J. Aust. Math. Soc. **107** (2019), 289–301
doi:10.1017/S1446788719000508

CHEBYSHEV SUBSETS OF A HILBERT SPACE SPHERE

THEO BENDIT

(Received 23 September 2018; accepted 8 June 2019)

Communicated by B. Sims

In memory of Jonathan M. Borwein, 1951–2016.

Abstract

The Chebyshev conjecture posits that Chebyshev subsets of a real Hilbert space X are convex. Works by Asplund, Ficken and Klee have uncovered an equivalent formulation of the Chebyshev conjecture in terms of uniquely remotal subsets of X. In this tradition, we develop another equivalent formulation in terms of Chebyshev subsets of the unit sphere of $X \times \mathbb{R}$. We characterise such sets in terms of the image under stereographic projection. Such sets have superior structure to Chebyshev sets and uniquely remotal sets.

2010 *Mathematics subject classification*: primary 41A65; secondary 41A50, 46B20.

Keywords and phrases: Chebyshev sets, Chebyshev conjecture, stereographic projection, proximal points, furthest points.

1. Introduction

The Chebyshev conjecture is a long-standing open problem, spanning fields of functional and convex analysis, as well as optimisation. The conjecture states that Chebyshev sets, sets with unique closest points, in a real Hilbert space must be convex. It is well known that Chebyshev sets must be closed and nonempty, and that closed convex sets are Chebyshev. The Chebyshev conjecture therefore aims to characterise all Chebyshev subsets of real Hilbert spaces.

The conjecture was originally posed by Efimov and Stečkin [4] in 1958, although the first work towards the conjecture appeared over 20 years prior [3, 9, 10] (for a good historical account, see [7]). Despite the simplicity of the conjecture's statement and significant interest from the mathematical community, the conjecture remains open.

One of the more novel results came from Asplund, Ficken and Klee, who established, with sphere inversions, a nontrivial equivalence between nonconvex Chebyshev sets and nontrivial uniquely remotal sets. Their result leads to an equivalent formulation of the Chebyshev conjecture in terms of uniquely remotal sets.

This paper is intended to be part one of a two-part paper. In this part, we use a similar idea, substituting stereographic projection for sphere inversion. Stereographic projection, like sphere inversion, generically maps balls to balls (without necessarily preserving the centres of such balls). This similarly leads to an equivalent formulation of the Chebyshev conjecture in terms of Chebyshev subsets of the unit sphere of a Hilbert space, which is the main result of this part of the paper.

While the unit sphere lacks the linear structure of Hilbert spaces, the closed and open balls are complementary in the sphere. This fact implies that Chebyshev sets and uniquely remotal sets coincide on the sphere. We examine, particularly in the sequel, such sets when mapped under inverse stereographic projection. The geometric properties of stereographic projection show that such sets are Chebyshev in the Hilbert space, but often also uniquely remotal. As such, these sets, which we call *sphere-Chebyshev sets*, will be shown to have superior structure to Chebyshev sets.

As we are dealing with Chebyshev subsets of a (nonlinear) metric space, we must define Chebyshev and uniquely remotal sets in terms of general metric spaces.

Suppose that (M, ρ) is a metric space and let $x \in M$ and $r > 0$. Throughout, we will denote by

$$\begin{aligned} B[x; r] &= \{y \in M : \rho(x, y) \le r\}, \\ B(x; r) &= \{y \in M : \rho(x, y) < r\}, \\ S[x; r] &= \{y \in M : \rho(x, y) = r\}, \end{aligned}$$

respectively, the closed ball, open ball and sphere, centred at x with radius r.

Suppose that (M, ρ) is a metric space with $C \subseteq M$. For $x \in M$, let

$$\begin{aligned} d_C(x) &= \inf_{c \in C} \rho(x, c), \\ P_C(x) &= S[x; d_C(x)] \cap C, \\ r_C(x) &= \sup_{c \in C} \rho(x, c), \\ F_C(x) &= S[x; r_C(x)] \cap C, \end{aligned}$$

the *distance*, *projection*, *radial* and *furthest point* maps of C, respectively. If P_C is single-valued everywhere, we say that C is *Chebyshev*. Similarly, if C is bounded and F_C is single-valued everywhere, we say that C is *uniquely remotal* (note that assuming C is bounded is necessary to properly define r_C and F_C).

Suppose that X is a normed linear space, $C \subseteq X$ is Chebyshev and $X \setminus C$ is convex, nonempty and bounded. Then C is a *Klee cavern*.

As such, Klee caverns are nonconvex Chebyshev sets with extra structure. The following remarkable result connects Chebyshev sets, uniquely remotal sets and Klee caverns.

Theorem 1.1 (Asplund, Ficken, Klee). *Suppose that X is a real inner product space. Then the following are equivalent:*

(1) *every Chebyshev subset of X is convex;*

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(2) *every uniquely remotal subset of X is a singleton;*
(3) *there does not exist a Klee cavern in X.*

The first step in proving this theorem is to construct from a nonconvex Chebyshev set, a nontrivial uniquely remotal set. This was published by Klee in 1961 [8], but Klee attributed the idea to Ficken, though Ficken did not publish it. The second step is to construct a Klee cavern from a nontrivial uniquely remotal set, which Asplund did in 1969 [1].

Ficken's idea involved sphere inversions. Given a normed linear space X, the map

$$\iota_{y;r} : X \setminus \{y\} \to X \setminus \{y\} : x \mapsto y + \frac{r^2}{\|x-y\|^2}(x-y)$$

represents inversion in the sphere $S[y; r]$. Note that this map is a continuous involution and hence a homeomorphism.

