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AN $n + 1$ MEMBER DECOMPOSITION FOR SETS WHOSE Lnc POINTS FORM n CONVEX SETS

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1. Introduction. Let S be a subset of R^d. A point x in S is a *point of local convexity of* S if and only if there is some neighborhood N of x such that, if $y, z \in N \cap S$, then $[y, z] \subseteq S$. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) *of* S.

Several interesting results have been proved for a set S whose lnc points Q may be represented as a finite union of convex sets. (See Valentine [**5**], Guay and Kay [**2**].) In particular, in [**2**] it is proved that for S closed, connected, $S \sim Q$ connected, and Q having cardinality n, S is expressible as a union of $n + 1$ or fewer closed convex sets. Since the natural generalization of the Guay-Kay Theorem fails when Q is merely decomposable into n convex sets [**1**], this paper is concerned with obtaining sufficient conditions under which an analogue of the theorem might be proved.

The notation and terminology, following that used in [**1**], are introduced for completeness: Throughout the paper, S is a closed subset of R^d, where $d = \dim \operatorname{aff} S$, the dimension of the affine hull of S. Q denotes the set of lnc points of S, and $S \sim Q$ is connected. We assume that $Q \subseteq \ker S \neq \emptyset$ (so S is connected) and that $Q = \bigcup_{i=1}^{n} C_i$ where each C_i is convex. Since Q is closed, without loss of generality, we consider each C_i to be closed. Further, we assume that n is minimal in the following sense: For every i, there are points of C_i which do not belong to any C_j for $j \neq i$, $1 \leqq i, j \leqq n$.

2. Preliminary results. We begin with a sequence of lemmas which will be useful in proving the main theorem of the paper. The first is a variation of a result by Valentine [**5**, Corollary 2].

LEMMA 1 (Valentine). *If* $[x, y] \cup [y, z] \subseteq S$ *and no point of* Q *lies in* $\operatorname{conv}\{x, y, z\} \sim [x, z]$, *then* $\operatorname{conv}\{x, y, z\} \subseteq S$.

The second lemma is proved in [**1**].

LEMMA 2. *For* s *in* S, *every neighborhood of* s *contains points in* $\operatorname{int} S$. *Hence* $S = \operatorname{cl}(\operatorname{int} S)$.

LEMMA 3. *If* $p \in Q$ *and* N *is any convex neighborhood of* p, $(N \cap S) \sim Q$ *is connected.*

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Proof. We assert that $N \cap \text{int } S$ is connected. Since $S \sim Q$ is connected and locally convex, it is polygonally connected [**4**], and by standard arguments, since $S = \text{cl}(\text{int } S)$, $(\text{int } S) \sim Q = \text{int } S$ is also polygonally connected. Thus for x, y in $N \cap \text{int } S$, there is a polygonal path λ in $\text{int } S$ from x to y. Since $p \in \ker S$ and $\lambda \subseteq \text{int } S$, for every z in λ, $(p, z] \subseteq \text{int } S$. Then there is a path λ_0 in $(\bigcup\{(p, z] : z \text{ in } \lambda\}) \cap N \subseteq (\text{int } S) \cap N$ from x to y. We have $N \cap \text{int } S = (N \cap \text{int } S) \sim Q$ polygonally connected and hence connected. Since $(N \cap \text{int } S) \sim Q \subseteq (N \cap S) \sim Q \subseteq \text{cl}((N \cap \text{int } S) \sim Q)$, $(N \cap S) \sim Q$ is also connected, and the lemma is proved.

COROLLARY. *For each C_i, $1 \leqq i \leqq n$,* dim aff $C_i = d - 2$. *Moreover,* if $Q = C$ *is convex, then S may be represented as a union of two closed convex sets.*

Proof. By Lemma 3, each C_i satisfies the hypotheses of Theorems 1, 2, and 3 in [**1**]. Hence the corollary follows immediately from these results.

Finally, the following theorem by Lawrence, Hare and Kenelly [**3**, Theorem 2] will be helpful.

