LMS J. Comput. Math. 17 (1) (2014) 273–281
doi:10.1112/S1461157014000011

On the Bessel function $J_\nu(x)$ in the transition region

Ilia Krasikov

Abstract

We give an approximation for the value of the Bessel function $J_\nu(x)$ in the transition region with an explicit sharp error term.

1. *Introduction*

In this paper we will obtain an approximation with an explicit asymptotically sharp error term for the Bessel function $J_\nu(x)$ in the transition region, starting from appropriate upper bounds on $|J_\nu(x)|$ and $|J'_\nu(x)|$. All basic formulas and asymptotic expressions for special functions we use without references can be found in [4]. To write down error terms in a compact form we will use $\theta, \theta_1, \theta_2, \ldots,$ to denote quantities with the absolute value not exceeding one.

The standard asymptotic expansion for the Bessel function $J_\nu(x)$ in the transition region is

$$J_\nu(\nu + \nu^{1/3} z) \sim \frac{2^{1/3}}{\nu^{1/3}} \mathrm{Ai}(-2^{1/3} z) \sum_{k=0}^{\infty} \frac{V_k(z)}{\nu^{2k/3}} + \frac{2^{2/3}}{\nu} \mathrm{Ai}'(-2^{1/3} z) \sum_{k=0}^{\infty} \frac{U_k(z)}{\nu^{2k/3}},$$

where it is assumed that $\nu \to \infty,\ z \in \mathbb{C}$. Here $\mathrm{Ai}(x)$ is the Airy function and $V_k(z),\ U_k(z)$ are some polynomials in z of degree growing with k (see [4, 10.19.8]). One of the shortcomings of this formula is that it makes little sense for z depending on ν. It also gives no insight into how large the transition region is. Our main result is the following theorem.

Theorem 1. *Let $\nu \geqslant 1/2$, then for $0 \leqslant z \leqslant \nu^{4/15}$,*

$$J_\nu(\nu + \nu^{1/3} z) = \frac{2^{1/3}}{\nu^{1/3}} \mathrm{Ai}(-2^{1/3} z) + \theta \frac{4z^{9/4} + 21}{7\nu}. \tag{1}$$

The value of z here is restricted to $\nu^{4/15}$ since then the error and the main term become of the same order. For sufficiently large ν the error term in (1) can be about twice as large as the actual value. More precisely, we prove the following.

Theorem 2. *For $z \geqslant 0,\ z = o(\nu^{4/15})$,*

$$\limsup_{\nu \to \infty} \limsup_{z \to \infty} \left| J_\nu(\nu + \nu^{1/3} z) - \frac{2^{1/3}}{\nu^{1/3}} \mathrm{Ai}(-2^{1/3} z) \right| z^{-9/4} \nu = \frac{3}{5 \cdot 2^{1/3} \sqrt{\pi}}. \tag{2}$$

To prove Theorems 1 and 2 we first shall establish the following results which may be of independent interest.

Theorem 3. *Let $\nu \geqslant 1/2$ and $x \geqslant \nu + \nu^{1/3}(\sqrt{7} - 1)/2^{2/3}$, then*

$$\frac{x(4(x^2 - \nu^2)^3 - 3x^4 - 10x^2\nu^2 + \nu^4)^{1/4}}{x^2 - \nu^2} |J'_\nu(x)| < \frac{2}{\sqrt{\pi}}. \tag{3}$$

Received 29 July 2013.