We can also naturally extend the inversion map to $X \cup \{\infty\}$, X with a point at ∞ adjoined, by setting $\iota_{y,s}(y) = \infty$ and $\iota_{y,s}(\infty) = y$.

Although $\iota_{y;r}$ can be defined on any normed linear space, they are most interesting when X is an inner product space. Such maps predictably map balls to balls, half-spaces or complements of balls, depending on the location of y relative to the ball. Ficken used this property to show that $\iota_{y,r}(C)$ is uniquely remotal whenever C is Chebyshev and $y \in \overline{\text{conv}} C \setminus C$.

2. Stereographic projection

As mentioned previously, we substitute stereographic projection in the role of sphere inversion in Ficken's argument. In order to define stereographic projection, we first consider the inner product space $X \times \mathbb{R}$ with the inner product $\langle (x, r), (y, s)\rangle = \langle x, y\rangle + rs$.

In particular, in this paper, we consider $S_{X\times\mathbb{R}}$, the unit sphere of $X \times \mathbb{R}$, as a metric subspace of $X \times \mathbb{R}$.

We define *stereographic projection* as follows:

$$\sigma : S_{X\times\mathbb{R}} \to X \cup \{\infty\} : \begin{cases} (x, r) \mapsto \dfrac{x}{1-r} & \text{if } (x, r) \neq (0, 1), \\ \infty & \text{if } (x, r) = (0, 1). \end{cases}$$

It is straightforward to verify the inverse of stereographic projection,

$$\sigma^{-1} : X \cup \{\infty\} \to S_{X\times\mathbb{R}} : \begin{cases} x \mapsto \dfrac{1}{\|x\|^2+1}(2x, \|x\|^2 - 1) & \text{if } x \neq \infty, \\ (0, 1) & \text{if } x = \infty, \end{cases}$$

either by trigonometry or by composition with σ.

We define the following term in order to avoid certain degenerate cases in the results ahead.

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Suppose that M is a metric space and $C \subseteq M$ is a sphere or a ball (open or closed). We say that C is *nondegenerate* if C and $M \setminus C$ contain at least two elements.

For example, $B((0,-1);2) \subseteq S_{X\times\mathbb{R}}$ and $S[0;0] \subseteq X$ are degenerate. Nondegenerate spheres $C \subseteq S_{X\times\mathbb{R}}$ have the advantageous property that $S_{X\times\mathbb{R}} \setminus C$ has two connected components, each a (nondegenerate) open ball.

The following geometric fact lies at the heart of the connection between Chebyshev subsets of X and $S_{X\times\mathbb{R}}$.

PROPOSITION 2.1. *Let X be a real inner product space. Fix $B \subseteq S_{X\times\mathbb{R}}$. Then B is a nondegenerate ball in $S_{X\times\mathbb{R}}$ if and only if $\sigma(B)$ is a nondegenerate ball, a half-space or the complement of a nondegenerate ball in $X \cup \{\infty\}$, depending on whether $(0,1)$ lies in the interior, exterior or boundary of B.*

In particular, B is a closed (respectively open) nondegenerate ball if and only if one of the following is true:

- *$\sigma(B)$ is a closed (respectively open) nondegenerate ball and $(0,1) \notin \operatorname{cl} B$;*
- *$\sigma(B) = X \cup \{\infty\} \setminus C$, where C is an open (respectively closed) nondegenerate ball and $(0,1) \in \operatorname{int} B$;*
- *$\sigma(B) = H \cup \{\infty\}$ (respectively H), where H is a closed (respectively open) half-space and $(0,1) \in \operatorname{bdry} B$.*

For an illustration of Proposition 2.1, see Figures 1, 2 and 3.

Proposition 2.1 is a piece of mathematical folklore. We omit the proof, as the formulae for how balls map under σ are long and unenlightening enough not to state here. If the reader wishes to fill in the gaps, it is the author's recommendation to follow the following steps with the aid of a computer algebra system.

(1) Show that σ is the restriction of the sphere inversion map $\iota_{(0,1);\sqrt{2}}$ in $X \times \mathbb{R}$ to $S_{X\times\mathbb{R}}$.
(2) Compute the formulae for the mapping of balls under stereographic projection. See [5, Proposition 3.19] for a good start.
(3) Show that balls and half-spaces in $X \times \mathbb{R}$ intersect with X in balls and half-spaces, and with $S_{X\times\mathbb{R}}$ in balls.

Proposition 2.1 shows a correspondence between balls in $S_{X\times\mathbb{R}}$ and simple geometric subsets of X. As metric projection problems can be expressed geometrically in terms of maximal open balls that fail to intersect a given set, this suggests a connection between Chebyshev subsets of $S_{X\times\mathbb{R}}$ and certain subsets of $X \cup \{\infty\}$. We explore this connection in Section 3, as well as throughout the sequel.

PROPOSITION 2.2. *Suppose that X is a real inner product space. For all $x, y \in X$,*

$$\|\sigma^{-1}(x) - \sigma^{-1}(y)\| = \frac{2\|x-y\|}{\sqrt{(\|x\|^2+1)(\|y\|^2+1)}}.$$

In particular, σ^{-1} is Lipschitz on bounded subsets of X and σ is Lipschitz on subsets of $S_{X\times\mathbb{R}}$ bounded away from $(0,1)$.