LEMMA 4 (Lawrence, Hare, Kenelly). *Let T be a subset of a linear space such that for each finite subset $F \subseteq T$, F is a union of k sets $F_1, \ldots, F_k$, where* conv $F_i \subseteq T$, $1 \leqq i \leqq k$. *Then T is a union of k convex sets.*

3. The decomposition theorem.

THEOREM 1. *Let S be a closed subset of R^d, Q the points of local nonconvexity of S, with $S \sim Q$ connected. If $Q \subseteq$* ker $S \neq \emptyset$ *and Q is expressible as a union of n convex sets, then S is a union of $n + 1$ or fewer convex sets.*

Proof. We assert that, without loss of generality, we may assume S to be a finite union of sets of the form $\text{conv}(T \cup Q)$, where T is a finite subset of S: For F any finite subset of S, define

$$S_F = \{x : x \in \text{conv}(T \cup Q) \subseteq S \text{ for some } T \subseteq F\}.$$

Clearly each finite subset F' of S may be extended to a finite subset F of S for which S_F is a full d-dimensional and $S_F \sim Q$ is connected. Also, the set of lnc points of S_F lies in Q, and by an appropriate choice of F, this set of lnc points will be exactly Q. (For each C_i, select $x_i \in (\text{rel int } C_i) \sim \bigcup_{j \neq i} C_j$ and let N be a neighborhood of x_i disjoint from $\bigcup_{i \neq j} C_i$. By adapting an argument in [**1**, Theorem 3], we may select p_i, q_i in $N \cap S$ so that for $p' \in \text{conv}(\{p_i\} \cup C_i) \sim C_i$ and $q' \in \text{conv}(\{q_i\} \cup C_i) \sim C_i$, $[p', q'] \not\subseteq S$. Then if $p_i, q_i \in F$, each point of C_i will be an lnc point for S_F.) Clearly $Q \subseteq \ker S_F$, so S_F satisfies the hypothesis of Theorem 1. Now by the Lawrence, Hare, Kenelly Theorem, we need only show that F' is a union of $n + 1$ convex sets, each having its convex hull in $S_F \subseteq S$. Therefore, it suffices to prove that S_F is a union of $n + 1$ convex sets, so throughout the proof, we assume that S is a finite union of sets of the form $\text{conv}(T \cup Q)$, where T is a finite subset of S.

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The proof of the theorem will be by induction. For $n = 0$, $Q = \emptyset$, and the result is an immediate consequence of a theorem by Tietze [**4**]. In case $n = 1$, Q is convex, and the result follows from the corollary to Lemma 3. Inductively, for some $n > 1$, we assume that the theorem is true whenever Q is expressible as a union of fewer than n convex sets.

Select some point $p \in (\text{rel int } C_1) \sim (\bigcup_{i=2}^{n} C_i)$, and let N be a convex neighborhood of p such that $(\text{cl } N) \cap (\bigcup_{i=2}^{n} C_i) = \emptyset$. Letting $T = \text{cl}[(N \cap S) \sim Q]$, the set Q_T of lnc points for T is exactly $T \cap C_1$. Using Lemma 3, it is clear that $T \sim Q_T$ is connected, and since Q_T is convex, by Theorem 3 in [**1**], T is expressible as a union of two closed convex sets. Moreover, by the proof of that theorem, there is a hyperplane M containing C_1 such that $\text{cl}(T \cap M_1)$, $\text{cl}(T \cap M_2)$ are convex sets whose union is T (where M_1, M_2 represent distinct open halfspaces determined by M), and $[(\ker T) \cap M] \sim \text{aff } C_1 \neq \emptyset$.

Now let H denote a hyperplane supporting $\text{cl}(T \cap M_1)$ which contains C_1 and which also contains some point x in $[N \cap (\text{bdry } S) \cap \text{cl}(T \cap M_1)] \sim \text{cl}(T \cap M_2)$. (Clearly since T is not convex, the set $[N \cap (\text{bdry } S) \cap \text{cl}(T \cap M_1)] \sim \text{cl}(T \cap M_2)$ is not empty, and by our opening assumption concerning S, H may be obtained by rotating M about the $(d - 2)$ flat aff C_1.) Assume that $\text{cl}(T \cap M_1) \subseteq \text{cl } H_1$. We assert that $\text{cl}(T \cap H_1)$, $\text{cl}(T \cap H_2)$ are also convex sets whose union is T. The proof follows:

If $H = M$, there is nothing to prove, so assume H, M are distinct. Now for y in $T \cap H_2$, $y \notin \text{cl}(T \cap M_1)$, and $y \in T \cap H_2 \cap M_2$. Thus $\text{cl}(T \cap H_2) = \text{cl}(T \cap M_2 \cap H_2)$, which is convex. To see that $T \cap H_1$ is convex, recall that there is some w in $[(\ker T) \cap M] \sim \text{aff } C_1$. Now $w \in (\ker T) \cap M \subseteq \text{cl}(T \cap M_1) \subseteq \text{cl } H_1$; also $w \notin \text{aff } C_1 = H \cap M$. Thus $w \in M \cap H_1$. For points y, z in $T \cap H_1$, $[y, w] \cup [z, w] \subseteq T$, and since $C_1 \subseteq H$, there can be no point of C_1 in $\text{conv}\{y, w, z\}$. Hence by Valentine's lemma, $[y, z] \subseteq T$. Then $[y, z] \subseteq T \cap H_1$, $T \cap H_1$ is convex, and $\text{cl}(T \cap H_1)$ is convex. Since $S = \text{cl}(\text{int } S)$, $T = \text{cl}(T \cap H_1) \cup \text{cl}(T \cap H_2)$, and the assertion is proved.

Furthermore, no point of Q may lie in H_2: Otherwise, for y in $H_2 \cap Q \subseteq \ker S$ and x the member of H selected above, $(x, y] \subseteq H_2$, and since x is interior to N, there would be a sequence (x_n) in $T \cap H_2 \subseteq T \cap H_2 \cap M_2 \subseteq T \cap M_2$ converging to x. But then $x \in \text{cl}(T \cap M_2)$, clearly impossible by our choice of x.

Define $A_1 = S \cap H_1$, $A_2 = S \cap H_2$. We will show that $\text{cl } A_2$ is convex and that $\text{cl } A_1$ is a set satisfying our induction hypothesis with its lnc points expressible as a union of $n - 1$ or fewer convex sets.

To see that $\text{cl } A_2$ is convex, let $y, z \in A_2 = S \cap H_2$. Then $[y, p]$, $[z, p] \subseteq S$ and each of these segments contains points of $N \cap S \cap H_2 \subseteq T \cap H_2$. Select y', z' in $T \cap H_2$ for which $[y, y']$, $[z, z'] \subseteq S$. Since $T \cap H_2$ is convex, $[y', z'] \subseteq S \cap H_2$, and since no lnc points of S lie in H_2, by repeated use of Valentine's lemma, $[y, z] \subseteq S \cap H_2$. Therefore A_2 is convex, as in $\text{cl } A_2$.

It remains to show that $\text{cl } A_1$ satisfies our induction hypothesis. Clearly $\text{cl } A_1$ is connected since $[a, p] \subseteq \text{cl } A_1$ for every $a \in \text{cl } A_1$. To see that $\text{cl } A_1 \sim Q$ is connected, let $y, z \in (S \cap H_1) \sim Q$ and let U, V be neighborhoods

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of y, z respectively, with $U \cap S$, $V \cap S$ convex (and hence disjoint from Q). We assert that $U \cap S$ contains some y_1 for which $[y_1, p) \subseteq (S \cap H_1) \sim Q$: Each C_i has dimension $d - 2$, so for each i, $1 \leqq i \leqq n$, aff$(\{p\} \cup C_i)$ determines a flat of dimension at most $d - 1$. Since $S = \text{cl}(\text{int}\, S)$, we may select y_1 in $U \cap S \cap H_1$ and in none of these flats. Then $[y_1, p)$ is disjoint from Q. Similarly, there is some z_1 in $V \cap S \cap H_1$ with $[z_1, p) \subseteq (S \cap H_1) \sim Q$.

Now select y_2 on (y_1, p), z_2 on (z_1, p), with y_2, z_2 in $(S \cap N) \cap H_1 \subseteq T \cap H_1$. Since $T \cap H_1$ is convex and disjoint from Q, the path $[y, y_1] \cup [y_1, y_2] \cup [y_2, z_2] \cup [z_2, z_1] \cup [z_1, z]$ lies in $(S \cap H_1) \sim Q$. Thus the set $(S \cap H_1) \sim Q = A_1 \sim Q$ is polygonally connected and hence connected. Since $A_1 \sim Q \subseteq (\text{cl}\, A_1) \sim Q \subseteq \text{cl}(A_1 \sim Q)$, $(\text{cl}\, A_1) \sim Q$ is also connected. Trivially, if Q_A denotes the set of lnc points of cl A_1, $(\text{cl}\, A_1) \sim Q_A$ is connected.