2010 Mathematics Subject Classification 33C10, 41A60 (primary).

https://doi.org/10.1112/S1461157014000011 Published online by Cambridge University Press

THEOREM 4. *For $x_1, x_2 \geqslant 0$ and $|\nu| \leqslant 1/2$,*

$$\sqrt{x_1 x_2}|J_{-\nu}(x_1)J_\nu(x_2) - J_{-\nu}(x_2)J_\nu(x_1)| \leqslant \frac{2}{\pi}\sin\pi\nu. \tag{4}$$

THEOREM 5. *Let $\gamma = 2^{-1/3}a_1 = 1.855757\ldots$, where a_1 is the least-positive zero of $\mathrm{Ai}(-x)$, then for $\nu \geqslant 1/2$,*

$$J_\nu(\nu + \gamma\nu^{1/3}) < \frac{7}{6\nu}. \tag{5}$$

In the last theorem the point $\nu + \gamma\nu^{1/3}$ is just an approximation to the least-positive zero $j_{\nu 1}$ of J_ν. More precisely, it is known that the sth-positive zero $j_{\nu s}$ is given by [**5**]

$$j_{\nu s} = \nu + 2^{-1/3}a_s\nu^{1/3} + \theta^2\frac{3\cdot 2^{-2/3}a_s^2}{10}\nu^{-1/3}, \quad \nu > 0, \tag{6}$$

where a_s is the sth-positive zero of the Airy function $\mathrm{Ai}(-x)$. It is also worth comparing (5) with the following result [**1**]

$$J_\nu(\nu) = \frac{2^{1/3}}{3^{2/3}\Gamma(2/3)(\nu + \theta^2\alpha)^{1/3}}, \quad \nu > 0, \tag{7}$$

where $\alpha = 0.09434980\ldots$.

The idea behind the proof of Theorem 1 is rather simple and can be applied to other special functions satisfying a second-order ordinary differential equation (ODE). Suppose that $f(x)$ is a solution of

$$f'' + b^2(x)f = 0,$$

then an asymptotics of f with an explicit error term in the transition region, that is around a zero of $b(x)$, can be obtained as follows. Let

$$b(\alpha) = 0, \qquad d = \frac{d}{dx}b^2(x)\bigg|_{x=\alpha} \neq 0,$$

then in a vicinity of α we can write

$$b^2(\alpha + d^{-1/3}t) = d^{2/3}t + \delta(t),$$

where $\delta(t)$ is small. The function $y(t) = f(\alpha + d^{-1/3}t)$ satisfies the inhomogeneous Airy-type ODE

$$y''(t) + ty(t) = -\delta(t)d^{-2/3}y(t).$$

Suppose now that we know an *a priori* upper bound on $|y(x)|$. Then solving it as an inhomogeneous equation one obtains an explicit error term (see Lemma 6 below). However, one still has to use some initial or boundary conditions to fix the integration constants. Even if they are known or can be derived, as in the case of J_ν, this step may require some quite involved calculations.

2. *Upper bounds*

In this section, using so-called Sonin's function $S(x)$, we prove Theorems 3 and 4, thus establishing the upper bounds we need in the following.

Let $y(x)$ be a solution of

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0.$$

https://doi.org/10.1112/S1461157014000011 Published online by Cambridge University Press

Then $S(x) = y^2 + y'^2/b$ is just an envelope of y^2 coinciding with it in all maxima. The sign of $S' = -(2ab + b')y'^2/b^2$, depends only on a and b, what in many cases enables one to find the global maximum of y^2.

In what follows it will be convenient to use the following parameters:

$$\mu = \left|\nu^2 - \frac{1}{4}\right|, \qquad \omega_\nu = \frac{(2\nu+1)\pi}{4}.$$

Let $f(x)$ be $J_\nu(x)$ or $Y_\nu(x)$, the Bessel functions of the first or the second kind, respectively. Then $f(x)$ is a solution of

$$x^2 f'' + x f' + (x^2 - \nu^2) f = 0. \tag{8}$$

We need the following classical bound [6, Theorem 7.31.2]:

$$|f(x)| \leqslant \sqrt{\frac{2}{\pi x}}, \qquad |\nu| \leqslant 1/2,\ x > 0. \tag{9}$$

Note that in [6, Theorem 7.31.2] the result is stated for J_ν only, however the proof is still valid for Y_ν as well (see [3]). For $\nu \geqslant 1/2$ we have the following result [3],

$$|x^2 - \mu|^{1/4} |J_\nu(x)| < \sqrt{2/\pi}, \quad x > 0. \tag{10}$$

The constant $\sqrt{2/\pi}$ in (9) and (10) is the best possible.

