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The Heisenberg covering of the Fermat curve

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Abstract. For N integer ≥ 1, K. Murty and D. Ramakrishnan defined the Nth Heisenberg curve, as the compactified quotient X'_N of the upper half-plane by a certain non-congruence subgroup of the modular group. They ask whether the Manin–Drinfeld principle holds, namely, if the divisors supported on the cusps of those curves are torsion in the Jacobian. We give a model over $\mathbf{Z}[\mu_N, 1/N]$ of the Nth Heisenberg curve as covering of the Nth Fermat curve. We show that the Manin–Drinfeld principle holds for $N = 3$, but not for $N = 5$. We show that the description by generator and relations due to Rohrlich of the cuspidal subgroup of the Fermat curve is explained by the Heisenberg covering, together with a higher covering of a similar nature. The curves X_N and the classical modular curves $X(n)$, for n even integer, both dominate $X(2)$, which produces a morphism between Jacobians $J_N \to J(n)$. We prove that the latter has image 0 or an elliptic curve of j-invariant 0. In passing, we give a description of the homology of X'_N.

1 Introduction

Let Γ be a subgroup of finite index of $\mathrm{SL}_2(\mathbb{Z})$. This subgroup acts by homographies on the complex upper half-plane $\mathfrak{H}$. Consider the corresponding modular curve $Y_\Gamma = \Gamma\backslash\mathfrak{H}$, and its compactification obtained by adding the cusps X_Γ. We say that X_Γ *satisfies the Manin–Drinfeld* principle if any cuspidal (i.e., supported on the cusps) divisor of degree 0 is torsion in the Jacobian of X_Γ. Manin and Drinfeld proved that it is the case when Γ is a congruence subgroup.

For a subgroup of finite index (not necessarily a congruence subgroup), K. Murty and Ramakrishnan [13] give an analytic criterion for the Manin–Drinfeld principle to be satisfied. As an illustrative example, Murty and Ramakrishnan consider modular curves attached to certain subgroups of $\Gamma(2)$: Fermat curves, and what they propose to call Heisenberg curves. We revisit those examples. In [2], we reconsider this question and give an analytic criterion of a different nature, but also based on Eisenstein series; this is unconnected to the present work, which is purely algebraic in nature.

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The Heisenberg curves are defined as follows from the complex analytic point of view. Let A and B be the classes of the matrices $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$, respectively, in $\bar{\Gamma}(2) = \Gamma(2)/\{\pm 1\}$. These matrices generate freely the group $\bar{\Gamma}(2)$. Let $C = ABA^{-1}B^{-1}$.

Let N be an integer > 0. Denote by Φ_N the kernel of the morphism $\bar{\Gamma}(2) \to (\mathbb{Z}/N\mathbb{Z})^2$ which to A associates $(1,0)$ and to B associates $(0,1)$. The corresponding modular curve is the *Fermat modular curve* and is denoted by X_N. It can be identified with the complex points of the Fermat curve F_N (see, for instance, [7, 17]). Let Φ'_N be the subgroup of Φ_N generated by A^N, B^N, C^N and the third term $[\bar{\Gamma}(2), [\bar{\Gamma}(2), \bar{\Gamma}(2)]]$ in the descending central series of $\bar{\Gamma}(2)$. An exact sequence of groups follows:

$$1 \to \Phi'_N \to \bar{\Gamma}(2) \to H_N \to 1,$$

where H_N is a central extension of $(\mathbb{Z}/N\mathbb{Z})^2$ by $\mathbb{Z}/N\mathbb{Z}$ (coinciding with the $\mathbb{Z}/N\mathbb{Z}$-points of the Heisenberg group). The *Nth Heisenberg modular curve*, in the sense of Murty and Ramakrishnan, is $X'_N = X_{\Phi'_N}$.

Let $\mathbb{Q}(\mu_N)$ be a cyclotomic extension of $\mathbb{Q}$ generated by Nth roots of unity. Denote by $\mathbb{Z}[\mu_N]$ its ring of integers. The covering $X'_N \to X_N$ extends to a morphism $F'_N \to F_N$ of curves over $\mathbb{Q}(\mu_N)$ that we call the *Heisenberg covering of the Fermat curve*.