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PROOF. Suppose that $x, y \in X$. Consider

$$\begin{aligned}
&(\|x\|^2 + 1)^2(\|y\|^2 + 1)^2\|\sigma^{-1}(x) - \sigma^{-1}(y)\|^2 \\
&\quad = \|(\|y\|^2 + 1)(2x, \|x\|^2 - 1) - (\|x\|^2 + 1)(2y, \|y\|^2 - 1)\|^2 \\
&\quad = 4\|(\|y\|^2 + 1)x - (\|x\|^2 + 1)y\|^2 \\
&\qquad + ((\|x\|^2 - 1)(\|y\|^2 + 1) - (\|y\|^2 - 1)(\|x\|^2 + 1))^2 \\
&\quad = 4(\|y\|^2 + 1)^2\|x\|^2 + 4(\|x\|^2 + 1)^2\|y\|^2 - 8(\|x\|^2 + 1)(\|y\|^2 + 1)\langle x, y\rangle \\
&\qquad + 4(\|x\|^2 - \|y\|^2)^2 \\
&\quad = 4\|x\|^4\|y\|^2 + 4\|x\|^2\|y\|^4 + 4\|x\|^4 + 4\|y\|^4 + 8\|x\|^2\|y\|^2 + 4\|x\|^2 + 4\|y\|^2 \\
&\qquad - 8(\|x\|^2 + 1)(\|y\|^2 + 1)\langle x, y\rangle.
\end{aligned}$$

One obtains the same expression after expanding

$$4(\|x\|^2 + 1)(\|y\|^2 + 1)(\|x\|^2 + \|y\|^2 - 2\langle x, y\rangle),$$

which yields the identity, as desired.

Our bounded subset of X can be assumed without loss of generality to be a ball $B[0; r]$, where $r > 0$. Then, for $x, y \in B[0; r]$,

$$\frac{2}{r^2 + 1}\|x - y\| \leq \|\sigma^{-1}(x) - \sigma^{-1}(y)\| \leq 2\|x - y\|.$$

Suppose that $C \subseteq S_{X\times\mathbb{R}}$ is bounded away from $(0, 1)$, that is, there exists some $s > 0$ such that $C \cap B((0, 1); s) = \emptyset$. As in Proposition 3.1, we have $B[(0, -1); r] = S_{X\times\mathbb{R}} \setminus B((0, 1); s)$, where $r = \sqrt{4 - s^2}$. Thus, without loss of generality, we may assume that $C = B[(0, -1); r]$, where $0 < r < 2$.

Suppose that $(x, t) \in S_{X\times\mathbb{R}} \setminus \{(0, 1)\}$. Then, by the identity proven above,

$$\begin{aligned}
&\|(x, t) - (0, -1)\|^2 \leq r^2 \\
&\iff \frac{4\|\sigma(x, t) - 0\|^2}{(\|\sigma(x, t)\|^2 + 1)(\|0\|^2 + 1)} \leq r^2 \\
&\iff \frac{4\|\sigma(x, t)\|^2}{\|\sigma(x, t)\|^2 + 1} \leq r^2 \\
&\iff \|\sigma(x, t)\|^2 \leq \frac{r^2}{4 - r^2}.
\end{aligned}$$

Thus, $\sigma(C) = B[0; r/s]$. Since σ^{-1} is bi-Lipschitz on this set, it follows easily that σ is bi-Lipschitz on C. □

We finish the section with an important fact about spheres in $S_{X\times\mathbb{R}}$, which, while not a property of stereographic projection, is easily proven using stereographic projection.

PROPOSITION 2.3. *Let X be an inner product space, and (M, ρ) be X or $S_{X\times\mathbb{R}}$. Further suppose that S_1 and S_2 are nondegenerate spheres in M that intersect in at most one point. If $(x_n)_{n\in\mathbb{N}} \in S_1$ and $(y_n)_{n\in\mathbb{N}} \in S_2$ are such that $\rho(x_n, y_n) \to 0$, then $S_1 \cap S_2 = \{z\}$ for some $z \in M$, and x_n and y_n converge to z.*

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Consequently, if $S_1 \cap S_2 = \emptyset$*, then*

$$\inf\{\rho(p,q) : p \in S_1, q \in S_2\} > 0.$$

If instead $S_1 \cap S_2 = \{z\}$ *and* $\varepsilon > 0$ *is small enough so that* $S_1, S_2 \nsubseteq B[z;\varepsilon]$*, then*

$$\inf\{\rho(p,q) : p \in S_1 \setminus B[z;\varepsilon], q \in S_2\} > 0.$$

Proof. If $\dim X \le 1$, then the result is trivial, due to the fact that the spheres are finite. Therefore, we may assume that $\dim X > 1$ (possibly where X is infinite dimensional). In particular, spheres are connected, so if two spheres $S[x;r], S[y;s] \subseteq M$ fail to intersect, or if they intersect uniquely, then one of three cases occurs:

(1) $S[x;r] \subseteq B[y;s]$ ($S[y;s]$ envelops $S[x;r]$);
(2) $S[y;s] \subseteq B[x;r]$ ($S[x;r]$ envelops $S[y;s]$); or
(3) $B[x;r] \cap B[y;s] = S[x;r] \cap S[y;s]$ (neither sphere envelops the other).