We start with proving Theorem 3. It requires some rather involved calculations which seems difficult to perform without a symbolic package, we used Mathematica.

Proof of Theorem 3. Set $\psi(x) = 4(x^2-\nu^2)^3 - 3x^4 - 10x^2\nu^2 + \nu^4$, under the assumptions of the theorem we have to prove that $|z(x)| < 2/\sqrt{\pi}$, where

$$z(x) = \frac{x\psi^{1/4}(x)}{x^2-\nu^2} J_\nu'(x).$$

First note that $\psi(x) > 0$ for $\nu \geqslant 1/2$ and $x \geqslant \nu + \nu^{1/3}(\sqrt{7}-1)/2^{2/3}$. Indeed, one can check that the substitutions

$$x = y + \nu + \frac{\sqrt{7}-1}{2^{2/3}}\nu^{1/3}, \qquad \nu = (2^{-1/3} + n)^3, \tag{11}$$

transform ψ into a polynomial in n and y with non-negative coefficients. The function z satisfies the differential equation

$$z'' - \frac{4u^3\nu^2 + 3u^3 + 32u^2\nu^2 + 52u\nu^4 + 24\nu^6}{ux(4u^3 - 3u^2 - 16u\nu^2 - 12\nu^4)} z' + \frac{Q(x)}{x^2(x^2-\nu^2)^2\psi^2(x)} z = 0,$$

where $u = x^2 - \nu^2$, and

$$\begin{aligned} Q(x) = {} & u^6(16u^3 - 36u^2 + 45u - 9) - 4u^5\nu^2(42u^2 - 155u + 45) \\ & - u^4\nu^4(124u^2 - 2064u + 963) + 24u^3\nu^6(102u - 107) \\ & + 48u^2\nu^8(20u - 73) - 576v^{10}(4u - v^2). \end{aligned}$$

Consider the corresponding Sonin function

$$S(x) = z^2(x) + \frac{x^2(x^2-\nu^2)^2\psi^2(x)}{Q(x)} z'^2(x),$$

https://doi.org/10.1112/S1461157014000011 Published online by Cambridge University Press

and its derivative
$$S'(x) = \frac{6x^3(x^2-\nu^2)^4\psi(x)P(x)}{Q^2(x)}\, z'^2(x),$$
where
$$\begin{aligned}P(x) = {} & 48u^7 + 9u^6 + 4u^4\nu^2(16u^3+338u^2+9) + 2u^3\nu^4(40u^3+3136u^2+1348u-105)\\ & + 8u^2\nu^6(1204u^2+1432u-129) + 16u\nu^8(294u^2+1172u-87)\\ & + 608\nu^{10}(24u-1) + 4608\nu^{12}.\end{aligned}$$
Substitution (11) transforms P and Q into polynomials in y and n with non-negative coefficients. Hence, $S(x)$ is increasing and $z^2(x) \leqslant S(x) < S(\infty)$. Finally, using the asymptotics
$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\cos(x-\omega_\nu), \qquad J'_\nu(x) \sim -\sqrt{\frac{2}{\pi x}}\sin(x-\omega_\nu),$$
we obtain $z^2(x) < \lim_{x\to\infty} S(x) = 4/\pi$, and the result follows. □