Theorem 1.1 *Suppose N is an odd integer. The Heisenberg modular curve X'_N extends to a smooth projective scheme $\mathcal{F}'_N$ of relative dimension one over* $\mathrm{Spec}(\mathbb{Z}[\mu_N, 1/N])$ *given by the following model:*

$$X^N + Y^N = Z^N$$

and, for every primitive Nth root of unity ζ in $\mathbb{Q}(\mu_N)$

$$\prod_{j=1}^{(N-1)/2} (Y - \zeta^{-j}Z)^j T_\zeta^N = \prod_{j=1}^{(N-1)/2} (Y - \zeta^j Z)^j U_\zeta^N.$$

It seems to have been known to Deligne (see a comment in [13]) that the generic fiber F'_N of $\mathcal{F}'_N$ can be defined over $\mathbb{Q}$, an assertion for which we provide a proof.

Rohrlich [17] (see also Vélu [18]) has determined the cuspidal subgroup of F_N. In particular, he has shown that any cuspidal divisor on F_N is of order dividing N. This description plays a key role in justifying the existence of the Heisenberg covering. We show that, by going further in the descending central series of $\Gamma(2)$, X'_N is covered by a modular curve X''_N, in such a way that X''_N is still an abelian covering of the Fermat curve X_N. We do not describe algebraically X''_N.

We note that there has been a considerable interest in the cuspidal group of the Fermat curve. For instance, in [10, p. 39], Mazur draws (or rather "stretches") an analogy between Fermat curves and modular curves. Such an analogy is somewhat strengthened by the fact that the Heisenberg covering is analogous to the familiar Shimura covering $X_1(N) \to X_0(N)$ between modular curves.

Like Murty and Ramakrishnan, our goal had been to illustrate our study of non-congruence subgroups by examining Heisenberg curves. We can not determine in general whether such curves satisfy the Manin–Drinfeld principle. But we can show easily that the principle holds for $N = 3$. Furthermore, for $N = 3$, we study

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the connection between F_3' and various modular curves. By contrast, we have the following theorem for $N = 5$:

Theorem 1.2 *There exists a cuspidal divisor on X_5' whose class in the Jacobian of X_5' is of infinite order.*

Let $\bar{\Gamma}'(2)$ be the subgroup of index 3 of $\bar{\Gamma}(2)$ obtained by pulling back the 2-Sylow subgroup of $\mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$. Let Γ be a congruence subgroup of $\bar{\Gamma}(2)$. Consider the correspondence $X_N' \to X_\Gamma$ obtained by combining pulling back to the modular curve $X_{\Gamma\cap\Phi_N'}$ with pushing to X_Γ. It produces a morphism of abelian varieties between the Jacobians or Riemann surfaces $\theta_{N,\Gamma}\colon J_N' \to J_\Gamma$. In view of the following statement, we can hardly have any hope of establishing a limited form of the Manin–Drinfeld principle for Heisenberg curves using the classical Manin–Drinfeld theorem for congruence subgroups.

Theorem 1.3 *The morphism $\theta_{N,\Gamma}$ is zero if and only if either $3 \nmid N$ or Γ is not contained in $\bar{\Gamma}'(2)$. If $3 \mid N$ and Γ is contained in $\bar{\Gamma}'(2)$, the image of $\theta_{N,\Gamma}$ is isogenous to an elliptic curve with j-invariant 0. Furthermore, when Γ is contained in $\bar{\Gamma}'(2)$, $\theta_{3,\Gamma}$ has finite kernel.*

The proof of Theorem 1.3 is a translation of a group theoretic statement: any term of the lower central series of $\bar{\Gamma}(2)$ is essentially dense in adelic completions of $\bar{\Gamma}(2)$ (see Proposition 2.10). In addition to these results of algebraic nature, we give a combinatorial description of the homology of the Riemann surface X_N', by a method similar to Manin's presentation, but following the variant introduced in [12]. This might have an interest of its own. But it does not help us for establishing our other results.

