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A PROBLEM ON GROWTH SEQUENCES OF GROUPS

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Abstract

The aim of this paper is to consider Problem 1 posed by Stewart and Wiegold in [6]. The main result is that if G is a finitely generated perfect group having non-trivial finite images, then there exists a finite image B of G such that the growth sequence of B is eventuallly faster than that of every finite image of G. Moreover we investigate the growth sequences of simple groups of the same order.

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Introduction

Let G be a finitely generated group, and G^n the nth direct power of G. The growth sequence of G is the sequence $\{d(G^n)\}$, where $d(G^n)$ is the minimum number of generators of G^n. Wiegold gave a very tight description on the growth sequences of finite groups in [5,7,8,9,10]. The rough picture is that if G is perfect, the growth sequence of G increases roughtly logarithmically in n and if G is not perfect, then $d(G^n) = nd(G/G')$ for large enough n. For infinite groups, the situation is less clear. If G is perfect, the growth sequence of G is bounded above by a logarithmic function of n and if G is not perfect, then again we have $d(G^n) = nd(G/G')$ for large enough n (see [11]). There are several difficult problems left in the case of infinite groups; in particular, when G is a finitely generated perfect group.

The present article considers Problem 1 posed by Stewart and Wiegold in [6], as follows:

PROBLEM. Let G be a finitely generated group having a non-trivial finite image. Is the growth sequence of G eventually the same as that of a finite image of G?

This is certainly true if G is not perfect. For, by the above remarks, $d(G^n) = nd(G/G')$ for large n, in this case. However, as a finitely generated abelian group, G/G' has a finite image X of prime-power order with $d(G/G') = d(X)$, and for all n, $d(X^n) = nd(X) = nd(G/G')$. In this paper, we prove that there exists a finite image B of G such that the growth sequence of B is eventually as large as that of every finite image of G. This question was left undecided in [6]. So the problem shortens to this: Is the growth sequence of G eventually the same as that of B? We have not been able to resolve this, but we believe that the growth sequence of G could be faster than B in some cases. We prove the following result, which is an improvement of [6, Theorem A].

THEOREM A$'$. *Let G be a finitely generated perfect group having non-trivial finite images. Then there is a finite image B of G such that the growth sequence of B is eventually as large as that of every finite image of G; that is to say, there exists a positive integer K depending only on G such that for every finite image H of G, $d(B^n) \geq d(H^n)$ for $n \geq K$.*

PROOF. Suppose that S is a non-trivial finite image of G of the smallest order. From the classification of the finite simple groups and a theorem of Artin [1], there are up to isomorphism at most two possibilities for S. If S is unique, then by Theorem A in [6] we can choose $B = S^\lambda$, where S^λ is the hightest power of S that is an image of G. So let us consider the case when S is not unique. Suppose that S and T are two non-trivial images of G of the smallest order, and let S^λ and T^μ be the highest powers of S and T respectively that are images of G. Suppose that H is a non-trivial finite image of G; let $S_1, S_2, \dots, S_r$ be the simple images of H, and $S_1^{\lambda_1}, S_2^{\lambda_2}, \dots, S_r^{\lambda_r}$ the highest powers of $S_1, S_2, \dots, S_r$ that are images of H. There are three cases for S and T, as follows:

CASE 1. Suppose that S and T occur among $S_1, S_2, \dots, S_r$; say $S = S_1$ and $T = S_2$. By a result of Gaschütz [3] (See [8, 10]),

$$d(H^n) = \max\left\{d(H), d(S_1^{\lambda_1 n}, \dots, d(S_r^{\lambda_r n})\right\}$$

for all n. By the second part of the proof of Theorem A in [6], there is a number L depending only on G such that $d(H^n) = \max\left\{d(H), d(S^{\lambda_1 n}), d(T^{\lambda_2 n})\right\}$ for all $n \geq L$. Thus $d(H^n) = \max\left\{d(S^{\lambda_1 n}), d(T^{\lambda_2 n})\right\}$ provided $n \geq L$, $d(S^{\lambda_1 n}) \geq d(G)$ and $d(T^{\lambda_2 n}) \geq d(G)$. Since this holds whenever $\log_s \lambda_1 n \geq d(G)$ and $\log_s \lambda_2 n \geq d(G)$ where $s = |S|$, and also

$$d(S^{\lambda n} \times T^{\mu n}) \geq d(S^{\lambda_1 n} \times T^{\lambda_2 n}) \geq d(H^n)$$

by [7], we can therefore choose $B = S^\lambda \times T^\mu$ in this case.