Proof of Theorem 4. The function
$$F = F(x_1,x_2) = \sqrt{x_1x_2}(J_{-\nu}(x_1)J_\nu(x_2) - J_{-\nu}(x_2)J_\nu(x_1))$$
satisfies the differential equations
$$\frac{\partial^2 F}{\partial x_i^2} + b(x_i)F = 0, \quad i = 1,2, \tag{12}$$
where $b(x) = 1 + \mu/x^2 > 0$. We consider the majorant of F^2 given by Sonin's function
$$F^2(x_1,x_2) \leqslant S(x_1,x_2) = F^2 + \frac{(\partial F/\partial x_2)^2}{b(x_2)}.$$
By (12) we have
$$\frac{\partial S}{\partial x_2} = -\left(\frac{(\partial/\partial x_2)\, b(x_2)}{b^2(x_2)}\right)\left(\frac{\partial F}{\partial x_2}\right)^2.$$
Since
$$\frac{\partial}{\partial x_2} b(x_2) = -\mu/x_2^3 < 0$$
for $|\nu| < 1/2$, the Sonin function increases in x_2. By using the asymptotics
$$\begin{aligned}J_\nu(x) &= \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{(2\nu+1)\pi}{4}\right) + o(x^{-1/2}),\\ J'_\nu(x) &= -\sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{(2\nu+1)\pi}{4}\right) + o(x^{-1/2}),\end{aligned}$$
taking the limit and applying (9) we obtain
$$\begin{aligned}F^2(x_1,x_2) \leqslant S(x_1,x_2) \leqslant \lim_{x_2\to\infty} S(x_1,x_2) &= \frac{2x_1}{\pi}(J_\nu^2(x_1) - 2J_\nu(x_1)J_{-\nu}(x_1)\cos\pi\nu + J_{-\nu}^2(x_1))\\ &= \frac{2x_1\sin^2\pi\nu}{\pi}(J_\nu^2(x_1) + Y_\nu^2(x_1)) \leqslant \frac{4\sin^2\pi\nu}{\pi^2}.\end{aligned}$$
This completes the proof. □

3. *Approximation in the transition region*

Our estimates in the transition region are based on the following observation.

https://doi.org/10.1112/S1461157014000011 Published online by Cambridge University Press

LEMMA 6. *Let*

$$y''(x) + q^2 xy(x) = u(x), \quad q \in \mathbb{R}, \tag{13}$$

then for $x \geqslant 0$, *provided that the integral exists,*

$$y(x) = \sqrt{x}(c_1 J_{-1/3}(\xi_x) + c_2 J_{1/3}(\xi_x)) + \theta q^{-1} x^{-1/4} \int_0^x |u(t)| t^{-1/4}\, dt, \tag{14}$$

where $\xi_x = 2qx^{3/2}/3$.

Proof. Let y_1 and y_2 be two linearly independent solutions of the corresponding homogeneous equation $y''(x) + q^2 xy(x) = 0$, and let

$$U(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}.$$

Then the general solution of (13) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \int^x U(x,t)u(t)\, dt,$$

and choosing $y_1(x) = \sqrt{x}\, J_{-1/3}(\xi_x)$, $y_2(x) = \sqrt{x} J_{1/3}(\xi_x)$, we find

$$y_1(t)y_2'(t) - y_1'(t)y_2(t) = \frac{3\sqrt{3}}{2\pi}.$$

Applying (4) we obtain

$$U(x,t) = 2\pi\sqrt{\frac{tx}{27}}(J_{-1/3}(\xi_t)J_{1/3}(\xi_x) - J_{-1/3}(\xi_x)J_{1/3}(\xi_t)) = \theta q^{-1} t^{-1/4} x^{-1/4}.$$

Hence,

$$\left|\int_0^x U(x,t)u(t)\, dt\right| \leqslant q^{-1} x^{-1/4} \int_0^x |u(t)| t^{-1/4}\, dt,$$

and the result follows. □

To prove Theorem 1 we have to find the constants of integration c_1, c_2 in (14). To that end we shall prove first Theorem 5. Then we will know the value of $J_\nu(x)$ at two points: $x = \nu + \gamma\nu^{1/3}$ and $x = \nu$, where the last is given by (7).