One of the referees noted connections to the work of Anderson and Ihara [1], as well as unpublished computations of Deligne. We hope that all this will be made explicit in the future.

2 Heisenberg groups and the lower central series of $\bar{\Gamma}(2)$

2.1 The Heisenberg group

The Heisenberg group is the algebraic group of 3×3 unipotent upper triangular matrices. Set: $x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, and $z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Those elements satisfy the relations $xz = zx$, $yz = zy$, and $z = xyx^{-1}y^{-1} = [x, y]$. Thus, one obtains a presentation of the Heisenberg group over the integers. Note the formula, for $a, b, c \in \mathbf{Z}$,

$$x^a z^b y^c = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

From that perspective, the group law is given by

$$x^a z^b y^c x^{a'} z^{b'} y^{c'} = x^{a+a'} z^{b+b'+a'c} y^{c+c'}.$$

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The Heisenberg group $H_{\mathbf{Z}}$ over $\mathbf{Z}$ can be identified with $\mathbf{Z}^3$ with the previous group law. The abelianization of $H_{\mathbf{Z}}$ is freely generated by the images of x and y and is thus isomorphic to $\mathbf{Z}^2$. Thus, $H_{\mathbf{Z}}$ is a central extension of $\mathbf{Z}^2$ by $\mathbf{Z}$.

Let M and N be natural integers. Let L be a common divisor of M and N. Let $H_{M,N,L}$ be the quotient group of $H_{\mathbf{Z}}$ spanned by x and y with relations $xz = zx$, $yz = zy$, $z = xyx^{-1}y^{-1}$, and $x^N = y^M = z^L = 1$. Such a group can be identified (as a set) with $\mathbf{Z}/M\mathbf{Z} \times \mathbf{Z}/L\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$ via the inverse of the map $(a, b, c) \mapsto x^a z^b y^c$.

Note the map $H_{M,N,L} \to \mathbf{Z}/M\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$ is coming from the abelianization.

Let M', N', and L' be integer ≥ 1 such that $M|M'$, $N|N'$, and $L|L'$. The canonical group homomorphism $H_{M',N',L'} \to H_{M,N,L}$ is surjective. Its kernel is generated by $\{x^M, y^N\}$. Since $[x^M, y^N] = z^{NM}$, this kernel is abelian if and only if $L'|NM$.

Proposition 2.1 *Let T be the lower common multiple of M and N. The group $H_{M,N,L}$ is of exponent T if T is odd or T/L is even. It is of exponent $2T$ otherwise. In particular $H_{N,N,N}$, for N odd, and $H_{N,N,N/2}$, for N even, are of exponent N.*

Proof Let $\alpha, \beta \in H_{M,N,L}$ in the subgroups generated by x and y, respectively. Let $e = T$ if T is odd or T/L is even. Let $e = 2T$ otherwise. Since $[\alpha, \beta]$ belongs to the center of $H_{M,N,L}$, it is of order dividing L. Moreover, one has $(\alpha\beta)^n = \alpha^n [\alpha, \beta]^{n(n-1)/2} \beta^n$, which vanishes if $n = e$. By this calculation, one has $(xy)^n = x^n z^{n(n-1)/2} y^n$, which shows that xy is of order T. ∎

Since the group $\bar{\Gamma}(2) = \Gamma(2)/\{\pm 1\}$ is freely generated by A and B, one gets a surjective group homomorphism $\bar{\Gamma}(2) \to H_{\mathbf{Z}}$ which sends A and B to x and y, respectively. Its kernel is $[\bar{\Gamma}(2), [\bar{\Gamma}(2), \bar{\Gamma}(2)]]$. Every (necessarily nilpotent) finite quotient group of $\bar{\Gamma}(2)$ which factorizes through $\bar{\Gamma}(2)/[\bar{\Gamma}(2), [\bar{\Gamma}(2), \bar{\Gamma}(2)]]$ is isomorphic to one of the groups $H_{M,N,L}$.

Denote by $\Gamma_{M,N,L}$ the kernel of the map: $\bar{\Gamma}(2) \to H_{\mathbf{Z}} \to H_{M,N,L}$.

