8

Quantising fields: QED

We turn now to the quantisation of the electrodynamic fields introduced in Chapter 7. So far we have treated the electromagnetic field and the Dirac field as classical fields (though we were compelled in Chapter 7 to recognise that Dirac fields anticommute). On quantisation, these fields become operator fields, acting on the states of a system. The classical total field energy becomes the Hamiltonian operator, which determines the dynamics of the system. We shall use the formalism of annihilation and creation operators; this formalism is reviewed briefly in Appendix C for readers not already familiar with it.

Quantum electrodynamics, or QED, is an important component of the Standard Model. It is also the foundation of our understanding of the material world at the atomic level. However, we do not wish to enter into the technical complications of electrons in atoms or in material media. In this chapter we shall only consider more simple situations of a few interacting photons, electrons and positrons, at energies sufficiently high for bound systems of electrons and positrons to be ignored. In these situations, the free field approximation to QED provides a sound basis for understanding the interactions of particles as perturbations on their free behaviour.

This is not a text on quantum field theory, and our outline of perturbation theory in this chapter is necessarily sketchy. But our intention is to try to give some insight into how the results of calculations, presented in later chapters, are arrived at. We shall attempt to explain the necessity of *renormalisation*, which is an important concept in the formulation of the Standard Model.

8.1 Boson and fermion field quantisation

The simplest classical field we have introduced is that of a massive free scalar particle. It satisfies the Klein–Gordon equation (3.19). In the field expansion (3.21) we have so far regarded the classical wave amplitudes $a_{\mathbf{k}}$ and $a^*_{\mathbf{k}}$ as ordinary complex

https://doi.org/10.1017/9781009401685.010 Published online by Cambridge University Press

numbers. We now quantise the theory. We interpret a_{k} as an annihilation operator and a_{k}^* becomes the creation operator $a_{\mathbf{k}}^\dagger$, the Hermitian conjugate of a_{k}. These operators are to obey the commutation relations

$$\left[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger\right] = \delta_{\mathbf{k}\mathbf{k}'}, \qquad [a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0, \qquad \left[a_{\mathbf{k}}^\dagger, a_{\mathbf{k}'}^\dagger\right] = 0. \tag{8.1}$$

The total field energy (3.30) becomes the Hamiltonian operator

$$H = \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \omega_{\mathbf{k}} = \sum_{\mathbf{k}} N_{\mathbf{k}} \omega_{\mathbf{k}}, \tag{8.2}$$

where $\omega_{\mathbf{k}} = \sqrt{(\mathbf{k}^2 + m^2)}$ and it follows from the commutation relations that $N_{\mathbf{k}} = a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ is the number operator (Appendix C). As in Chapter 3, we shall in this chapter confine all particles to a cube of side l, volume $V = l^3$, and use periodic boundary conditions. By defining the Hamiltonian to be of the form (8.2), rather than the more symmetrical form

$$\frac{1}{2} \sum_{\mathbf{k}} \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}} a_{\mathbf{k}}^\dagger\right) \omega_{\mathbf{k}} = \sum_{\mathbf{k}} \left(N_{\mathbf{k}} + \frac{1}{2}\right) \omega_{\mathbf{k}} \tag{8.3}$$

we discard 'zero-point energy' contributions and hence make the energy of the vacuum state $|0\rangle$ to be zero. The excited energy eigenstates of the Hamiltonian can then be interpreted as assemblies of particles (π^0 mesons, say, or Higgs particles) with an integer number $n_{\mathbf{k}}$ of particles in the state $\mathbf{k}$, where $n_{\mathbf{k}}$ is the eigenvalue of the number operator $N_{\mathbf{k}}$. The particles will obey Bose–Einstein statistics.

In the radiation gauge of Section 4.1, the electromagnetic field in free space is quantised in a very similar way to the Klein–Gordon field. The wave amplitudes $a_{\mathbf{k}\alpha}$ and $a_{\mathbf{k}\alpha}^*$ which appear in the expansion (4.15), become the annihilation and creation operators $a_{\mathbf{k}\alpha}$ and $a_{\mathbf{k}\alpha}^\dagger$, and the total field energy (4.25) becomes the Hamiltonian operator

$$H_{em} = \sum_{\mathbf{k},\alpha} a_{\mathbf{k}\alpha}^\dagger a_{\mathbf{k}\alpha} \omega_{\mathbf{k}} \tag{8.4}$$

where $\omega_{\mathbf{k}} = |\mathbf{k}|$. The operators $a_{\mathbf{k}\alpha}$ and $a_{\mathbf{k}\alpha}^\dagger$ annihilate and create *photons* of wave vector $\mathbf{k}$ and polarisation α, and satisfy commutation relations

$$\left[a_{\mathbf{k}\omega}, a_{\mathbf{k}'\alpha'}^\dagger\right] = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\alpha\alpha'}, \qquad [a_{\mathbf{k}\alpha}, a_{\mathrm{k}'\alpha'}] = 0, \qquad \left[a_{\mathbf{k}\alpha}^\dagger, a_{\mathbf{k}'\alpha'}^\dagger\right] = 0. \tag{8.5}$$

$N(\mathbf{k}, \alpha) = a_{\mathbf{k}\alpha}^\dagger a_{\mathbf{k}\alpha}$ is the number operator. The energy eigenstates of the radiation field correspond to assemblies of photons. Photons, like scalar particles, obey Bose–Einstein statistics. (See Problem 8.1.)