Consider first $M = X$ with case (3). Fix $S_1 = S[x;r]$ and $S_2 = S[y;s]$. Suppose that $(x_n) \in S_1$ and $(y_n) \in S_2$ are such that $\|x_n - y_n\| \to 0$. We have

$$\begin{aligned} 0 &\le \|x-y\| - (r+s) \\ &= \|x-y\| - \|x_n - x\| - \|y_n - y\| \\ &\le \|y_n - x\| - \|x_n - x\| \\ &\le \|x_n - y_n\| \to 0 \end{aligned}$$

and hence $\|x-y\| = r+s$ and $\|y_n - x\| \to r$. Let

$$z = \frac{r}{r+s}y + \frac{s}{r+s}x \in S_1 \cap S_2.$$

Consequently,

$$\begin{aligned} \|y_n - z\|^2 &= \|y_n - y\|^2 + \|z-y\|^2 - \langle y_n - y, z - y\rangle \\ &= 2s^2 - \frac{s}{r+s}(\|x-y\|^2 + \|y_n - y\|^2 - \|x - y_n\|^2) \\ &\to 2s^2 - \frac{s}{r+s}((r+s)^2 + s^2 - r^2) = 0. \end{aligned}$$

Thus, y_n, and hence x_n, converge to z.

Next, consider $M = S_{X\times\mathbb{R}}$. Suppose that $S_1, S_2 \subseteq M$ are spheres. Note that such spheres pertain to precisely two balls each, centred at antipodal points. Without loss of generality, we may choose balls B_1 and B_2 corresponding respectively to S_1 and S_2 in such a way that case (3) occurs. By applying a surjective isometry to $S_{X\times\mathbb{R}}$, without loss of generality, we may assume that $(0,1) \notin B_1 \cup B_2$.

Consider $\sigma(B_1)$ and $\sigma(B_2)$. Using Proposition 2.1, these sets are balls, whose spheres are $\sigma(S_1)$ and $\sigma(S_2)$. Therefore,

$$\sigma(B_1) \cap \sigma(B_2) = \sigma(B_1 \cap B_2) = \sigma(S_1 \cap S_2) = \sigma(S_1) \cap \sigma(S_2)$$

and hence case (3) applies to $\sigma(S_1)$ and $\sigma(S_2)$.

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If we take $(x_n, t_n) \in S_1$ and $(y_n, s_n) \in S_2$ such that $\|(x_n, t_n) - (y_n, s_n)\| \to 0$, then Proposition 2.2 implies that $\|\sigma(x_n, t_n) - \sigma(y_n, s_n)\| \to 0$. By the argument above, $\sigma(S_1)$ and $\sigma(S_2)$ intersect uniquely at the common limit z of $\sigma(x_n, t_n)$ and $\sigma(y_n, s_n)$. Clearly, $\sigma^{-1}(z) \in S_1 \cap S_2$ and, by the continuity of σ^{-1} on X, we have $(x_n, t_n), (y_n, s_n) \to \sigma^{-1}(z)$, as required.

Finally, we prove the more general $M = X$ case using the previous arguments. Suppose that $S_1, S_2 \subseteq X$ are spheres with at most one point of intersection, and $(x_n) \in S_1$ and $(y_n) \in S_2$ are such that $\|x_n - y_n\| \to 0$. Then $\sigma^{-1}(S_1)$ and $\sigma^{-1}(S_2)$ are spheres in $S_{X\times\mathbb{R}}$ with at most one point of intersection. Moreover, applying Proposition 2.2, $\|\sigma^{-1}(x_n) - \sigma^{-1}(y_n)\| \to 0$. Thus, $\sigma^{-1}(x_n)$ and $\sigma^{-1}(y_n)$ converge to the unique point $(z, r) \in \sigma^{-1}(S_1) \cap \sigma^{-1}(S_2)$ and so $x_n, y_n \to \sigma(z, r) \in S_1 \cap S_2$. □

3. Sphere-Chebyshev sets

Suppose that X is a real inner product space. We call a subset $C \subseteq S_{X\times\mathbb{R}}$ *sphere-Chebyshev* if it is Chebyshev in the metric space $S_{X\times\mathbb{R}}$.

We additionally call a subset $C \subseteq X \cup \{\infty\}$ sphere-Chebyshev if $\sigma^{-1}(C)$ is sphere-Chebyshev in $S_{X\times\mathbb{R}}$.

One may ask, why transfer the Chebyshev problem to $S_{X\times\mathbb{R}}$? This is a question that will be more completely answered in the sequel. For now, we present one helpful geometric property of $S_{X\times\mathbb{R}}$: nondegenerate open and closed balls are complementary and hence the Chebyshev and uniquely remotal concepts coincide.

PROPOSITION 3.1. *Suppose that X is an inner product space and $C \subseteq S_{X\times\mathbb{R}}$. Then, for all $(x, t) \in S_{X\times\mathbb{R}}$,*

$$r_C^2(x, t) + d_C^2(-x, -t) = 4.$$

Therefore, C is sphere-Chebyshev if and only if C is uniquely remotal.