Proof of Theorem 5. By (6) we have

$$j_{\nu 1} = \nu + \gamma\nu^{1/3} + \theta^2 \frac{3\gamma^2}{10}\nu^{-1/3}, \tag{15}$$

yielding the following Tailor expansion

$$J_\nu(\nu + \gamma\nu^{1/3}) = -\theta^2 \frac{3\gamma^2}{10}\nu^{-1/3} J_\nu'\left(\nu + \gamma\nu^{1/3} + \theta_1^2 \frac{3\gamma^2}{10}\nu^{-1/3}\right).$$

Setting

$$\delta = \gamma + \theta_1^2 \frac{3\gamma^2}{10}\nu^{-2/3}, \qquad \epsilon = \delta\nu^{-2/3},$$

by (3) and $\gamma > (\sqrt{7} - 1)/2^{2/3}$ we can write

$$|J_\nu'(\nu + \delta\nu^{1/3})| < \frac{2\epsilon}{\sqrt{\pi}}\phi(\epsilon, \delta),$$

https://doi.org/10.1112/S1461157014000011 Published online by Cambridge University Press

where

$$\phi(\epsilon,\delta) = \frac{2+\epsilon}{1+\epsilon}(4(8\delta^3-3)+16(3\delta^3-2)\epsilon+4(6\delta^3-7)\epsilon^2+4(\delta^3-3)\epsilon^3-3\epsilon^4)^{-1/4}.$$

Note that $\phi(\epsilon,\delta) < \phi(\epsilon,\gamma)$ since $\delta > \gamma$ and

$$\frac{\partial}{\partial\delta}\,\phi(\epsilon,\delta) = -\frac{3\delta^2(1+\epsilon)^4}{2+\epsilon}\phi^5(\epsilon,\delta) < 0.$$

Moreover,

$$\epsilon = \gamma\nu^{-2/3} + \frac{3\gamma^2}{10}\nu^{-4/3} < 6,$$

whereas, as one can check, $(\partial/\partial\epsilon)\phi(\epsilon,\gamma) < 0$ in the interval $0 \leqslant \epsilon \leqslant 7$. Thus, $\phi(\epsilon,\delta) < \phi(0,\gamma)$, yielding

$$|J'_\nu(\nu+\delta\nu^{1/3}| < \frac{2\epsilon}{\sqrt{\pi}}\phi(0,\gamma) = \frac{2^{3/2}\gamma}{\sqrt{\pi}(8\gamma-3)^{1/4}}\nu^{-2/3},$$

and the result follows. □

Now we are in a position to prove Theorem 1. We will need the following constants

$$a_{\max} = \max_{z\geqslant 0}|\mathrm{Ai}(-z)| = 0.53565\ldots < 15/28, \tag{16}$$

$$b_{\max} = \max_{z\geqslant 0}\sqrt{z}|J_{1/3}(\sqrt{2}\zeta)| = 0.768507\ldots < 10/13, \qquad \zeta = \frac{2}{3}z^{3/2}. \tag{17}$$

In both cases the maximum is attained at the first local extremum since the functions $\mathrm{Ai}(-2^{1/3}z)$ and $\sqrt{z}J_{1/3}(\sqrt{2}\zeta)$ are solutions of the same differential equation $f''(z)+2zf(z) = 0$, with the decreasing Sonin function $S = f^2 + f'^2/2z,\ S' = -f'^2/2z^2$.

Proof of Theorem 1. Consider the function

$$y(z) = \sqrt{\nu+\nu^{1/3}z}\,J_\nu(\nu+\nu^{1/3}z),$$

satisfying the differential equation

$$y''(z) + 2zy(z) = \frac{8z^3+12\nu^{2/3}z^2-1}{4(\nu^{2/3}+z)^2}y(z) := u(z). \tag{18}$$