2.2 The lower central series

Recall that the *lower central series* $(G_k)_{k\geq 1}$ of a group G is defined recursively by $G_1 = G$ and $G_{k+1} = [G_k, G]$. The quotient G_k/G_{k+1} is then an abelian group. When G is a free group generated by the family $(t_i)_{i\in I}$, G_k/G_{k+1} is a free abelian group generated (not freely) by the classes of the commutators of weight k on the generators, i.e., by the $[t_{i_1}, \ldots, t_{i_k}]$, where the indices run through any sequence $\{1, 2, \ldots, k\} \to I$ [6, Theorem 10.2.3]. When, furthermore, I is finite of cardinality m, the rank r_k of G_k/G_{k+1} is given by Witt's formula, involving the necklace polynomial,

$$r_k(G) = \frac{1}{k} \sum_{d|k} \mu(d) m^{k/d},$$

where μ is the Möbius function. In particular, the lower central series $(\bar{\Gamma}(2)_k)_{k\geq 1}$ of $\bar{\Gamma}(2)$ satisfies $r_1(\bar{\Gamma}(2)) = 2$, $r_2(\bar{\Gamma}(2)) = 1$, $r_3(\bar{\Gamma}(2)) = 2$, and $r_4(\bar{\Gamma}(2)) = 3$. The corresponding generators of $\bar{\Gamma}(2)_k/\bar{\Gamma}(2)_{k+1}$ for $k = 1, 2, 3$ are the classes of $\{A, B\}$, $\{C\}$, and $\{[C, A], [C, B]\}$, respectively. Therefore, one has a surjective group morphism ϕ_1: $\bar{\Gamma}(2) \to \mathbb{Z}^2$ such that $\phi_1(A) = (1, 0)$ and $\phi_1(B) = (0, 1)$. Its kernel is $\bar{\Gamma}(2)_2$. Moreover, one gets a group isomorphism: $\bar{\Gamma}(2)_1/\bar{\Gamma}(2)_3 \simeq H_{\mathbf{Z}}$.

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Next, we have the surjective group morphism $\phi_2\colon \bar{\Gamma}(2)_2 \to \mathbb{Z}$, such that $\phi_2(C) = 1$. Its kernel is $\bar{\Gamma}(2)_3$. We can now describe $\phi_3\colon \bar{\Gamma}(2)_3 \to \mathbb{Z}^2$ such that $\phi_3([C,A]) = (1,0)$ and $\phi_3([C,B]) = (0,1)$. Something interesting happens at that stage.

For k integer ≥ 2, the extension $1 \to G_k/G_{k+1} \to G_{k-1}/G_{k+1} \to G_{k-1}/G_k \to 1$ is central. Consequently, since $r_2(\bar{\Gamma}(2)) = 1$, the group $\bar{\Gamma}(2)_2/\bar{\Gamma}(2)_4$ is abelian, and free of rank 3. Of course, the extension $1 \to \bar{\Gamma}(2)_2/\bar{\Gamma}(2)_4 \to \bar{\Gamma}(2)_1/\bar{\Gamma}(2)_4 \to \bar{\Gamma}(2)_1/\bar{\Gamma}(2)_2 \to 1$ is not central.

Proposition 2.2 *There exists a group isomorphism $\psi\colon \bar{\Gamma}(2)_2/\bar{\Gamma}(2)_4 \simeq \mathbb{Z}^3$ given by $C \mapsto (0,0,1)$, $[C,A] \mapsto (1,0,0)$, and $[C,B] \mapsto (0,1,0)$. One has, for $\gamma \in \bar{\Gamma}(2)$ and $\delta \in \bar{\Gamma}(2)_2$, the formula*

$$\psi(\gamma\delta\gamma^{-1}) = (-\phi_1(\gamma)\phi_2(\delta), 0) + \psi(\delta),$$

or equivalently

$$\psi([\delta,\gamma]) = (\phi_1(\gamma)\phi_2(\delta), 0).$$

In particular, one has, for i, $j \in \mathbb{Z}$, the formula $\psi(A^iB^jC^kB^{-j}A^{-i}) = (-ki, -kj, k)$.