PROOF. Note that, by simple circle geometry, given $(x, t) \in S_{X\times\mathbb{R}}$ and $0 \le r \le 2$,

$$B[(x, t); r] = S_{X\times\mathbb{R}} \setminus B\Big((-x, -t);\ \sqrt{4 - r^2}\Big).$$

In particular,

$$S[(x, t); r] = S\Big[(-x, -t);\ \sqrt{4 - r^2}\Big].$$

Suppose that $C \subseteq S_{X\times\mathbb{R}}$ and fix $(x, t) \in S_{X\times\mathbb{R}}$. Note that

$$C \subseteq B[(x, t); r_C(x, t)] = S_{X\times\mathbb{R}} \setminus B\Big((-x, -t);\ \sqrt{4 - r_C(x, t)^2}\Big)$$

and hence $d_C(-x, -t) \ge \sqrt{4 - r_C(x, t)^2}$. On the other hand, if $r < r_C(x, t)$, then

$$C \nsubseteq B[(x, t); r] = S_{X\times\mathbb{R}} \setminus B\Big((-x, -t);\ \sqrt{4 - r^2}\Big),$$

which implies that $d_C(-x, -t) \le \sqrt{4 - r^2}$. Taking the limit as $r \uparrow r_C(x, t)$,

$$r_C^2(x, t) + d_C^2(-x, -t) = 4,$$

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as required. This yields the following equivalence:

$$
\begin{aligned}
& C \subseteq S_{X\times\mathbb{R}} \text{ is sphere-Chebyshev,} \\
\iff\quad & \forall (x,t) \in S_{X\times\mathbb{R}}, S\,[(x,t); d_C(x,t)] \cap C \text{ is a singleton} \\
\iff\quad & \forall (x,t) \in S_{X\times\mathbb{R}}, S\Big[(x,t); \sqrt{4 - r_C^2(-x,-t)}\Big] \cap C \text{ is a singleton} \\
\iff\quad & \forall (x,t) \in S_{X\times\mathbb{R}}, S\,[(-x,-t); r_C(-x,-t)] \cap C \text{ is a singleton} \\
\iff\quad & \forall (x,t) \in S_{X\times\mathbb{R}}, S\,[(x,t); r_C(x,t)] \cap C \text{ is a singleton} \\
\iff\quad & C \text{ is uniquely remotal in } S_{X\times\mathbb{R}}. \qquad \square
\end{aligned}
$$

While it is more natural to define sphere-Chebyshev sets in $S_{X\times\mathbb{R}}$, we primarily consider their stereographic projections in $X \cup \{\infty\}$, to recover the rich linear structure of X.

The following is our main result: a categorisation theorem of sphere-Chebyshev subsets of $X \cup \{\infty\}$.

Theorem 3.2 (Sphere-Chebyshev sets in $X \cup \{\infty\}$). *Suppose that X is a real inner product space. A subset $C \subseteq X \cup \{\infty\}$ is sphere-Chebyshev if and only if $C \setminus \{\infty\}$ is Chebyshev in X and one of the following is true:*

(1) *$\infty \notin C$, C is uniquely remotal and, for every $f \in X^* \setminus \{0\}$, f attains its supremum uniquely on C; or*

(2) *$\infty \in C$ and no functional $f \in X^* \setminus \{0\}$ achieves a maximum on $C \setminus \{\infty\}$.*

Proof. We first outline the proof with details to follow.

Outline. The proof of Theorem 3.2 is essentially an application of Proposition 2.1. A Chebyshev subset $C \subseteq S_{X\times\mathbb{R}}$ is characterised by how the set intersects with boundary spheres of maximal balls B whose interiors miss the set. Depending on where $(0,1)$ lies relative to B, $\sigma^{-1}(B)$ will be a ball, a complement of a ball or a half-space.

When $\sigma^{-1}(B)$ is a ball, this corresponds to a projection problem in X and hence $\sigma^{-1}(C)$ must be Chebyshev; see Figure 1.

When $\sigma^{-1}(B)$ is a complement of a ball, that is, when $(0,1) \in \operatorname{int} B$, then this corresponds to a furthest point problem around the centre of $X \setminus \sigma^{-1}(B)$; see Figure 2.

When $\sigma^{-1}(B)$ is a half-space, then $(0,1) \in \operatorname{bdry} B$, so we must consider whether or not $(0,1) \in C$. If so, then no other point on the corresponding boundary hyperplane can intersect with the set. If not, then this boundary hyperplane must be uniquely intersected; see Figure 3.

Throughout the proof, we will use Propositions 2.1, 3.1 and 2.3 multiple times without explicit reference.

Details. Suppose that $C \subseteq X \cup \{\infty\}$ is sphere-Chebyshev. We begin by acknowledging that the result holds true whenever $C = \emptyset$, C is a singleton or $C = X \cup \{\infty\}$. We therefore assume without loss of generality that this is not the case.

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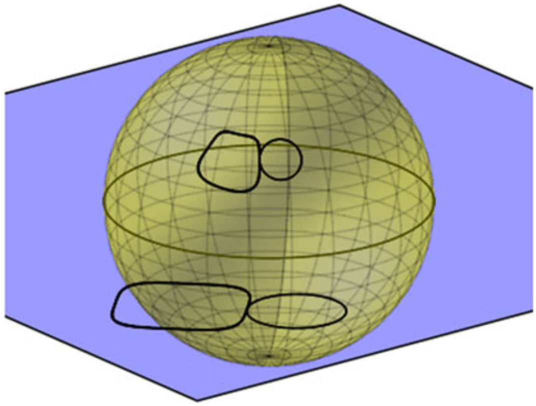

FIGURE 1. A projection problem.

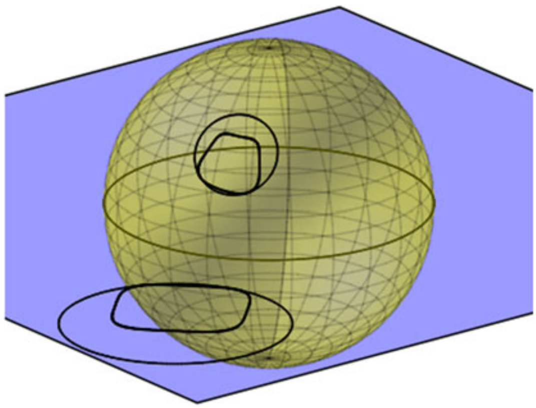

FIGURE 2. A furthest point problem.