For $z > 0$ by (10) we have

$$y(z) \leqslant \frac{2\nu^{1/6}\sqrt{\nu^{2/3}+z}}{\sqrt{\pi}(4\nu^{2/3}z^2+8\nu^{4/3}z+1)^{1/4}} \leqslant \frac{2^{1/4}\sqrt{\nu^{2/3}+z}}{\sqrt{\pi}\nu^{1/6}z^{1/4}},$$

and

$$|u(z)| \leqslant \frac{|8z^3+12\nu^{2/3}z^2-1|}{2^{7/4}\sqrt{\pi}\nu^{1/6}z^{1/4}(\nu^{2/3}+z)^{3/2}} \leqslant \frac{12\nu^{2/3}z^2+1}{2^{7/4}\sqrt{\pi}\nu^{7/6}z^{1/4}}.$$

Now Lemma 6 yields

$$\begin{aligned} y(z) &= \sqrt{z}(c_1J_{-1/3}(\sqrt{2}\zeta)+c_2J_{1/3}(\sqrt{2}\zeta)) + \theta R(z) \\ &= c_3\mathrm{Ai}(-2^{1/3}z) + c_4\sqrt{z}J_{1/3}(\sqrt{2}\zeta) + \theta R(z), \end{aligned} \tag{19}$$

$$|R(z)| \leqslant \frac{(12\nu^{2/3}z^2+5)z^{1/4}}{10\cdot 2^{1/4}\sqrt{\pi}\nu^{7/6}}. \tag{20}$$

https://doi.org/10.1112/S1461157014000011 Published online by Cambridge University Press

It is left to find the constants c_3, c_4. We have by (7)

$$y(0) = c_3\mathrm{Ai}(0) = \frac{c_3}{3^{2/3}\Gamma(2/3)} = \sqrt{\nu}J_\nu(\nu) = \frac{2^{1/3}\sqrt{\nu}}{3^{2/3}\Gamma(2/3)(\nu+\theta^2\alpha)^{1/3}},$$

hence,

$$c_3 = \frac{2^{1/3}\sqrt{\nu}}{(\nu+\theta^2\alpha)^{1/3}} = 2^{1/3}\nu^{1/6}\left(1-\theta^2\frac{\alpha}{3\nu}\right),$$

and

$$c_3\mathrm{Ai}(-2^{1/3}z) = (4\nu)^{1/6}\mathrm{Ai}(-2^{1/3}z) + \theta\frac{2^{1/3}\alpha}{3\nu^{5/6}}a_{\max} = (4\nu)^{1/6}\mathrm{Ai}(-2^{1/3}z) + \frac{\theta}{47\nu^{5/6}}.$$

Now $\mathrm{Ai}(-2^{1/3}\gamma) = 0$ and c_4 can be found from

$$y(\gamma) = c_4\sqrt{\gamma}\,J_{1/3}\left(\frac{(2\gamma)^{3/2}}{3}\right) + \theta_1 R(\gamma) = \sqrt{\nu+\gamma\nu^{1/3}}\,J_\nu(\nu+\gamma\nu^{1/3}).$$

Combining the above estimate on R with (5) and (17) one obtains

$$c_4\sqrt{z}\left|J_{1/3}\left(\frac{(2\gamma)^{3/2}}{3}\right)\right| < 11.1/\sqrt{\nu}.$$

This yields

$$|y(z) - (4\nu)^{1/6}\mathrm{Ai}(-2^{1/3}\,z)| \leqslant \frac{3\cdot 2^{3/4}z^{9/4}}{5\sqrt{\pi\nu}} + \frac{z^{1/4}}{2^{5/4}\sqrt{\pi}\nu^{7/6}} + \frac{1}{47\nu^{5/6}} + \frac{11.1}{\sqrt{\nu}} := \phi(\nu,z),$$

giving

$$\begin{aligned} J_\nu(\nu+\nu^{1/3}z) &= \frac{(4\nu)^{1/6}\mathrm{Ai}(-2^{1/3}z)}{\sqrt{\nu+\nu^{1/3}z}} + \theta\frac{\phi(\nu,z)}{\sqrt{\nu+\nu^{1/3}z}} \\ &= \frac{2^{1/3}}{\nu^{1/3}}\mathrm{Ai}(-2^{1/3}\,z) + \theta_1\frac{z}{2^{2/3}\nu}\mathrm{Ai}(-2^{1/3}z) + \theta\frac{\phi(\nu,z)}{\sqrt{\nu}}. \end{aligned}$$