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CASE 2. Suppose that S is one of $S_1, S_2, \ldots, S_r$, say $S = S_1$, but T is not. Again by the same method as in Case 1, there is a constant K depending only on G such that $d(S^{\lambda n}) \geq d(H^n)$. It is clear that $d(S^{\lambda n} \times T^{\mu n}) \geq d(H^n)$, so we can choose $B = S^\lambda \times T^\mu$ here.

CASE 3. Suppose that S and T are not among $S_1, S_2, \ldots, S_r$. As in the first part of proof of Theorem A in [6], we see that $d(S^{\lambda n}), d(T^{\mu n}) \geq d(H^n)$ for large n. So there are three possibilities for λ and μ as follows:

(i) $\lambda < \mu$: It is clear that $d(S^{\lambda n}) \geq d(T^{\lambda n})$ and $d(T^{\mu n}) \geq d(s^{\lambda n} \times T^{\lambda n})$, because T^λ and $S^\lambda \times T^\lambda$ are images of G. So we can choose $B = T^\mu$.

(ii) $\mu < \lambda$: As in (i), we see that S^λ works.

(iii) $\lambda = \mu$: Here we can choose $B = S^\lambda \times T^\mu$ and the proof of Theorem A′ is complete.

It is possible that S^λ or T^λ can be chosen for B in case (iii), but we have been unable to check this. Let us consider the case $\lambda = \mu = 1$ as an example. Then we have two finite images S and T of G of the smallest order (which must, of course, be simple). It follows from the classification of the finite simple groups and a theorem of Artin [1] that the possibilities for S and T are as follows:

(a) $S = A_8, T = PSL(3, 4)$.

(b) $S = PSp(2m, q), T = O'(2m+1, q)$ where $m \geq 3$ and q is an odd prime-power.

THEOREM B. *Suppose that $S = A_8$ and $T = PSL(3, 4)$. Then $d(S^n) \leq d(T^n)$ for large enough n.*

PROOF. For any finite group U, set $h(m, U) = \max\{n : d(U^n) \leq m\}$. By [7], we have $h(m, S) = |\operatorname{Aut} S|^{-1}|S|^m(1 - \epsilon(m))$, and $h(m, T) = |\operatorname{Aut} T|^{-1}|T|^m(1 - \eta(m))$, where $\eta(m), \epsilon(m) \to 0$ as $m \to \infty$. Thus, as $m \to \infty$, $h(m, S)/h(m, T) \to |\operatorname{Aut} T|/|\operatorname{Aut} S| = 6$ by [2]. Thus $h(m, S) > h(m, T)$ for large enough m and then $d(S^n) \leq d(T^n)$ for large n.

COROLLARY. *For each λ in* (iii) *of Case* 3, *S and T as Theorem B, the growth sequence of T^λ is faster than that of every finite image of G in Theorem A′.*

We can say less about the second possibility for S and T. However, the difference in the growth sequence is very small indeed:

THEOREM C. *Suppose that $S = PSp(2m, q)$ and $T = O'(2m + 1, q)$ with $m \geq 3$, q an odd prime-power. Then $|d(S^n) - d(T^n)| = 0$ or 1 for large enough n.*

PROOF. We know that $|\operatorname{Aut} S| = |\operatorname{Aut} T|$ by Liebeck, Praeger and Saxl [4]. Set $s = |S|$ and $a = |\operatorname{Aut} S|$. We have (see [7]) for sufficiently large n,

$$\log_s n + \log_s a < d(S^n), d(T^n) \leq \log_s n + \log_s a + 1 + \varphi(n)$$

where $\varphi(n) \to 0$ as $n \to \infty$. In considering upper and lower bounds, two cases arise.

(1) $\log_s n + \log_s a$ is an integer. For large n, it is clear that $d(S^n) = d(T^n) = \log_s n + \log_s a + 1$.

(2) $\log_s n + \log_s a$ is not an integer. We see easily that for large n, $d(S^n)$ and $d(T^n)$ are both one of the two smallest integers greater than $\log_s n + \log_s a$. Thus $|d(S^n) - d(T^n)| \leq 1$ and the proof of the theorem is complete.

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