On quantising the Dirac field of a free electron, the wave amplitudes appearing in the expansion (6.24), and their complex conjugates likewise become operators: $b_{\mathbf{p}\varepsilon}$ and ${b_{\mathbf{p}\varepsilon}}^\dagger$ annihilate and create electrons of momentum $\mathbf{p}$, helicity ε; $d_{\mathbf{p}\varepsilon}$ and ${d_{\mathbf{p}\varepsilon}}^\dagger$

https://doi.org/10.1017/9781009401685.010 Published online by Cambridge University Press

annihilate and create positrons of momentum $\mathbf{p}$, helicity ε. Electrons and positrons are fermions, and these operators obey anticommutation relations, for example

$$b_{\mathbf{p}\varepsilon} b^{\dagger}_{\mathbf{p}'\varepsilon'} + b^{\dagger}_{\mathbf{p}'\varepsilon'} b_{\mathbf{p}\varepsilon} = \left\{ b_{\mathbf{p}\varepsilon}, b^{\dagger}_{\mathrm{p}'\varepsilon'} \right\} = \delta_{\mathbf{pp}'} \delta_{\varepsilon\varepsilon'}, \quad \{ b_{\mathbf{p}\varepsilon}, b_{\mathbf{p}'\varepsilon'} \} = 0, \quad \left\{ b^{\dagger}_{\mathbf{p}\varepsilon}, b^{\dagger}_{\mathbf{p}'\varepsilon'} \right\} = 0 \tag{8.6}$$

$d_{\mathbf{p}\varepsilon}$ and $d^{\dagger}_{\mathbf{p}'\varepsilon'}$ obey similar rules. Also all electron operators anticommute with all positron operators. The electron number operator $N_{\mathbf{e}}(\mathbf{p}, \varepsilon) = b^{\dagger}_{\mathbf{p}\varepsilon} b_{\mathbf{p}\varepsilon}$ and the positron number operator $N_{\mathbf{p}}(\mathbf{p}, \varepsilon) = d^{\dagger}_{\mathbf{p}\varepsilon} d_{\mathbf{p}\varepsilon}$ have possible eigenvalues restricted to 0 and 1, in accord with the Pauli exclusion principle (Appendix C). Electrons and positrons obey Fermi–Dirac statistics. (See Problem 8.2.)

After second quantisation, the difficulties that were associated with the interpretation of the Dirac equation as a single particle wave equation disappear. Electrons and positrons are now on a similar footing and the 'sea' of filled negative energy states is no longer needed. The total field energy (6.25) becomes the Hamiltonian

$$H = \sum_{\mathbf{p},\varepsilon} \left(b^{\dagger}_{\mathbf{p}\varepsilon} b_{\mathbf{p}\varepsilon} - d_{\mathbf{p}\varepsilon} d^{\dagger}_{\mathbf{p}\varepsilon} \right) E_{\mathbf{p}}.$$

Using an anticommutation relation, we can replace this by

$$H = \sum_{\mathbf{p},\varepsilon} \left(b^{\dagger}_{\mathbf{p}\varepsilon} b_{\mathbf{p}\varepsilon} + d^{\dagger}_{\mathbf{p}\varepsilon} d_{\mathbf{p}\varepsilon} - 1 \right) E_{\mathbf{p}}.$$

We shall discard the constant zero-point energy term (which we note is negative for fermions) and take

$$H = \sum_{\mathbf{p},\varepsilon} \left(b^{\dagger}_{\mathbf{p}\varepsilon} b_{\mathbf{p}\varepsilon} + d^{\dagger}_{\mathbf{p}\varepsilon} d_{\mathbf{p}\varepsilon} \right) E_{\mathbf{p}}. \tag{8.7}$$

The energy of the vacuum state is then zero, and the excited states correspond to assemblies of electrons and positrons.

Similarly, the field momentum (6.26) becomes the momentum operator

$$\mathbf{P} = \sum_{\mathbf{p},\varepsilon} \left(b^{\dagger}_{\mathbf{p}\varepsilon} b_{\mathbf{p}\varepsilon} + d^{\dagger}_{\mathbf{p}\varepsilon} d_{\mathbf{p}\varepsilon} \right) \mathbf{p}. \tag{8.8}$$

The conserved particle number (Problem 7.1) becomes the time independent operator

$$\int \mathrm{P}\left(x^0, \mathbf{x} \right) \mathrm{d}^3\mathbf{x} = \sum_{\mathbf{p},\varepsilon} \left(b^{\dagger}_{\mathbf{p}\varepsilon} b_{\mathbf{p}\varepsilon} + d_{\mathbf{p}\varepsilon} d^{\dagger}_{\mathbf{p}\varepsilon} \right). \tag{8.8}$$

which we replace by:

$$\text{conserved number operator} = \sum_{\mathbf{p},\varepsilon} \left(b^{\dagger}_{\mathbf{p}\varepsilon} b_{\mathbf{p}\varepsilon} + d^{\dagger}_{\mathbf{p}\varepsilon} d_{\mathbf{p}\varepsilon} \right). \tag{8.9}$$

https://doi.org/10.1017/9781009401685.010 Published online by Cambridge University Press

This operator counts the number of electrons minus the number of positrons, a number which is therefore constant in quantum electrodynamics.