By $|\mathrm{Ai}(-x)| < \pi^{-1/2}x^{-1/4}$ (see [3]) the error term here does not exceed

$$\left|\frac{z^{3/4}}{2^{3/4}\sqrt{\pi\nu}} + \frac{\phi(\nu,z)}{\sqrt{\nu}}\right|.$$

Elementary calculations show that the maximum of this function in the region $\nu \geqslant 1/2,\ 0 \leqslant z \leqslant \nu^{4/15}$ is less than $(4z^{9/4}+21)/7$. This completes the proof. □

To prove Theorem 2 we repeat the arguments of Lemma 6 with the approximation to $J_\nu(\nu+\nu^{1/3}z)$ given by (1) instead of inequality (4). We need the following standard asymptotics:

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos(x-\omega_\nu) + O(x^{-3/2}), \tag{21}$$

$$\mathrm{Ai}(-x) = \frac{\cos(\zeta-\pi/4)}{\sqrt{\pi}x^{1/4}} + O(x^{-7/4}), \qquad \zeta = \frac{2x^{3/2}}{3}. \tag{22}$$

Proof of Theorem 2. By Theorem 1 we have

$$y(z) = \sqrt{\nu + \nu^{1/3} z} J_\nu(\nu + \nu^{1/3} z) = (4\nu)^{1/6} \mathrm{Ai}(-2^{1/3} z) + \mathcal{R}_\nu(z),$$

where $\mathcal{R}_\nu(z) = O(z^{9/4}\nu^{-1/2})$. Writing the error term $\mathcal{R}_\nu$ as in the proof of Lemma 6 and splitting the range of integration into two intervals, we have

$$\mathcal{R}_\nu(z) = \int_0^1 U(z,t)u(t)\,dt + \int_1^z U(z,t)u(t)\,dt = I_1 + I_2,$$

where u is defined by (18) and

$$U(z,t) = 2\pi\sqrt{\frac{tz}{27}}(J_{-1/3}(\zeta_t)J_{1/3}(\zeta_z) - J_{-1/3}(\zeta_z)J_{1/3}(\zeta_t)), \qquad \zeta_x = \frac{(2x)^{3/2}}{3}.$$

First, we estimate I_1. Using the classical inequality [4, equation (10.14.4)],

$$|J_\nu(x)| \leqslant \frac{|x|^\nu}{2^\nu \Gamma(\nu+1)}, \quad x \in \mathbb{R},\ \nu > -\frac{1}{2},$$

we obtain that $U(z,t) = O(1)$ for $0 \leqslant t \leqslant z \leqslant 1$. As well, $|y(t)| = O(\nu^{1/6})$ by Theorem 1, and for the factor at $y(z)$ in the right-hand side of (18) we have

$$\frac{8t^3 + 12\nu^{2/3}t^2 - 1}{4(\nu^{2/3}+t)^2} = 3t^2\nu^{-2/3} + O(t^3\nu^{-4/3}), \tag{23}$$

hence $I_1 = O(\nu^{-1/2})$.

Now we estimate I_2. By (21) for $1 \leqslant t \leqslant z$, we have

$$\begin{aligned} U(z,t) &= \frac{\sin(\zeta_z - \zeta_t)}{\sqrt{2} t^{1/4} z^{1/4}} + O(t^{-7/4} z^{-1/4}); \\ y(t) &= (4\nu)^{1/6}\mathrm{Ai}(-2^{1/3}t) + O(t^{9/4}\nu^{-1/2}) \\ &= \frac{2^{1/4}\nu^{1/6}\cos(\zeta_t - \pi/4)}{\sqrt{\pi} t^{1/4}} + O(t^{9/4}\nu^{-1/2} + t^{-7/4}\nu^{1/6}); \end{aligned}$$