Proof Given that $\{[C,A],[C,B]\}$ is basis of the $\mathbb{Z}$-module $\bar{\Gamma}(2)_3/\bar{\Gamma}(2)_4$, and C is a basis of the $\mathbb{Z}$ module $\bar{\Gamma}(2)_2/\bar{\Gamma}(2)_3$, any lifting of C modulo $\bar{\Gamma}(2)_3$, to a class C' modulo $\bar{\Gamma}(2)_4$ gives a basis $\{[C,A],[C,B],C'\}$ of $\bar{\Gamma}(2)_2/\bar{\Gamma}(2)_4$. The choice $C = C'$ is evidently suitable, but is somewhat arbitrary.

One has

$$\psi(\gamma\delta\gamma^{-1}) = \psi(\gamma\delta C^{-\phi_2(\delta)}\gamma^{-1}C^{\phi_2(\delta)}\delta^{-1}\delta C^{-\phi_2(\delta)}\gamma C^{\phi_2(\delta)}\gamma^{-1}).$$

Since $\delta C^{-\phi_2(\delta)} \in \bar{\Gamma}(2)_3$, the factor $\gamma\delta C^{-\phi_2(\delta)}\gamma^{-1}C^{\phi_2(\delta)}\delta^{-1}$ belongs to $\bar{\Gamma}(2)_4$; hence its image by ψ vanishes. So we get

$$\psi(\gamma\delta\gamma^{-1}) = \psi(\delta) - \phi_2(\delta)\psi(C) + \phi_2(\delta)\psi(\gamma C\gamma^{-1}), \tag{2.1}$$

which translates into

$$\psi([\delta,\gamma]) = \phi_2(\delta)\psi([C,\gamma]).$$

It remains to determine $\psi([C,\gamma])$. Let $\gamma_1, \gamma_2 \in \bar{\Gamma}(2)$. One has

$$(\gamma_1\gamma_2)\delta(\gamma_1\gamma_2)^{-1} = [\gamma_1,\gamma_2][\gamma_2,[\gamma_1,\delta]][\gamma_1,\delta][\gamma_2,\delta]\delta[\gamma_2,\gamma_1]. \tag{2.2}$$

Since $[\gamma_2,[\gamma_1,\delta]]$ is a commutator of degree 4, it is in the kernel of ψ. It follows that the map $\gamma \mapsto \psi(\gamma\delta\gamma^{-1})$ is a group homomorphism from $\bar{\Gamma}(2)$ to $\mathbb{Z}^2$. For $\delta = C$, $\gamma = A$ (resp. $\gamma = B$), one has $\psi(\gamma\delta\gamma^{-1}) = -(\psi_1(\gamma), 0)$, hence the latter equality is true for all $\gamma \in \bar{\Gamma}(2)$. Thus, one gets

$$\psi([C,\gamma]) = (\phi_1(\gamma), 0),$$

which gives the main formula. It remains to apply this to $\gamma = A^iB^j$ and $\delta = C^k$ to obtain the final formula. ∎

The exact sequence $1 \to \bar{\Gamma}(2)_3/\bar{\Gamma}(2)_4 \to \bar{\Gamma}(2)_1/\bar{\Gamma}(2)_4 \to \bar{\Gamma}(2)_1/\bar{\Gamma}(2)_3 \to 1$ identifies to a central group extension

$$0 \to \mathbb{Z}^2 \to H'_{\mathbb{Z}} \to H_{\mathbb{Z}} \to 1.$$

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2.3 The groups Φ_N, Φ'_N, and Φ''_N

We denote the group $\Gamma_{N,N,1}$ by Φ_N. It is the kernel of the group homomorphism $\bar{\phi}_1$: $\bar{\Gamma}(2) \to (\mathbb{Z}/N\mathbb{Z})^2$, which maps A to $(1,0)$ and B to $(0,1)$ (and thus C to $(0,0)$). A system of representatives for the cosets $\Phi_N \backslash \bar{\Gamma}(2)$ is given by $A^i B^j$ with $i, j \in \{0, 1 \dots (N-1)\}$. The structure of Φ_N is probably well-known, but we could not find a complete reference for a presentation Φ_N.