8.2 Time dependence

In the Schrödinger picture, a system described by a Hamiltonian H evolves in time from a state $|t_0\rangle$ at time t_0 to a state $|t\rangle$ at time t, where

$$|t\rangle = \mathrm{e}^{-\mathrm{i}H(t-t_0)}|t_0\rangle.$$

Thus time displacements are generated by the unitary operator $\mathrm{e}^{-\mathrm{i}Ht}$.

The expectation value of a time independent operator $\hat{O}$ at time t is

$$\begin{aligned}\langle t|\hat{O}|t\rangle &= \langle t_0|\mathrm{e}^{\mathrm{i}H(t-t_0)}\hat{O}\mathrm{e}^{-\mathrm{i}H(t-t_0)}|t_0\rangle \\ &= \langle t_0|\hat{O}_H(t-t_0)|t_0\rangle\end{aligned}$$

where

$$\hat{O}_H(t) = \mathrm{e}^{\mathrm{i}Ht}\hat{O}\mathrm{e}^{-\mathrm{i}Ht} \tag{8.10}$$

depends on t.

These last equations give the so-called Heisenberg picture, in which the states of a system remain fixed and the operators become time dependent. In the case of free fields, the time dependence of the annihilation and creation operators is very simple. For example, in the case of a scalar field (see (3.21)),

$$a_{\mathbf{k}}(t) = e^{-i\omega_{\mathbf{k}}t}a_{\mathbf{k}}, \quad a^{\dagger}_{\mathbf{k}}(t) = e^{\mathrm{i}\omega_{\mathbf{k}}t}a^{\dagger}_{\mathbf{k}}, \tag{8.11}$$

as may be seen by considering the effect of the operators on a state $|n_{\mathrm{k}}\rangle$ (Appendix C). It is usual in quantum field theory to work in the Heisenberg picture.

In the case of interacting fields, the basic free field states we have defined are no longer eigenstates of the total Hamiltonian. In QED we may write

$$H = H_0 + V, \tag{8.12}$$

where

$$H_0 = H\,(\text{photons}) + H\,(\text{electrons}) + H\,(\text{positrons})$$

is given by (8.4) and (8.7). The eigenstates of H_{o} are just collections of freely moving photons, electrons, and positrons.

V comes from the term $-q\left(\bar{\psi}\gamma^{\mu}\psi\right)A_{\mu}$ in the Lagrangian density, (7.7), which we constructed in Chapter 7. We are here excluding external fields. Since V does not depend on derivatives of the fields, its contribution to the energy density T^0_0 is just $q(\bar{\psi}\gamma^{\mu}\psi)A_{\mu}$, and setting $q = -e$ for electrons we obtain at $t = t_0$

$$V(t_0) = -e \int \bar{\psi}(\mathbf{r}, t_0)\gamma^{\mu}\psi(\mathbf{r}, t_0)A_{\mu}(\mathbf{r}, t_0)\mathrm{d}^3\mathbf{r}. \tag{8.13}$$

Note that the subsequent time development of the fields is not that of the free fields, since it is determined by the full Hamiltonian $H = H_0 + V$.

We can expand the fields A_μ and ψ at the initial time t_o using (4.15) and (6.24), replacing the wave amplitudes by appropriate operators. On expanding out V there will be several types of term. For example, setting $t_o = 0$ one can easily pick out a term

$$-\frac{em}{\sqrt{(2V\omega_{\mathbf{k}}E_{\mathbf{p}'}E_{\mathbf{p}''})}}[\bar{u}_{\varepsilon'}(\mathbf{p}')\gamma^{\mu}v_{\varepsilon''}(\mathbf{p}'')\varepsilon_{\mu}]d^{\dagger}_{\mathbf{p}'\varepsilon}d^{\dagger}_{\mathbf{p}''\varepsilon''}a_{\mathbf{k}\alpha}\delta_{(\mathbf{k}-\mathbf{p}'-\mathbf{p}''),0}. \tag{8.14}$$

This term annihilates a photon and creates an electron–positron pair. The condition $\mathbf{k} - \mathbf{p}' - \mathbf{p}'' = 0$ comes from the integration over space of the exponential factors, and explicitly conserves momentum.

Dynamical calculations in a quantum field theory can be viewed as the calculation of the unitary operator $\mathrm{e}^{-\mathrm{i}Ht}$ acting on some initial specified state. In QED, the coupling (8.13) between the radiation field and the Dirac field is determined by the charge on the electron e. It is natural to introduce the dimensionless parameter α, the *fine structure constant*:

$$\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}.$$

α characterises the strength of the coupling, and is small. Much progress has been made in QED by the construction of the operator $\mathrm{e}^{-\mathrm{i}Ht}$ as an expansion of the form

$$\mathrm{e}^{-\mathrm{i}Ht} = \mathrm{e}^{-\mathrm{i}H_0t}[1 + e\hat{O}_1(t) + e^2\hat{O}_2(t) + \ldots] \tag{8.15}$$

where the $\hat{O}_n(t)$ are time-dependent operators.