Finally, we show that Q_A is expressible as a union of $n - 1$ or fewer convex sets, each in ker(cl A_1). However, the following preliminary result will be needed: For $i \neq j$, $1 \leqq i, j \leqq n$, if (rel int C_i) $\cap$ aff $C_j \neq \emptyset$, then aff C_i = aff C_j. The proof is given below.

For simplicity of notation, we will prove the result for $j = 1$. Recall that p is an arbitrary point in rel int C_1 and in no C_i, $i \neq 1$, N is a convex neighborhood of p disjoint from C_i, $i \neq 1$, and H a hyperplane supporting cl$(T \cap M_1)$, H containing C_1 and some x in $[N \cap (\text{bdry}\, S) \cap \text{cl}(T \cap M_1)] \sim \text{cl}(T \cap M_2)$, cl$(T \cap M_1) \subseteq$ cl H_1. Similarly, let J be a hyperplane supporting cl$(T \cap M_2)$, J containing C_1 and some point in $[N \cap (\text{bdry}\, S) \cap \text{cl}(T \cap M_2)] \sim \text{cl}(T \cap M_1)$, cl$(T \cap M_2) \subseteq$ cl J_2. By previous remarks, no point of Q may lie in H_2 or in J_1. Hence $Q \subseteq \text{cl}\, H_1 \cap \text{cl}\, J_2$. For $2 \leqq i \leqq n$, if C_i contains a point in $[\text{cl}\, H_1 \cap \text{cl}\, J_2] \sim (\text{aff}\, C_1)$, then certainly (rel int C_i) $\cap$ aff $C_1 = \emptyset$. Otherwise, $C_i \subseteq$ aff C_1, and aff C_i = aff C_1.

Using this result, it is not hard to show that no point of $C_1 \sim \bigcup_{i=2}^n C_i$ is in Q_A. Let $u \in C_1 \sim \bigcup_{i=2}^n C_i$, $u \neq p$. Then $(u, p] \subseteq$ rel int C_1. If $[u, p]$, contains any point of C_i, $2 \leqq i \leqq n$, then rel int $C_1 \cap C_i \neq \emptyset$, and by our earlier result, $C_i \subseteq$ aff C_1. We assert that for each v on $[u, p]$ there is a convex neighborhood N_v of v such that $N_v \cap Q \subseteq C_1$: Since $u \in C_1 \sim \bigcup_{i=2}^n C_i$, select N_u disjoint from each C_i, $2 \leqq i \leqq n$. For $v \in (u, p]$, it is simple to select a neighborhood N_v of v disjoint from every C_i not containing v. Also, since $v \in$ rel int C_1, N_v may be selected so that $N_v \cap$ aff $C_1 \subseteq$ rel int C_1. If N_v contains a point q of some C_i, $i \neq 1$, then $v \in C_i$, $v \in$ (rel int C_1) $\cap C_i \neq \emptyset$, and $C_i \subseteq$ aff C_1. Hence $q \in N_v \cap C_i \subseteq N_v \cap$ aff $C_1 \subseteq$ rel int C_1, and $N_v \cap Q \subseteq$ rel int C_1. Thus the assertion is proved.

By Lemma 3, $(N_v \cap S) \sim Q$ is connected for each neighborhood N_v selected above. Reduce to a finite subcollection $N_1, \ldots, N_j$ of the N_v sets which covers $[u, p]$. Choose a convex cylinder U' so that cl $U' \subseteq N_1 \cup \ldots \cup N_j$, and define

$$U \equiv (U' \cap S) \sim Q.$$

Clearly the lnc points for cl U are exactly $C_1 \cap$ cl U, $C_1 \cap$ cl $U = C_1 \cap$ cl U' is convex, cl U is closed, connected, and using Lemma 3, it is easy to see that

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$(\text{cl } U) \sim C_1$ is connected. Hence our previous argument for cl T may be adapted to cl U to show that each of the sets $\text{cl}(U \cap H_1)$, $\text{cl}(U \cap H_2)$ is convex. Thus u cannot be an lnc point for cl $A_1 = \text{cl}(S \cap H_1)$, since U' is a neighborhood of u whose intersection with cl A_1 is convex. Then $u \notin Q_A$, the desired result.

