr{V}$ is mapped bijectively. Now for an event with $x_0 > 0$,

$$\begin{aligned} x_0 \to x_0' &= a_{00}x_0 + a_{01}x_1 + a_{02}x_2 + a_{03}x_3 \\ &\geq a_{00}x_0 - (a_{01}^2 + a_{02}^2 + a_{03}^2)^{1/2}(x_1^2 + x_2^2 + x_3^2)^{1/2} \end{aligned}$$

by the Cauchy-Schwarz inequality and so equation (6) together with the inequality for timelike lines implies that $x_0' \geq a_{00}x_0 - (a_{00}^2 - 1)^{1/2}x_0$. Thus x_0 and its image x_0' are both positive and since $\|\cdot\|^2$ is invariant, each event on the fixed timelike line is invariant. Q.E.D.

This theorem implies the existence of special types of isotropy mappings which are described in Sections 3.1-2 below.

3.1. Isotropy mappings with invariant timelike line Q

The timelike line **Q** has a 4-velocity vector $(1, 0, 0, 0)$: equations (ii) and (iii) of the preceding Theorem 1 imply that the transformation matrix has the form

$$a_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & a_{\alpha\beta} \end{bmatrix}$$

where the sub-matrix $a_{\alpha\beta}$ represents an orthogonal transformation in the $X_1X_2X_3$ subspace. Isotropy mappings of this type will be called *orthogonal transformations*.

3.2. Isotropy mappings about the invariant timelike line with 4-velocity $(\cosh r, \sinh r, 0, 0)$

A second special type of isotropy mapping has the transformation matrix

$$b_{ij} = \begin{bmatrix} \cosh 2r & -\sinh 2r & 0 & 0 \\ \sinh 2r & -\cosh 2r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which has an invariant timelike line with 4-velocity vector $(\cosh r, \sinh r, 0, 0)$, as is readily verified by showing that equations (ii) and (iii) of Theorem 1 are satisfied and that the stated 4-velocity vector is invariant. This invariant timelike line has a rapidity r in the X_1-direction which is equivalent to a speed $v = \tanh r$.

https://doi.org/10.1017/S0334270000006767 Published online by Cambridge University Press

3.3. Lorentz boost in X_1-direction with speed v

An important transformation, which is not an isotropy mapping, is obtained from the composition of two isotropy mappings of the type specified in Section 3.2 but with rapidities in the X_1-direction of (firstly) $r/2$ and (secondly) r. It is called a "*Lorentz boost in the X_1-direction with rapidity r and speed v*" where $v = \tanh r$.

3.4. Hyperbolic velocity space

Although a detailed discussion is beyond the scope of this paper it is worth remarking that the set of timelike lines $\mathscr{V}$ through the origin of $\mathbf{R}^4$ may be interpreted as the "points" of a geometry, which turns out to be a non-euclidean hyperbolic or Bolyai-Lobachevskian geometry with relative rapidity as an intrinsic metric [9], known as "velocity space". Isotropy mappings are isometries of velocity space. The isotropy mapping of Section 3.2 (above) is a "reflection in a plane". The Lorentz boosts of Section 3.3, which are generated by the composition of two of these "reflection" mappings, are "translations" of velocity space.

4. The orthochronous Poincaré group

The orthogonal transformations and Lorentz boosts of the previous section have been obtained as isotropy mappings or as compositions of isotropy mappings. Since orthogonal transformations and Lorentz boosts are generators of the orthochronous Lorentz group, it follows that any orthochronous Lorentz transformation may be represented as a composition of isotropy mappings.

For any given event $e(e_0, e_1, e_2, e_3)$ and for each orthochronous Lorentz transformation a, there is a corresponding orthochronous Poincaré transformation $\tau^{-1} \circ a \circ \tau$ which leaves e invariant, where τ is the translation $\tau\colon x_i \to x_i' = x_i - e_i$. Thus isotropy mappings with invariant events other than $O(0,0,0,0)$ can be composed to generate spacelike translations, timelike translations, and then more general translations. It follows that isotropy mappings generate the orthochronous Poincaré group. An immediate consequence is that Minkowski space-time is isotropic (in the sense defined in Section 2.3 above).

4.1. Coordinate frames

Since each isotropy mapping may be considered to specify a mapping of space-time coordinates it follows that for any timelike line and any event on the line, there is a space-time coordinate system which has the line and the event, respectively, as the origins of position-space and space-time. A coordinate system obtained in this way is called a *coordinate frame*. The position-space is euclidean and the previous concepts of 4-velocity, 3-velocity and rapidity apply, so the speed of lightlike lines is constant and equal to unity in every coordinate frame. Theorem 1 implies that any two coordinate frames are related by a transformation which belongs to the orthochronous Poincaré group.

5. Conclusion

It has been shown that the property of isotropy applies to every observer, and that isotropy mappings generate the orthochronous Poincaré group. The set of all isotropy mappings then maps the single assumed coordinate frame onto a set of coordinate frames related by the transformations of the orthochronous Poincaré group. Readers interested in characterisations of Minkowski space-time are referred to the survey article by Guts [5] and a recent characterisation by Schutz [10] in which Minkowski space-time is characterised by affine structure, the finiteness of the speed of light, and the property of isotropy which is assumed to apply only for a single observer.

References

[1] A. D. Alexandrov, "On Lorentz transformations", *Usp. Mat. Nauk* **5:3** (1950) 187.

[2] ——, "Mappings of Spaces with Families of Cones and Space-Time Transformations", *Annali di Mathematica Pura ed Applicata* **103** (1975) 229–257.

[3] H. J. Borchers and G. C. Hegerfeldt, "The structure of space-time transformations", *Commun. Math. Phys.* **28** (1972) 259–266.

[4] A. Einstein, "Zur Elektrodynamic bewegter Körper", Ann. der Phys. **17** (1905) 891–921. ["On the electrodynamics of moving bodies", in A. Einstein, H. A. Lorentz, H. Minkowski and H. Weyl, '*The Principle of Relativity*', translated by W. Perrett and G. B. Jeffrey (Methuen, 1923). (Reprinted. Dover, New York, 1958).].

[5] A. K. Guts, "Axiomatic relativity theory", *Usp. Mat. Nauk* **37:2** (1982) 39–79. [*Russian Math. Surveys* **37:2** (1982) 41–89.].

[6] A. Lenard, "A characterization of Lorentz transformations", *J. Math. Phys.* **19** (1978) 157.

[7] J. A. Lester and M. A. McKiernan, "On null cone preserving mappings", *Math. Proc. Cambridge Philos. Soc.* **81** (1977) 455–462.

[8] A. A. Robb, *The absolute relations of time and space* (Cambridge University Press, Cambridge, 1921).

[9] J. W. Schutz, *Foundations of special relativity: kinematic axioms for Minkowski space-time*, Lecture Notes in Mathematics **361** (Springer, Berlin-Heidelberg-New York, 1973).

[10] ——, "Affine structure and isotropy imply Minkowski space-time and the orthochronous Poincaré group", *J. Math. Phys.* **30** (1989) 635–638.

[11] E. C. Zeeman, "Causality implies the Lorentz group", *J. Math. Phys.* **5** (1964) 490–493.