8.3 Perturbation theory

To construct the perturbation expansion (8.15), one can start by considering

$$\mathrm{e}^{-\mathrm{i}Ht} = [\mathrm{e}^{-\mathrm{i}H\delta t}]^n \text{with } \delta t = t/n.$$

For large enough n (small enough δt), one can take

$$\mathrm{e}^{-\mathrm{i}H\delta t} = 1 - \mathrm{i}H\delta t$$

and discard higher order terms in the Taylor expansion. Then

$$\mathrm{e}^{-\mathrm{i}Ht} = [1 - \mathrm{i}(H_0 + V)\delta t]^n .$$

https://doi.org/10.1017/9781009401685.010 Published online by Cambridge University Press

In the lowest order of perturbation theory only the terms linear in V are kept, so that

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}H_0 t} e\hat{O}_1(t) &= -\mathrm{i}\sum_{r=0}^{n-1}[1-\mathrm{i}H_0\delta t]^{n-1-r}\, V\delta t\,[1-\mathrm{i}H_0\delta t]^r \\ &= -\mathrm{i}\sum_{r=0}^{n-1}\mathrm{e}^{-\mathrm{i}H_0\left(t-t'\right)}V\delta t\,\mathrm{e}^{-\mathrm{i}H_0 t'}\end{aligned}$$

with $t' = r\delta t$ and n large.

In the limit of $\delta t \to 0$, we can replace the sum by an integral, so that

$$e\hat{O}_1(t) = -\mathrm{i}\int_0^t \mathrm{d}t'\mathrm{e}^{\mathrm{i}H^0 t'}V\mathrm{e}^{-\mathrm{i}H_0 t'}. \tag{8.16}$$

The operator $\mathrm{e}^{-\mathrm{i}H_0 t'}$ is the simple free field time evolution operator. If we take V to be given at $t = 0$ by (8.13), we can write

$$\hat{O}_1(t) = \mathrm{i}\int_0^t \bar{\psi}(\mathbf{r}', t')\gamma^\mu\psi(\mathbf{r}', t')A_\mu(\mathbf{r}', t')\,\mathrm{d}t'\mathrm{d}^3\mathbf{r}' \tag{8.17}$$

where the fields have the time dependence of free unperturbed fields. A term like (8.14), for example, will have time dependence (see equation (8.11)).

$$\mathrm{e}^{-\mathrm{i}\left(\omega_{\mathbf{k}}-E_{\mathbf{p}'}-E_{\mathbf{p}''}\right)t'} \tag{8.18}$$

The evolution of a state from time $-t/2$ in the past to time $t/2$ in the future corresponds to taking the integral in (8.17) from $-t/2$ to $t/2$. This more symmetrical form is appropriate to the description of particle scattering processes. For example, if the initial state at time $-t/2$ consists of a photon in the state $(\mathbf{k}, \alpha)$, the operators in (8.14) annihilate this photon and create an electron in a state $(\mathbf{p}', \varepsilon')$ and a positron in the state $(\mathbf{p}'', \varepsilon'')$. Taking the limit $t \to \infty$ in the time factor (8.18) gives

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}(\omega_{\mathbf{k}}-\mathrm{E}_{\mathbf{p}'}-E_{\mathbf{p}''})t'}\mathrm{d}t' = 2\pi\delta(\omega_{\mathbf{k}} - E_{\mathbf{p}'} - E_{\mathbf{p}''}).$$

Thus energy conservation, as well as momentum conservation, is explicit. In free space it is impossible to satisfy both these conservation laws in the case of pair production from a photon (Problem 8.3), so that first-order perturbation theory contributes nothing. (In the presence of an external electromagnetic field, for example the Coulomb field of a nucleus, momentum conservation between electrons and photons is lost, and pair production is possible if $\omega_{\mathbf{k}} > 2m$.)

https://doi.org/10.1017/9781009401685.010 Published online by Cambridge University Press

When the first-order transition amplitude at time t does not vanish, we have, using (8.16),

$$\langle \text{final state}|e\hat{O}_1(t)|\text{initial state}\rangle = \langle \text{f}|V(0)|i\rangle \int_{-t/2}^{t/2} \mathrm{e}^{-\mathrm{i}\Delta E t'} \mathrm{d}t',$$

where $\Delta E = E_\text{i} - E_\text{f}$ and E_i and E_f are the energies of the initial state $|i\rangle$ and final state $|f\rangle$. It is shown in textbooks on quantum mechanics that the time dependence can be interpreted as a transition probability per unit time, from the initial state i to the final state f, given by

transition probability $= 2\pi\,|\langle \text{f}|V(0)|i\rangle|^2 \rho(E_\text{f})$,
where$\rho(E_\text{f})$is the density of final energy states at $E_\text{f} = E_\text{i}$.

It is straightforward to extract higher order terms of the perturbation expansion. For example

$$\hat{O}_2(t) = \int_{-t/2}^{t/2} \mathrm{d}^4 x_2\, \bar{\psi}(x_2)\gamma^\mu \psi(x_2)\, A_\mu(x_2) \int_{-t/2}^{t_2} \mathrm{d}^4 x_1\, \bar{\psi}(x_1)\,\gamma^\mu \psi(x_1)\, A_\mu(x_1) \tag{8.19}$$

where $x_1 = (t_1, \mathbf{r}_1)$, $x_2 = (t_2, \mathbf{r}_2)$ and $-t/2 < t_2 < t/2$.