It is a simple matter to show that for each i, $1 \leqq i \leqq n$, $C_i \cap \text{cl } A_1 = C_i$, and hence $C_i \cap \text{cl } A_1$ is convex: Let $z \in C_i$, to prove $z \in \text{cl } A_1$. By previous remarks, $z \notin C_i \cap H_2 = \emptyset$, and if $z \in C_i \cap H_1 \subseteq S \cap H_1 = A_1$, the result is immediate. Therefore, we need only consider the case for $z \in C_i \cap H$. Since $S = \text{cl}(\text{int } S)$, there is a sequence in $S \sim H$ converging to z. Moreover, since z cannot be an lnc point for the convex set cl $A_2 = \text{cl}(S \cap H_2)$, there must be a sequence in $S \cap H_1$ converging to z, and $z \in \text{cl } A_1$. Thus $C_i \cap \text{cl } A_1 = C_i$, and the set is convex.

Furthermore, for $2 \leqq i \leqq n$, either $C_i \subseteq Q_A$ or $C_i \sim \bigcup_{j \neq i} C_j$ is disjoint from Q_A. The proof follows: Since we have already proved the result for $i = 1$, suppose that for some $2 \leqq i \leqq n$, $C_i \not\subseteq Q_A$. For convenience, relabel the C_j sets so that $i = 2$. Then clearly $C_2 \subseteq H$. There is some point r in rel int C_2 with $r \notin Q_A$, and for some convex neighborhood W of r, $\text{cl}(W \cap H_1)$, $\text{cl}(W \cap H_2)$ are convex. For t in $C_2 \sim \bigcup_{j \neq 2} C_j$, $(t, r] \subseteq \text{rel int } C_2$, and a previous argument may be repeated to select a convex neighborhood of $[t, r]$ whose intersection with cl A_1 is convex. Thus $t \notin Q_A$ and $C_2 \sim \bigcup_{j \neq 2} C_j$ is disjoint from Q_A.

The set Q_A is the union of some of the $n - 1$ convex sets $C_2, \ldots, C_n$. Moreover, each lnc point for cl A_1 is in ker(cl A_1): For q in Q, s in cl A_1, there is a sequence (s_n) in $S \cap H_1$ converging to s, $(q, s_n] \subseteq S \cap H_1$, and $[q, s] \subseteq \text{cl}(S \cap H_1) = \text{cl } A_1$. Hence $Q \subseteq \ker(\text{cl } A_1)$ and certainly $Q_A \subseteq \ker(\text{cl } A_1)$.

Therefore, the set cl A_1 satisfies our induction hypothesis and is expressible as a union of $(n - 1) + 1 = n$ or fewer convex sets. Then $S = \text{cl } A_1 \cup \text{cl } A_2$ is a union of $n + 1$ or fewer convex sets, finishing the proof of Theorem 1.

Clearly the bound of $n + 1$ in Theorem 1 is best possible for $n = 0$ and for $n = 1$. For $n \geqq 2$, the bound is best possible provided $S \subseteq R^d$, $d \geqq 3$, as the following example reveals.

Example 1. Let P be a prism in R^3 whose basis is a $2n$-gon, $n \geqq 2$. Remove disjoint wedges $W_1, \ldots, W_n$ from non-adjacent, non-basis facets of P to produce the convex sets of lnc points $C_1, \ldots, C_n$. Each wedge W_j should be removed so that the corresponding C_j intersects both bases of P, and so that for $1 \leqq i < j \leqq n$, no hyperplane containing C_j contains C_i. This may be done in such a way that the resulting set S satisfies the hypothesis of Theorem 1, and S is not expressible as a union of fewer than $n + 1$ convex sets.

The example may be generalized to $d > 3$.

In case $d \leqq 1$, n must be zero, and the theorem is trivial. Thus the only other interesting case occurs when $d = 2$, and we have the following theorem.

THEOREM 2. *Let S be a closed subset of the plane, Q the set of* lnc *points of S*

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with $S \sim Q$ connected. If $Q \subseteq \ker S \neq \emptyset$, then S is expressible as a union of three or fewer convex sets.

Proof. If card $Q = 0$, S is convex, and if card $Q = 1$, S is a union of two convex sets by Theorem 1. For card $Q = 2$, it is easy to see that the line determined by Q yields the desired decomposition. Similarly, in case $Q = \{x, y, z\}$, it is not hard to show that the points in Q cannot be collinear. Hence these points determine three lines, each pair of which yield a convex subset of S for the decomposition.

For card $Q \geqq 4$, an argument similar to that used by Valentine in Lemma 5 of [**6**] may be applied to show that S is 3-convex. Then S is expressible as a union of three or fewer convex sets by Theorem 2 of [**6**].

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