Fix $x \in X \setminus C$, let $B = B(x; d_{C\setminus\{\infty\}}(x))$ and note that B is nondegenerate. Then $\sigma^{-1}(B)$ is a nondegenerate open ball $B((y, s); r)$, which does not intersect $\sigma^{-1}(C \setminus \{\infty\})$. Since $(0, 1) \notin B[(y, s); r]$, it follows that $\sigma^{-1}(B) \cap \sigma^{-1}(C) = \emptyset$. Therefore, $d_{\sigma^{-1}(C)}(y, s) \geq r$.

Suppose that this inequality were strict. Choose $r' \in (r, d_{\sigma^{-1}(C)}(y, s))$ small enough so that $(0, 1) \notin B[(y, s); r']$. Let $B' = \sigma(B((y, s); r'))$. Note that $B' \cap C = \emptyset$. Because the spheres $S[(y, s); r]$ and $S[(y, s); r']$ are disjoint and nondegenerate, the spheres of B and B' have a strictly positive infimal distance. Hence, for $\varepsilon > 0$ smaller than this distance, $B(x; d_{C\setminus\{\infty\}}(x) + \varepsilon)$ is contained in B' and hence does not intersect C. This contradicts the definition of $d_{C\setminus\{\infty\}}(x)$ and hence $r = d_{\sigma^{-1}(C)}(y, s)$.

Since $\sigma^{-1}(C)$ is sphere-Chebyshev, we have that $S[(y, s); r] \cap \sigma^{-1}(C)$ is a singleton and hence so is $S[x; d_{C\setminus\{\infty\}}(x)] \cap (C \setminus \{\infty\})$. Therefore, $C \setminus \{\infty\}$ is Chebyshev.

Suppose now that $\infty \notin C$. Since $\sigma^{-1}(C)$ is sphere-Chebyshev and hence closed, $\sigma^{-1}(C)$ is bounded away from $(0, 1)$ by some nondegenerate ball centred at $(0, 1)$. This

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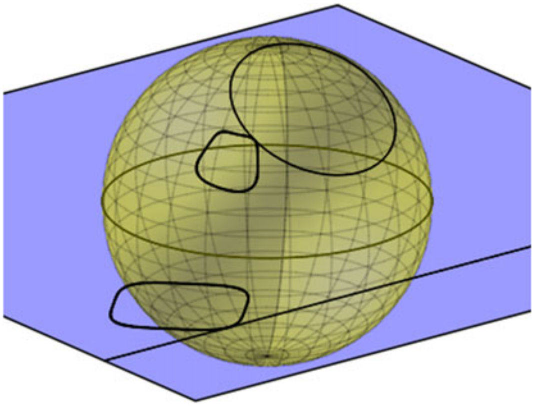

FIGURE 3. A functional maximisation problem.

ball maps to the complement of a ball. Therefore, C is bounded by this complementary ball.

Suppose that $x \in X$ and $B = B[x; r_C(x)] \supseteq C$. Then $\sigma^{-1}(B)$ is a nondegenerate closed ball $B[(y, s); r] \supseteq \sigma^{-1}(C)$ with $r > 0$. Therefore, $r_{\sigma^{-1}(C)}(y, s) \leq r$.

If this inequality were strict, then $S[(y, s); r]$ and $S[(y, s); r_{\sigma^{-1}(C)}(y, s)]$ would be disjoint nondegenerate spheres. Hence, $B' := \sigma(B[(y, s); r_{\sigma^{-1}(C)}(y, s)])$ is a nondegenerate ball containing C, and the spheres of B and B' have a strictly positive infimal distance. For $\varepsilon > 0$ smaller than this infimum, the ball $B[x; r_C(x) - \varepsilon]$ would still contain B' and hence C, contradicting the definition of $r_C(x)$. Thus, $r = r_{\sigma^{-1}(C)}(y, s)$.

Since $\sigma^{-1}(C)$ is uniquely remotal, the sphere $S[(y, s); r]$ intersects with $\sigma^{-1}(C)$ uniquely. Hence, $S[x; r_C(x)] \cap C$ is also a singleton and hence C is uniquely remotal.

Suppose that $f \in X^* \setminus \{0\}$. Let $\alpha = \sup f(C)$ and $H = f^{-1}(\alpha, \infty)$. Then $\sigma^{-1}(H) = B((y, s); r)$ for some $(y, s) \in S_{X\times\mathbb{R}}$ and $r > 0$. Note that $\sigma^{-1}(C) \cap B((y, s); r) = \emptyset$ and hence $r \leq d_{\sigma^{-1}(C)}(y, s)$.

Suppose that this inequality were strict. Let $B' := S_{X\times\mathbb{R}} \setminus B((y, s), d_{\sigma^{-1}(C)}(y, s))$. Then B' is a closed nondegenerate ball, containing $\sigma^{-1}(C)$, but not $(0, 1)$. Thus, $\sigma(B')$ is a nondegenerate ball containing C, but contained in H. The spheres of B' and $\sigma^{-1}(H)$ are nondegenerate and disjoint and hence the sphere of $\sigma(B')$ is of positive distance from the hyperplane $f^{-1}\{\alpha\}$. Consequently, $\sup f(C) \leq \max f(\sigma(B')) < \alpha$, against assumption. So, $r = d_{\sigma^{-1}(C)}(y, s)$.