8.4 Renormalisation and renormalisable field theories

In second-order perturbation theory, we can pick out terms corresponding to the creation of an electron–positron pair at a point x_1 in space-time and its destruction at a point x_2. They may be characterised by the diagrams of Fig. 8.1. In these diagrams time runs from left to right. Momentum is conserved at x_1 and x_2. Overall there is also conservation of energy and angular momentum, so that the ‘unperturbed’ photon that emerges at time t_2 is in the same state as the initial unperturbed photon.

We pointed out that in free space it is not possible to create a real $\mathrm{e}^-\mathrm{e}^+$ pair from a photon. The $\mathrm{e}^-\mathrm{e}^+$ pair of the diagram is a virtual pair, corresponding to a term in a mathematical expansion. The transition amplitude

$$\langle \mathbf{k}|\mathrm{e}^{-\mathrm{i}H_0 t}\,\hat{O}_2(t)\,|\mathbf{k}\rangle = \mathrm{e}^{-i\omega_\mathrm{k} t}\,\langle \mathbf{k}|\,\hat{O}_2(t)\,|\mathbf{k}\rangle$$

is non-vanishing. The ‘real’ photon is evidently a complex object. Calculations show that the effect of virtual $\mathrm{e}^-\mathrm{e}^+$ pairs is to make the vacuum behave like an electrically polarisable medium. In particular, the Coulomb interaction between two ‘bare’ electrons is screened. We can envisage this effect as resulting from a screening cloud of virtual positrons around each bare electron, the corresponding

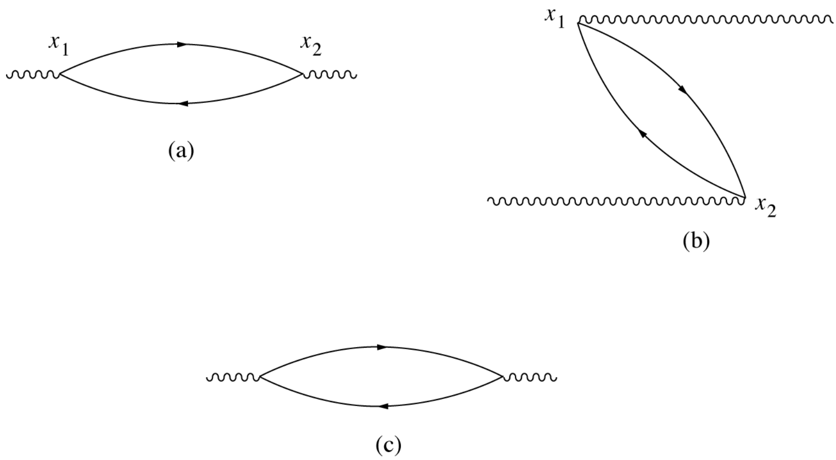

Figure 8.1 In these diagrams an unperturbed electron–positron pair is created at a point x_1 in space-time and destroyed at a point x_2. In (a) the initial unperturbed photon is destroyed at x_1 and recreated at x_2; vice versa in (b). In (a) and (b) time runs from left to right. As shown by Feynman it is convenient to characterise both processes by the single Feynman diagram (c). In all of these diagrams the arrows on the fermion lines follow the direction of electron number. (The arrows on positrons then run backwards in time.)

negative charge of the virtual e^-e^+ pairs appearing as charge at the surface of the confining volume.

What is measured experimentally as the charge $-e$ on an electron is the screened charge. To compensate for this screening effect, the parameter e that appears in the Lagrangian must be replaced by a 'bare' charge $e_o = e + \Delta e$. This gives 'counter terms' in the Lagrangian. Δe is chosen to cancel the screening effect. To second order the calculation gives $\Delta e = \alpha A_1 e$ where A_1 is a dimensionless quantity. With this adjustment and to this order, the screened charge on the electron becomes $-e$. In higher orders of perturbation theory one obtains

$$\Delta e = e[\alpha A_1 + \alpha^2 A_2 + \cdots].$$

To any order of perturbation theory an account must be kept of the readjustment of e, in order to extract from a calculation the significant physical effects which are also determined by terms in the perturbation expansion. The charge $-e$ on the electron is said to be *renormalised.* Δe itself can never be measured. Physical effects in atomic physics arising in part from vacuum polarisation terms have been calculated and measured with high precision. (See also Section 16.3.)

The other parameter appearing in electrodynamics is the mass of the electron. The bare mass of the electron is modified in second-order perturbation