Proposition 2.3 *The group Φ_N is generated by $U = \{A^N, B^N, A^i B^j C B^{-j} A^{-i} / 0 \le i, j \le N-1\}$. Moreover, Φ_N has presentation*

$$< A^N, B^N, T_{i,j}, 0 \le i, j \le N-1 | [A^N, B^N] = \prod_{i=0}^{N-1} \prod_{j=0}^{N-1} T_{N-1-i,j} > .$$

Proof The group Φ_N is generated by $U = \{A^N, B^N, A^i B^j C B^{-j} A^{-i} / 0 \le i, j \le N-1\}$ (see [7, 17]). By the Nielsen–Schreier theorem, Φ_N, being a free subgroup of index N^2 of a free group on two generators, is free on $N^2 + 1$ generators. A relation between the $N^2 + 2$ exhibited generators in U presents itself:

$$A^N B^N A^{-N} B^{-N} = \prod_{i=0}^{N-1} \prod_{j=0}^{N-1} A^{N-1-i} B^j C B^{-j} A^{i+1-N}. \tag{2.3}$$

One of the exhibited generators in U can be expressed in terms of the $N^2 + 1$ remaining elements of U. Thus, we get a presentation by generators and relations of Φ_N. ∎

Set $N' = N$ if N is odd, and $N' = N/2$ if N is even.

We set $\Phi'_N = \Gamma_{N,N,N'}$. It is the subgroup of Φ_N obtained as the kernel of the morphism $\bar{\phi}_2$: $\Phi_N \to \mathbb{Z}/N'\mathbb{Z}$ which vanishes on A^N and B^N and takes the value 1 on $A^i B^j C B^{-j} A^{-i}$ for i, j integers.

Alternately, Φ'_N is the kernel of the composed map $\bar{\Gamma}(2) \to H_{\mathbf{Z}} \to H_{N,N,N'}$. Thus, the composed map $[\bar{\Gamma}(2), \bar{\Gamma}(2)] \to \mathbb{Z} \to \mathbb{Z}/N'\mathbb{Z}$ extends to a map $\bar{\phi}_2$: $\Phi_N \to \mathbb{Z}/N'\mathbb{Z}$, with kernel Φ'_N. It vanishes on A^N and B^N.

Proposition 2.4 *The composed map $[\bar{\Gamma}(2), \bar{\Gamma}(2)] \to \mathbb{Z}^3 \to (\mathbb{Z}/N'\mathbb{Z})^3$ extends to a group homomorphism*

$$\bar{\psi} : \Phi_N \to (\mathbb{Z}/N'\mathbb{Z})^3,$$

which vanishes on A^N and B^N and, for $\gamma \in \bar{\Gamma}(2)$ and $\delta \in \Phi_N$, satisfies

$$\bar{\psi}(\gamma \delta \gamma^{-1}) = (-\bar{\phi}_1(\gamma) \bar{\phi}_2(\delta), 0) + \bar{\psi}(\delta). \tag{2.4}$$

In particular, for $i, j, k \in \mathbb{Z}$, one has

$$\bar{\psi}(A^i B^j C^k B^{-j} A^{-i}) = (-ik, -jk, k). \tag{2.5}$$

Proof It follows from the presentation of Φ_N that formula 2.5 defines a morphism $\Phi_N \to (\mathbb{Z}/N\mathbb{Z})^3$ (it vanishes on the exhibited relation 2.3 between the generators). Such a morphism coincides with the composed map $[\bar{\Gamma}(2), \bar{\Gamma}(2)] \to \mathbb{Z}^3 \to (\mathbb{Z}/N\mathbb{Z})^3$, by the formula we established on ψ.

