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ON THE PRODUCT OF THE PRIMES

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In memory of Leo Moser, a friend and teacher for many years

1. In recent years several attempts have been made to obtain estimates for the product of the primes less than or equal to a given integer n. Denote by $A(n)=\prod_{p\le n} p$ the above-mentioned product and define as usual

$$\Theta(n) = \sum_{p\le n} \log p \quad \text{and} \quad \Psi(n) = \sum_{p^\alpha \le n} \log p.$$

Analysis of binomial and multinomial coefficients has led to results such as $A(n) < 4^n$, due to Erdös and Kalmar (see [2]). A note by Moser [3] gave an inductive proof of $A(n) < (3.37)^n$, and Selfridge (unpublished) proved $A(n) < (3.05)^n$. More accurate results are known, in particular those in a paper of Rosser and Schoenfeld [4] in which they prove $\Theta(n) < 1.01624n$; however their methods are considerably deeper and involve the theory of a complex variable as well as heavy computations. Using only elementary methods we will prove the following theorem, which improves the results of [2] and [3] considerably.

THEOREM 1. *Let $B(n)$ denote the least common multiple of the integers* $1, 2, \ldots, n$. *Then $B(n) < 3^n$.*

Note that for a given prime p, if α_p is such that p^{α_p} is the highest power of p not exceeding n, then $B(n)$ is the product of the p^{α_p} taken over all primes $p \le n$. That is

$$B(n) = \prod_{p\le n} p^{\alpha_p} \quad \text{or} \quad B(n) = \prod_{p^\alpha \le n} p.$$

2. Before proving Theorem 1 we must first prove a number of preliminary lemmas.

LEMMA 1. *If $a_1, a_2, \ldots, a_k$ are positive integers such that*

$$\sum_{i=1}^{k} \frac{1}{a_i} \le 1 \textit{ and if } a_k > x \ge 1 \textit{ for } x \textit{ real, then}$$

$$[x] > \sum_{i=1}^{k} \left[\frac{x}{a_i}\right] \tag{2.1}$$

where the square brackets denote the greatest integer function.

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Proof. Using the fact that $[a/m]=[[a]/m]$ if m is a positive integer we have

$$\sum_{i=1}^{k}\left[\frac{x}{a_i}\right] = \sum_{i=1}^{k-1}\left[\frac{x}{a_i}\right] = \sum_{i=1}^{k-1}\left[\frac{[x]}{a_i}\right] \le \sum_{i=1}^{k-1}\frac{[x]}{a_i} \le [x]\left(1-\frac{1}{a_k}\right) < [x].$$

We now choose a particular set of a_i's defined as follows:

$$a_1 = 2, \qquad a_{n+1} = a_1a_2\ldots a_n+1.$$

A simple induction shows that the a_i's defined in this manner satisfy the following recurrence relation: $a_1=2$, $a_{n+1}=a_n^2-a_n+1$. It is easy to see that the a_i's also satisfy the conditions of lemma 1.

Define

$$C(n) = \frac{n!}{[n/a_1]\,!\,[n/a_2]\,!\,[n/a_3]\,!\ldots} \tag{2.2}$$

where the a_i's are as above. $C(n)$ may be seen to be an integer upon comparison to the appropriate multinomial coefficient.

LEMMA 2. *Let $\beta_p(n)$ be defined by $C(n)=\prod_{p\le n} p^{\beta_p(n)}$. Then $\beta_p(n)\ge[\log_p n]$.*

Proof. By Legendre's formula

$$\beta_p = \sum_{i=1}^{[\log_p n]}\left(\left[\frac{n}{p^i}\right]-\left[\frac{n}{a_1p^i}\right]-\left[\frac{n}{a_2p^i}\right]-\cdots\right).$$

That each term in this sum is at least 1 now follows from Lemma 1 by taking $x=n/p^i$. This proves Lemma 2.

LEMMA 3.

$$\frac{(n/a_i)^{n/a_i}}{[n/a_i]^{[n/a_i]}} < \left(\frac{en}{a_i}\right)^{(a_i-1)/a_i}, \quad n \ge a_i.$$

Proof. If $n=a_i$ the result is trivial. If $n>a_i$ we have

$$\begin{aligned}\frac{(n/a_i)^{n/a_i}}{[n/a_i]^{[n/a_i]}} &\le \frac{(n/a_i)^{n/a_i}}{((n-a_i+1)/a_i)^{(n-a_i+1)/a_i}}\\ &= \left(1+\frac{1}{(n-a_i+1)/(a_i-1)}\right)^{((n-a_i+1)/(a_i-1))\times((a_i-1)/a_i)}\left(\frac{n}{a_i}\right)^{(a_i-1)/a_i}\\ &< \left(\frac{en}{a_i}\right)^{(a_i-1)/a_i}.\end{aligned}$$

We will now proceed to obtain upper bounds for $C(n)$ using the preceding lemmas.

LEMMA 4.

$$C(n) < \frac{n^n}{[n/a_1]^{[n/a_1]}[n/a_2]^{[n/a_2]}\ldots[n/a_k]^{[n/a_k]}}$$

for a particular $k=k(n)$.

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Proof. If $n=n_1+n_2+\cdots+n_k$, where n and n_i, $i=1,2,\ldots,k$, are positive integers. Then by the multinomial theorem we know that

$$(n_1+n_2+\cdots+n_k)^n > (n_1,n_2,\ldots,n_k)n_1^{n_1}n_2^{n_2}\ldots n_k^{n_k}, \tag{2.3}$$

since the right-hand side of (2.3) is just one term in the expansion of

$$(n_1+n_2+\cdots+n_k)^n.$$

Let k be the least integer such that $a_{k+1}>n$ and let $\sum_{i=1}^{k}[n/a_i]=t\leq n$, then

$$C(n) = \frac{n(n-1)\ldots(t+1)t!}{[n/a_1]!\,[n/a_2]!\ldots[n/a_k]!} < \frac{n^{n-t}t^t}{[n/a_1]^{[n/a_1]}[n/a_2]^{[n/a_2]}\ldots[n/a_k]^{[n/a_k]}}$$

by (2.3), and the lemma follows.