Together with the assumption $z = o(\nu^{4/15})$ and (23) this yields

$$I_2 = A\int_1^z t^{3/2}\cos\left(\zeta_t - \frac{\pi}{4}\right)\sin(\zeta_z - \zeta_t)\,dt + O(z\nu^{-1/2} + z^{9/2}\nu^{-5/3}), \tag{24}$$

where $A = 3/(2z)^{1/4}\sqrt{\pi\nu}$. It is left to estimate the last integral. With a little trigonometry and integrating by parts we have

$$\begin{aligned} &\int_1^z t^{3/2}\cos\left(\zeta_t - \frac{\pi}{4}\right)\sin(\zeta_z - \zeta_t)\,dt \\ &\quad= \frac{1 - z^{5/2}}{5}\cos\left(\zeta_z + \frac{\pi}{4}\right) + \frac{1}{2}\int_1^z t^{3/2}\sin\left(\zeta_z - 2\zeta_t + \frac{\pi}{4}\right)dt \\ &\quad= \frac{1 - z^{5/2}}{5}\cos\left(\zeta_z + \frac{\pi}{4}\right) - \frac{1}{4\sqrt{2}}\int_1^z t\cos\left(\zeta_z - 2\zeta_t + \frac{\pi}{4}\right)dt + \frac{1}{4\sqrt{2}}\int_1^z \cos\left(\zeta_z - 2\zeta_t + \frac{\pi}{4}\right)dt \\ &\quad= -\frac{z^{5/2}}{5}\cos\left(\zeta_z + \frac{\pi}{4}\right) + O(z^2). \end{aligned}$$

https://doi.org/10.1112/S1461157014000011 Published online by Cambridge University Press

Finally, we obtain

$$I_2 = -\frac{z^{5/2}}{5} A \cos\left(\zeta_z + \frac{\pi}{4}\right) + O(z^{7/4}\nu^{-1/2} + z^{9/2}\nu^{-5/3}),$$

and

$$z^{-9/4}\sqrt{\nu}\mathcal{R}_\nu(z) = z^{-9/4}\sqrt{\nu}(I_1 + I_2) = -\frac{3\cos(\zeta_z + \pi/4)}{5 \cdot 2^{1/3}\sqrt{\pi}} + O(z^{-1/2} + z^{9/4}\nu^{-7/6}).$$

By the assumption $z < \nu^{4/15}$ the error term here is of order $O(z^{-1/2})$. Thus we conclude that

$$\limsup_{\nu\to\infty} \limsup_{z\to\infty} z^{-9/4}\sqrt{\nu}|\mathcal{R}_\nu(z)| = \frac{3}{5 \cdot 2^{1/3}\sqrt{\pi}}.$$

This completes the proof. □

References

1. Á. Elbert and A. Laforgia, 'A lower bound for $J_\nu(\nu)$', *Appl. Anal.* 19 (1985) 137–145.
2. S. Finch, 'Bessel function zeroes', *Mathematical constants (supplementary material)* (Cambridge University Press, Cambridge, 2003).
3. I. Krasikov, 'Approximations for the Bessel and Airy functions with an explicit error term', *LMS J. Comput. Math.* to appear.
4. F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (eds), *NIST handbook of mathematical functions* (Cambridge University Press, Cambridge, 2010).
5. C. K. Qu and R. Wong, '"Best possible" upper and lower bounds for the zeros of the Bessel function $J_\nu(x)$', *Trans. Amer. Math. Soc.* 351 (1999) 2833–2859.
6. G. Szegö, *Orthogonal polynomials*, Colloquium Publications 23 (American Mathematical Society, Providence, RI, 1975).

Ilia Krasikov
Department of Mathematical Sciences
Brunel University
Uxbridge UB8 3PH
United Kingdom

mastiik@brunel.ac.uk