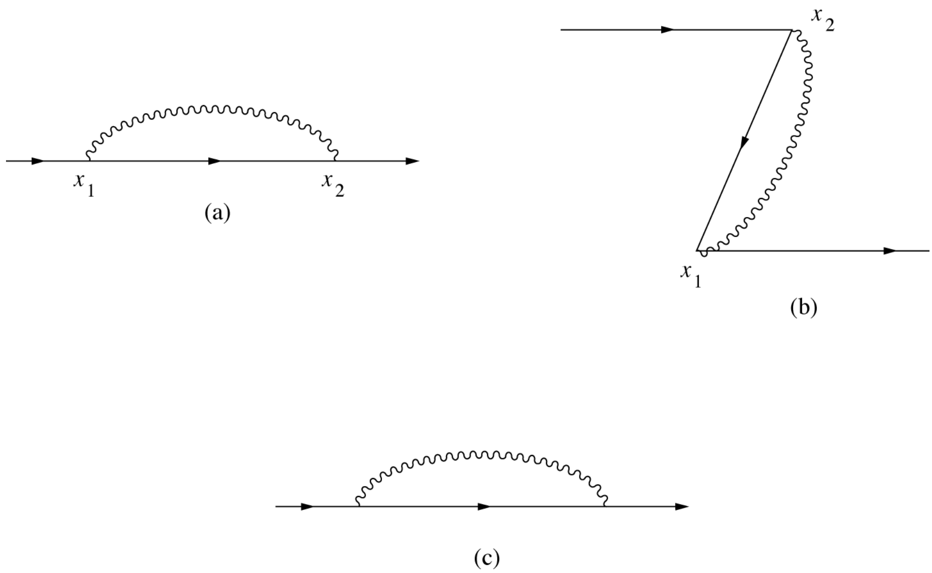

Figure 8.2 In these diagrams an unperturbed photon is created at a point x_1 in space-time and destroyed at a point x_2. In (a) the initial unperturbed electron is destroyed at x_1 and recreated at x_2; vice versa in (b). In (a) and (b) time runs from left to right. It is convenient to characterise both processes by the single Feynman diagram (c). In all of these diagrams the arrows on the fermion lines follow the direction of the electron number. (The arrows on positrons then run backwards in time.)

theory by the processes shown in Fig. 8.2. To compensate for these processes we must take $m_o = m - \Delta m$ in the Lagrangian where Δm is chosen to compensate for the shift in mass produced by the electron–photon interactions. We can think of the bare electron as 'dressed' by virtual photons. It is found that to second order $\Delta m = \alpha m B_1$, where B_1 is another dimensionless quantity, and more generally

$$\Delta m = m[\alpha B_1 + \alpha^2 B_2 + \cdots].$$

As with Δe, Δm has to be adjusted at each higher order of perturbation theory, and there is a systematic way of extracting physical answers from perturbation calculations. The physical mass m is the renormalised mass.

Diagrams like those of Fig. 8.3, in which virtual e^-e^+ pairs and virtual photons are created and annihilated together, give terms that modify the vacuum energy. Energy shifts in perturbation theory are to be expected, but since we have no unperturbed vacuum with which to compare, such shifts are not measurable. The cosmological constant of general relativity gives a measure of the vacuum energy

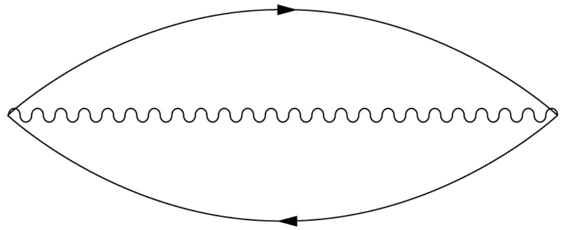

Figure 8.3 The vacuum state of quantum electrodynamics differs from the unperturbed vacuum by processes, one of which is illustrated in this figure.

density that is certainly very small, and is consistent with its being zero. We shall take the vacuum energy density, whatever its origin, to be zero.

It could have been anticipated without calculation that there would be perturbing effects of charge renormalisation and mass renormalisation. The unpalatable feature of quantum electrodynamics is that when the constants A_i, and B_i are calculated they all turn out to be infinite, as does the correction to the vacuum state energy. It is just as well that Δe and Δm have no physical significance. However, it is the case that an expansion in the small parameter α gives seemingly infinite corrections to quantities one cannot measure. An important feature of QED is that, leaving aside a scaling of the fields that is also part of the renormalisation scheme, infinities only appear in the renormalisation of the parameters of the theory, e, m and the vacuum energy. The only infinite counter terms that have to be added to the Lagrangian are contained in these parameters. Having made these adjustments, the remaining physical effects are calculable and finite.

QED is a local field theory, i.e. a theory in which the interaction terms involve a product of fields at the same point in space time. Infinities such as occur in QED are endemic in all local field theories. Field theories in which the infinities only appear in a finite number of parameters of the theory are said to be *renormalisable.*

The divergences in the coefficients A_i of Δe and B_i of Δm arise, for example, in the contribution from O_2 (see (8.19)), from the integration region where $x_2 \approx x_1$ and in particular where $\mathbf{r_2} \approx \mathbf{r_1}$. An important feature of QED is that the expansion parameter α and hence the coefficients, are dimensionless numbers. In Chapters 9 and 21 we will encounter theories in which the coupling constants and therefore the expansion parameters have the dimensions of inverse powers of mass. All the terms in perturbation expansions must have the same dimension, therefore the coefficients have a dimension to compensate those of the coupling constant. In the integration regions the integrands diverge with large inverse powers of $|\mathbf{r}_2 - \mathbf{r}_1|$ as $\mathbf{r}_2 \to \mathbf{r}_1$ to achieve the compensation, but they render the integrals infinite. Infinities occur for all multiparticle interactions, they can not be removed just by mass and coupling constant renormalisation. Such theories are unrenormalisable, they can not be taken seriously as quantum field theories.