Similarly to previous arguments, this implies that f achieves a unique maximum on C.

On the other hand, suppose that $\infty \in C$ and, for the sake of contradiction, that $f \in X^* \setminus \{0\}$ attains a maximum on $C \setminus \{\infty\}$ at a point $c \in C$. Let $H = \{x \in X : f(x) \leq f(c)\} \supseteq C$. Then $\sigma^{-1}(H \cup \{\infty\})$ is a ball $B[(y, s); r]$ that contains $\sigma^{-1}(C)$ and whose boundary contains $(0, 1)$. But $(0, 1) = \sigma^{-1}(\infty) \in \sigma^{-1}(C)$ and hence $r = r_{\sigma^{-1}(C)}(y, s)$. Further, it is straightforward to show that $\sigma^{-1}(c)$ also lies on this boundary, contradicting $\sigma^{-1}(C)$ being uniquely remotal.

Conversely, suppose that $C \subseteq X \cup \{\infty\}$ satisfies the conditions of the theorem. If C is a singleton or equal to $X \cup \{\infty\}$, then C is clearly sphere-Chebyshev, so we may assume without loss of generality that this is not the case. We first show that $\sigma^{-1}(C)$ is closed.

Suppose that $(x_n, t_n) \in \sigma^{-1}(C)$ converges to (x, t). If $(x, t) \neq (0, 1)$, then, by taking subsequences as necessary, $(x_n, t_n) \neq (0, 1)$ for all n. Then $\sigma(x_n, t_n) \in C \setminus \{\infty\}$ and, by the continuity of σ when restricted to $S_{X\times\mathbb{R}} \setminus \{(0, 1)\}$, it follows that $\sigma(x_n, t_n)$ converges to $\sigma(x, t)$. Since $C \setminus \{\infty\}$ is Chebyshev, it is closed and hence $(x, t) \in \sigma^{-1}(C)$.

Otherwise, $(x_n, t_n) \to (0, 1)$. Then $\sigma(x_n, t_n)$ is an unbounded sequence. By the uniform boundedness principle, there exists a functional $f \in X^* \setminus \{0\}$ such that $f(\sigma(x_n, t_n)) \to \infty$. Hence, f does not achieve a maximum on $C \setminus \{\infty\}$, so $\infty \in C$, that is, $(0, 1) \in \sigma^{-1}(C)$. Therefore, $\sigma^{-1}(C)$ is closed.

To show that $\sigma^{-1}(C)$ is sphere-Chebyshev, fix $(z, t) \in S_{X\times\mathbb{R}} \setminus \sigma^{-1}(C)$ and let $B = B((z, t), d_{\sigma^{-1}(C)}(z, t))$. Note that $\sigma^{-1}(C) \cap B = \emptyset$ and B is nondegenerate. We consider three cases: $(0, 1)$ is in the exterior, the interior or the boundary of B.

Suppose that $(0, 1)$ is in the exterior of B. Then $\sigma(B)$ is an open ball $B(x; r)$. Since $B(x; r) \cap C = \emptyset$, we have $r \leq d_{C\setminus\{\infty\}}(x)$.

Suppose that this inequality is strict. Then $S[x; d_{C\setminus\{\infty\}}(x)]$ and $S[x; r]$ are disjoint and nondegenerate, so $B' := \sigma^{-1}(B(x; d_{C\setminus\{\infty\}}(x)))$ is open with a sphere that has strictly positive infimal distance from the sphere of B. Therefore, for $\varepsilon > 0$ smaller than this distance, we have $B(x; d_C(x) + \varepsilon) \subseteq B'$ and hence it does not intersect C. This is a contradiction, so $r = d_{C\setminus\{\infty\}}(x)$.

Since $C \setminus \{\infty\}$ is Chebyshev, it follows that $B[x; r] \cap C$ is a singleton and hence so is $B[(z, t); d_{\sigma^{-1}(C)}(z, t)] \cap \sigma^{-1}(C)$. That is, $P_{\sigma^{-1}(C)}(z, t)$ is a singleton in this case.

Suppose instead that $(0, 1)$ is in the interior of B. Then $\sigma(B) = X \cup \{\infty\} \setminus B[x; r]$ for some $x \in X$ and $r > 0$. Since $B[x; r]$ contains C, C is bounded and $r_C(x) \leq r$.

Suppose that this inequality is strict. Then $S[x; r_C(x)]$ and $S[x; r]$ are disjoint and nondegenerate and hence $B' := \sigma^{-1}(X \cup \{\infty\} \setminus B[x; r_C(x)])$ is an open ball containing B such that the spheres of B and B' have positive infimal distance. Taking $\varepsilon > 0$ smaller than this distance, we have that $B((z, t); d_{\sigma^{-1}(C)}(z, t) + \varepsilon)$ is contained in B' and hence disjoint from $\sigma^{-1}(C)$. By contradiction, $r = r_C(x)$.

Note that since $C \subseteq B[x; r]$, we have $\infty \notin C$. Therefore, C is uniquely remotal, so $S[x; r] \cap C$ is a singleton. Therefore, $B[(z, t); d_{\sigma^{-1}(C)}(z, t)] \cap \sigma^{-1}(C)$ is a singleton and again $P_{\sigma^{-1}(C)}(z, t)$ is a singleton in this case.