The magnitude of k satisfies the following:

Lemma 5. *If* $a_k\leq n<a_{k+1}$, *then*

$$k < \log_2\log_2 n+2 \quad \text{for } k\geq 3. \tag{2.4}$$

Proof. We know $a_{k+1}=a_k^2-a_k+1$ and $a_3=7>2^{2^1}+1$. Therefore, inductively

$$a_{k+1} > 2^{2^{k-1}}+1$$

and

$$k < \log_2\log_2(a_k-1)+2 < \log_2\log_2 n+2.$$

Finally, applying Lemmas 3, 4 and 5 we have, if k is such that $a_k\leq n<a_{k+1}$,

$$C(n) < \frac{n^n(en/a_1)^{(a_1-1)/a_1}(en/a_2)^{(a_2-1)/a_2}\ldots(en/a_k)^{(a_k-1)/a_k}}{(n/a_1)^{n/a_1}(n/a_2)^{n/a_2}(n/a_3)^{n/a_3}\ldots} \tag{2.5}$$

since

$$[n/a_t] = 0,\ [n/a_t]! = 1 \quad\text{and}\quad \frac{1}{(n/a_t)^{n/a_t}} > 1 \quad \text{for all } t > k.$$

We observe that the product $a_1^{1/a_1}a_2^{1/a_2}\ldots a_k^{1/a_k}$ is monotonic increasing with k. Since

$$a_{i+1} = a_i^2-a_i+1, \quad a_i^2 > a_{i+1} > (a_i-1)^2 \text{ for } i\geq 1.$$

Therefore

$$\frac{\log a_{i+1}^{1/a_{i+1}}}{\log a_i^{1/a_i}} = \frac{a_i\log a_{i+1}}{a_{i+1}\log a_i} < \frac{2a_i}{a_{i+1}} < \frac{2a_i}{(a_i-1)^2} < \frac{1}{2} \quad \text{for } i\geq 3.$$

It now follows since $\log a_6^{1/a_6}<5\times 10^{-6}$ that

$$\sum_{i=1}^{\infty}\log a_i^{1/a_i} = \sum_{i=1}^{5}\log a_i^{1/a_i}+\sum_{i=6}^{\infty}\log a_i^{1/a_i} < 1.08240+10^{-5}.$$

That is, if we define

$$w = \lim_{k\to\infty}(a_1^{1/a_1}a_2^{1/a_2}\ldots a_k^{1/a_k})$$

that $w<2.952$.

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Observe that

$$\frac{a_1-1}{a_1}+\frac{a_2-1}{a_2}+\cdots+\frac{a_k-1}{a_k} = \left(1-\frac{1}{a_1}\right)+\cdots+\left(1-\frac{1}{a_k}\right)$$
$$= k-1+\frac{1}{a_{k+1}+1}.$$

It now follows from (2.5) that

$$C(n) < \frac{(en)^{k-1+1/(a_{k+1}+1)}w^n}{a_1^{(a_1-1)/a_1}a_2^{(a_2-1)/a_2}\ldots a_k^{(a_k-1)/a_k}} \tag{2.6}$$
$$< e^{k-3/2}n^{k-3/2}w^n, \quad k > 2 \text{ (since } n \le a_1a_2\ldots a_k).$$

A check of tables reveals $C(n)<3^n$ for $n>1300$ and a check of tables of $\Psi(n)$, such as those of Appel and Rosser [1], for $n \le 1300$ concludes the proof of Theorem 1.

3. Obtaining a lower bound for the product of the primes by similar methods leads to a less elegant result for small n. If we define

$$D(n) = \frac{n!}{[n/2]!\,[n/3]!\,[n/6]!}$$

it can be shown

$$\frac{(2^4 3^3)^{n/6}}{n^2} < D(n) < \prod_{p\le n} p \prod_{p \le n/5} pn^{n^{n/6}}$$

Theorem 1 now implies

$$\Theta(n) > 0.79169n-(2+n^{1/2})\log n > \tfrac{3}{4}n \quad \text{for } n > 8\times 10^4. \tag{3.1}$$

A simple check of tables shows that (3.1) holds for $n>13$.

Let $\pi(x)$ denote the number of primes less than or equal to x.

$$\pi(x) = \sum_{p\le x} 1 = \sum_{n=2}^{x} \frac{\Psi(n)-\Psi(n-1)}{\log n}.$$
$$= \sum_{n=2}^{x} \Psi(n)\left(\frac{1}{\log n}-\frac{1}{\log(n+1)}\right)+\frac{\Psi(x)}{\log x}.$$

It can be shown by Theorem 1 that

$$\pi(x) < \frac{x\log 3}{\log x}+\log 3\left(\frac{1}{\log^2 2}+\frac{1}{\log^2 x}+40\right)$$
$$< \frac{5}{4}\frac{x}{\log x} \quad \text{for } x \ge 25{,}000.$$

A direct check of tables (such as [1]) for values of $x<25{,}000$ implies

$$\pi(x) < \frac{5x}{4\log x}$$

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for $1<x<113$ and $x\geq 114$, and for $x=113$

$$\pi(x) = 1.25506\frac{x}{\log x}.$$

That is $\prod(x)/(x/\log x)$ is a maximum for $x=113$.

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