8.5 The magnetic moment of the electron

We shall now illustrate the remarkable success of QED in calculating quantities of physical significance by giving an account of the calculation of the electron's magnetic moment. In Chapter 7 we showed that the Dirac equation before second quantisation implies that the electron carries a magnetic moment of magnitude $\mu_{\rm B} = e\hbar/2m$ anti-aligned with its spin. The electron's magnetic moment has been measured with high precision: the experimental value $\mu_{\rm e}$ is

$$\mu_{\rm e} = \mu_{\rm B}\,(1 + a)$$

where the 'anomaly' $a = 0.001\,159\,652\,188\,4(43)$ (Van Dyck *et al.*, 1987).

After second quantisation, the perturbative corrections to the Dirac value can be calculated. The Dirac value is contained in the operator $\hat{O}_1$ of equation (8.16), and is associated with diagram (a) of Fig. 8.4. This lowest order calculation reproduces the Dirac result $\mu_{\rm e} = \mu_{\rm B}$.

Since $\mu_{\rm B}$ is the only combination of the parameters e, m_e and $\hbar$ which has the dimensions of magnetic moment, higher orders of perturbation theory will give terms of the form

$$\mu_{\rm e} = \mu_{\rm B}(1 + \alpha C_1 + \alpha^2 C_2 + \alpha^3 C_3 + \alpha^4 C_4 + \cdots),$$

where the C_i are dimensionless constants. To compare the theory with experiment we use the 1986 adjusted value of the fine structure constant,

$$\alpha^{-1} = 137.035\,9979\,(32)\,.$$

C_1 is associated with diagram (b) of Fig. 8.4; the calculation gives $C_1 = 1/(2\pi)$. Hence to this order

$$a = C_1\alpha = 0.001\,161\,409\,74,$$

which agrees with experiment to within five significant figures.

The next order correction, associated with diagrams (c) of Fig. 8.4, is

$$C_2 = \frac{1}{\pi^2}\left(\frac{197}{144} + \frac{3}{4}\zeta\,(3)\right) - \frac{1}{2}\ln 2 + \frac{1}{12}$$

where $\zeta(z)$ is the Riemann zeta function. To this order,

$$a = 0.001\,159\,637\,44,$$

in agreement to seven significant figures.

https://doi.org/10.1017/9781009401685.010 Published online by Cambridge University Press

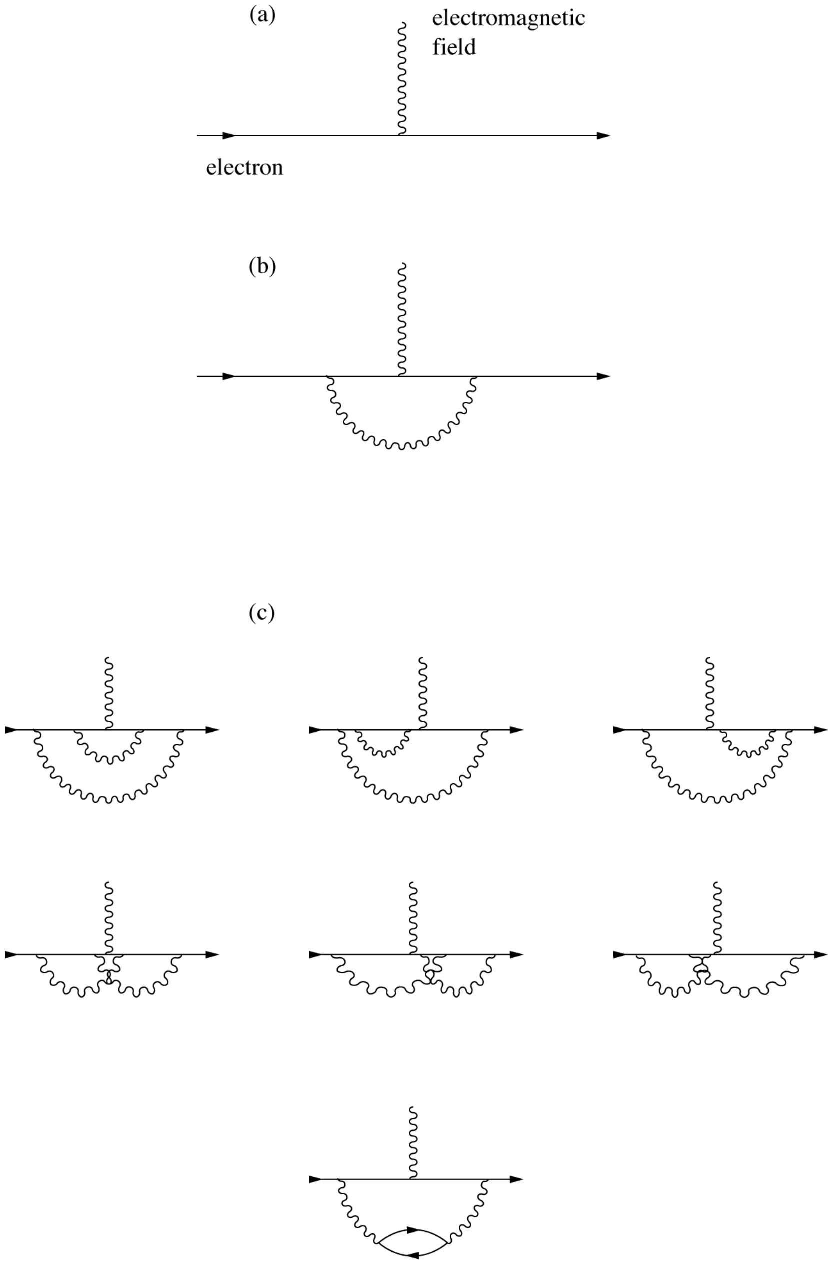

Figure 8.4 Perturbation theory Feynman diagrams that represent contnbutions to the electron magnetic moment. The anomalous moment, to order α^2, comes from calculations associated with diagrams (b) and (c).