Finally, suppose that $(0, 1)$ is on the boundary of B. Then there exist some $f \in X^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that $\sigma(B) = f^{-1}(\alpha, \infty)$. Note that $\alpha \geq \beta := \max f(C)$.

Suppose first that $\alpha > \beta$. Then $f^{-1}(\beta, \infty)$ does not intersect C and hence $B' := \sigma^{-1}(f^{-1}(\beta, \infty))$ contains B, but does not intersect $\sigma^{-1}(C)$. Let S_1 be the boundary of B and S_2 be the boundary of B'. Note that S_1 and S_2 are nondegenerate and $S_1 \cap S_2 = \{(0, 1)\}$.

If $(0,1) \notin \sigma^{-1}(C)$, then there exists some $\varepsilon > 0$ such that $B[(0,1);\varepsilon] \subseteq S_{X\times\mathbb{R}} \setminus \sigma^{-1}(C)$. Using Proposition 2.3, we can therefore extend the radius of B slightly while still missing $\sigma^{-1}(C)$. This contradicts the definition of B and hence $(0,1) \in P_{\sigma^{-1}(C)}(z,t) \subseteq \sigma^{-1}(C)$.

If there were another point in $P_{\sigma^{-1}(C)}(z,t)$, then this point would stereographically project to a point on $f^{-1}\{\alpha\} \cap C$, which would contradict $\alpha > \beta$.

Therefore, we may assume that $\alpha = \beta$. We have two possibilities: $\infty \in C$ or $\infty \notin C$. Supposing the former, then $(0,1) \in P_{\sigma^{-1}(C)}(z,t)$. No other points can be in the projection, as this would cause f to achieve its maximum on $C \setminus \{\infty\}$, against assumption.

If $\infty \notin C$, then f achieves its maximum uniquely on C. When mapped under σ^{-1}, this point forms $P_{\sigma^{-1}(C)}(z,t)$. □

Considering condition (b) in Theorem 3.2, we obtain the following corollary.

Corollary 3.3. *Suppose that X is a real inner product space, $C \subseteq X$ is Chebyshev and $X \setminus C$ is bounded (in particular, if C is a Klee cavern). Then $C \cup \{\infty\}$ is sphere-Chebyshev in $X \cup \{\infty\}$.*

Combining with Theorem 1.1, we obtain a condition equivalent to the Chebyshev conjecture.

Corollary 3.4. *Suppose that X is a real inner product space. Then every Chebyshev subset of X is convex if and only if the only sphere-Chebyshev subsets of $X \cup \{\infty\}$ are the singleton subsets and $X \cup \{\infty\}$ itself.*

Proof. ($\Longleftarrow$) Suppose that there exists a nonconvex Chebyshev subset of X. By Theorem 1.1, a Klee cavern C exists. By Corollary 3.3, $C \cup \{\infty\}$ is sphere-Chebyshev, while being proper and nontrivial.

($\Longrightarrow$) Suppose that every Chebyshev subset of X is convex and consider a sphere-Chebyshev subset $C \subseteq S_{X\times\mathbb{R}}$. Either $C = S_{X\times\mathbb{R}}$ (in which case we are done) or there exists some $(x,r) \in S_{X\times\mathbb{R}} \setminus C$. Choose a surjective isometry $\phi : S_{X\times\mathbb{R}} \to S_{X\times\mathbb{R}}$ that maps (x,r) to $(0,1)$. Then $\phi(C)$ is sphere-Chebyshev and $\sigma(\phi(C))$ is uniquely remotal. By Theorem 1.1, $\sigma(\phi(C))$ is a singleton and hence so is C. □

As a consequence of Corollary 3.4, we lose no generality in studying sphere-Chebyshev sets.

It is worth noting that, while the question of the existence of nonconvex Chebyshev sets in a real Hilbert space is yet to be answered, there is a positive answer known in certain incomplete inner product spaces. Johnson published a construction of a nonconvex Chebyshev subset of an incomplete real inner product space [6]. His construction contained significant errors, so a more geometric example in the same vein was constructed by Balaganskiĭand Vlasov [2] (for a more readable version of this construction, see Section 4 of [5]).

Therefore, by Corollary 3.4, there are certain incomplete real inner product spaces that contain proper, nontrivial sphere-Chebyshev sets.

https://doi.org/10.1017/S1446788719000508 Published online by Cambridge University Press

In the sequel, we show that, due to the additional structure imposed on sphere-Chebyshev sets, there is a larger group of maps that preserve the sphere-Chebyshev property. In particular, inversions in any sphere preserve the sphere-Chebyshev property. (Contrast this to Ficken's result, which uses inversion to map Chebyshev sets to uniquely remotal sets, but with restrictions on the Chebyshev set and the sphere used for inversion.)

This fact further allows us to transform freely between nearest point problems, furthest point problems and functional maximisation problems. The latter is a well-studied mainstay of convex analysis and Banach space geometry, and is the primary motivation for considering sphere-Chebyshev subsets of $X \cup \{\infty\}$.

In particular, we the use the dentability of the Hilbert space ball to prove some local structure results about sphere-Chebyshev sets. For example, we show that such sets cannot contain isolated points and the boundary cannot resemble a sphere or hyperplane on any neighbourhood.

Acknowledgements

The author wishes to thank Jon Borwein for his support, supervision and infectious enthusiasm about this beautiful problem in the early stages of this project. The author further wishes to thank Warren Moors for his continuing support, and his helping the author land on his feet after Jon's tragic death.

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THEO BENDIT, University of Newcastle,
Australia
e-mail: theo.bendit@uon.edu.au