The formula 2.4 is valid whenever $\delta \in \bar{\Gamma}(2)_2$, by 2.1. Since Φ_N is generated by $\bar{\Gamma}(2)_2$ together with A^N and B^N, it remains to prove 2.4 for $\delta = A^N$ and $\delta = B^N$. Consider the

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case $\delta = B^N$ for instance (the other case is similar). Let us use the formula 2.2 again. We get, for $\gamma_1, \gamma_2 \in \bar{\Gamma}(2)$, (leaving out opposite terms)

$$\bar{\psi}(\gamma_1\gamma_2 B^N \gamma_2^{-1}\gamma_1^{-1}) = \bar{\psi}([\gamma_2,[\gamma_1,B^N]]) + \bar{\psi}([\gamma_1,B^N]) + \bar{\psi}([\gamma_2,B^N]) + \bar{\psi}(B^N).$$

We have $\bar{\psi}(B^N) = 0$, and $\bar{\psi}([\gamma_2,[\gamma_1,B^N]])$ is the reduction modulo N of $\psi([\gamma_2,[\gamma_1,B^N]])$. One has $\psi([\gamma_2,[\gamma_1,B^N]]) = -(\phi_1(\gamma_2)\phi_2([\gamma_1,B^N],0)$. But $\phi_2([\gamma_1,B^N]) \in N\mathbb{Z}$. So we have proved that the map $\gamma \mapsto \bar{\psi}(\gamma B^N \gamma^{-1})$ is a group homomorphism. It remains to prove that the formula 2.4 holds for $\gamma = A$ and $\gamma = B$ (still in the configuration where $\delta = B^N$). Only the case $\gamma = A$ is of interest. We have

$$\bar{\psi}(AB^NA^{-1}) = \bar{\psi}(AB^NA^{-1}B^{-N}).$$

As $AB^NA^{-1}B^{-N} \in \bar{\Gamma}(2)_2$, $\bar{\psi}(AB^NA^{-1}B^{-N})$ is the reduction modulo N of $\psi(AB^NA^{-1}B^{-N})$. We use the identity

$$AB^NA^{-1}B^{-N} = (ABA^{-1})^NB^{-N} = (ABA^{-1}B^{-1}B)^NB^{-N} = CBCB^{-1}B^2CB^{-2}\dots B^{N-1}CB^{1-N}.$$

The right-hand side is a product of generators of Φ_N. We can apply 2.5

$$\psi(AB^NA^{-1}B^{-N}) = \sum_{i=0}^{N-1}\psi(B^iCB^{-i}) = \sum_{i=0}^{N-1}(0,-i,0) = (0,N(N-1)/2,0).$$

Since $N' = \gcd(N, N(N-1)/2)$, we have indeed that $\bar{\psi}(AB^NA^{-1}B^{-N}) = 0$. ∎

Remark 2.5 Let $N'' = N'$ if N is prime to 3, and $N'' = N'/3$ otherwise. The appearance of the denominator 2 (for N') and now 3 (for N'') is related to Bernoulli numbers. We suspect that ultimately it is related to the mixed Tate motives that have been discovered by Deligne in his study of the nilpotent completion of the fundamental group of the projective line deprived of three points [4].

We define Φ''_N as the kernel of the composed map $\Phi_N \to (\mathbb{Z}/N'\mathbb{Z})^2 \times (\mathbb{Z}/N'\mathbb{Z}) \to (\mathbb{Z}/N''\mathbb{Z})^2 \times (\mathbb{Z}/N'\mathbb{Z})$.

Corollary 2.6 *The exact sequence*

$$0 \to \Phi'_N/\Phi''_N \to \bar{\Gamma}(2)/\Phi''_N \to \bar{\Gamma}(2)/\Phi'_N \to 0$$

makes of the group $\bar{\Gamma}(2)/\Phi''_N$ *a central extension of the Heisenberg group* $H_{N,N,N'}$ *by* $(\mathbb{Z}/N''\mathbb{Z})^2$.

Proof It follows from formula 2.4, as $\bar{\phi}_2$ vanishes on Φ'_N. ∎

We denote by $H'_{\mathbb{Z}/N\mathbb{Z}}$ the group $\bar{\Gamma}(2)/\Phi''_N$.