https://doi.org/10.1017/9781009401685.010 Published online by Cambridge University Press

Calculations of higher orders of perturbation theory become rapidly more intractable. Numerical estimates give $C_3 \approx 0.03792$, $C_4 \approx -0.014$. At this level of accuracy, corrections have to be made for processes that come from other parts of the Standard Model, in particular from the muon. The most recent comprehensive calculations (Kinoshita and Lindquist, 1990) give

$$a = 0.001\,159\,652\,140\,0\,(41 + 53 + 271),$$

in agreement with experiment to ten significant figures. The largest error in the theory is from the uncertainty in α^{-1}.

Within its range of applicability, quantum electrodynamics provides an astonishingly exact model of Nature. One may have some confidence that the techniques of renormalisation in perturbation theory are valid.

8.6 Quantisation in the Standard Model

In this chapter we have outlined the ‘canonical quantisation’ techniques that have been particularly successful in quantum electrodynamics. Many books have been written on this subject, for example Itzykson and Zuber (1980); some will have to be consulted if one is to be competent and confident in making detailed calculations. However, many of the decay rates and cross-sections given in the following chapters, which are needed to compare the predictions of the Standard Model with experiment, are quite well approximated by the so-called ‘tree level’ of perturbation theory. The tree-level diagrams have no closed loops (see Fig. 8.4(a)) and require no renormalisation. It is a fortunate circumstance that in low orders of perturbation theory these can be calculated quite easily.

The particles and forces of the weak and the strong interactions are also described by local gauge field theories, which will be exhibited at the classical level in the chapters that follow. The quantisation procedures used in these extensions of QED have been most successfully pursued by the path integral method of quantisation (see, for example, Cheng and Li (1984)). Both the theory of the weak interaction and the theory of the strong interaction pose their own special problems, but the principles of gauge symmetry and renormalisability have been essential in the construction of the Standard Model as it is today.

Problems

8.1 A general two-particle state of scalar bosons (Section 8.1) can be written

$$|\text{state}\rangle = \sum_{\mathbf{k}_1,\mathbf{k}_2} f(\mathbf{k}_1, \mathbf{k}_2)\, a^\dagger_{\mathbf{k}1}\, a^\dagger_{\mathbf{k}2}|0\rangle,$$

where, apart from normalisation, $f(\mathbf{k}_1, \mathbf{k}_2)$ is any function of $\mathbf{k}_1$ and $\mathbf{k}_2$. (f can be called the wave function of the state.)

Show that this state may be written

$$|\text{state}\rangle = \sum_{\mathbf{k}_1,\mathbf{k}_2} g\,(\mathbf{k}_1, \mathbf{k}_2)\, a^{\dagger}_{\mathbf{k}1}\, a^{\dagger}_{\mathbf{k}2}|0\rangle$$

with $g(\mathbf{k}_1, \mathbf{k}_2) = \{f(\mathbf{k}_1, \mathbf{k}_2) + f(\mathbf{k}_2, \mathbf{k}_1)\}/2$, symmetric under the interchange of labelling.

8.2 A general two-particle state of fermions can be written

$$|\text{state}\rangle = \sum_{\mathbf{p}_1,\varepsilon_1,\mathbf{p}_2,\varepsilon_2} f\,(\mathbf{p}_1, \varepsilon_1, \mathbf{p}_2, \varepsilon_2)\, b^{\dagger}_{\mathbf{p}_1\varepsilon_1} b^{\dagger}_{\mathbf{p}_2\varepsilon_2}|0\rangle$$

where apart from normalisation f is any function of $\mathbf{p}_1$, ε_1 and $\mathbf{p}_2$, ε_2.

Show that this state can also be written

$$|\text{state}\rangle = \sum_{\mathbf{p}_1,\varepsilon_1,\mathbf{p}_2,\varepsilon_2} g\,(\mathbf{p}_1, \varepsilon_1, \mathbf{p}_2, \varepsilon_2)\, b^{\dagger}_{\mathbf{p}_1\varepsilon_1} b^{\dagger}_{\mathbf{p}_2\varepsilon_2}|0\rangle$$

with $g(\mathbf{p}_1, \varepsilon_1; \mathbf{p}_2, \varepsilon_2) = \{f(\mathbf{p}_1, \varepsilon_1; \mathbf{p}_2, \varepsilon_2) - f(\mathbf{p}_2, \varepsilon_2; \mathbf{p}_1, \varepsilon_1\}/2$, antisymmetric under the interchange of labelling.

8.2 Use energy and momentum conservation to show that pair creation by a single photon, $\gamma \to e^+ + e^-$, is impossible in free space.

8.3 The energy density of an electromagnetic field is given by equation (4.24). Show that the total electric field energy of a point charge q outside a sphere of radius R centred on the particle is

$$\text{energy} = q^2/(8\pi R).$$

Note that this classical contribution to the particle rest energy is infinite in the limit $R \to 0$.

https://doi.org/10.1017/9781009401685.010 Published online by Cambridge University Press