Proposition 2.7 *The group* $H'_{\mathbb{Z}/N\mathbb{Z}}$ *is of exponent N. In other words, for every* $\gamma \in \bar{\Gamma}(2)$, *one has* $\gamma^N \in \Phi''_N$.

Proof It relies on relations for commutators, that, we presume, are well-known. Let G be a group. Suppose G_4 is trivial. Then G_3 is contained in the center of G. Let α, $\beta \in G$. Set $\gamma = [\alpha,\beta]$, $\alpha' = [\gamma^{-1},\alpha]$, and $\beta' = [\gamma^{-1},\beta]$. Let $n \in \mathbb{N} \cup \{0\}$ be an integer. One has the relation

$$\beta^n\alpha = \alpha\gamma^{-n}\beta^n\alpha'^n\beta' - \frac{n(n-1)}{2}. \tag{2.6}$$

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We prove it by induction on n. Indeed, it holds for $n = 0$. Suppose it holds for some value of n. We have

$$\beta^{n+1}\alpha = \beta\alpha\gamma^{-n}\beta^n\alpha'^n\beta'^{-n(n-1)/2} = \gamma^{-1}\alpha\beta\gamma^{-n}\beta^n\alpha'^n\beta'^{-\frac{n(n-1)}{2}}.$$

We use the relations $\gamma^{-1}\beta = [\gamma^{-1}, \beta]\beta\gamma^{-1} = \beta'\beta\gamma^{-1}$, $\gamma^{-1}\alpha = [\gamma^{-1}, \alpha]\alpha\gamma^{-1} = \alpha'\alpha\gamma^{-1}$, and $\beta\gamma^{-n} = \beta'^{-n}\gamma^{-n}\beta$. Using the fact that $\alpha', \beta' \in Z(G)$, we get

$$\begin{aligned}\beta^{n+1}\alpha &= \gamma^{-1}\alpha\beta\gamma^{-n}\beta^n\alpha'^n\beta'^{-\frac{n(n-1)}{2}} \\ &= \alpha'\alpha\gamma^{-1}\beta\gamma^{-n}\alpha'^n\beta'^{-\frac{n(n-1)}{2}} \\ &= \alpha\gamma^{-n-1}\beta^{n+1}\alpha'^{n+1}\beta'^{-n-n(n-1)/2}\end{aligned}$$

which is the desired formula. We pass to the next step. We now claim that

$$(\alpha\beta)^n = \alpha^n\gamma^{-\binom{n}{2}}\beta^n\alpha'^{\binom{n+1}{3}}\beta'^{-\binom{n}{3}}. \tag{2.7}$$

We proceed again by induction on n. This is true for $n = 1$. We suppose the formula holds for a certain value of n. We get

$$(\alpha\beta)^{n+1} = (\alpha\beta)^n \cdot (\alpha\beta) = \alpha^n\gamma^{-\binom{n}{2}}\beta^n \cdot (\alpha\beta)\alpha'^{\binom{n+1}{3}}\beta'^{-\binom{n}{3}}.$$

Using the formula 2.6, we get

$$(\alpha\beta)^{n+1} = \alpha^n\gamma^{-\binom{n}{2}}\alpha\gamma^{-n}\beta^n\alpha'^n\beta'^{-\binom{n}{2}}\beta\alpha'^{\binom{n+1}{3}}\beta'^{-\binom{n}{3}}.$$

We now use the formula $\gamma^{-\binom{n}{2}}\alpha = \alpha'^{\binom{n}{2}}\alpha\gamma^{-\binom{n}{2}}$ and the centrality of α' and β' to get

$$(\alpha\beta)^{n+1} = \alpha^n\gamma^{-\binom{n}{2}-n}\beta^n\alpha'^{\binom{n+1}{3}+\binom{n}{2}+n}\beta'^{-\binom{n}{3}-\binom{n}{2}} = \alpha^n\gamma^{-\binom{n+1}{2}}\beta^n\alpha'^{\binom{n+2}{3}}\beta'^{-\binom{n+1}{3}}$$

which establishes 2.7.

The Nth powers of both A and B belong to Φ_N'' (Proposition 2.4). The N'th, and therefore the $\binom{N}{2}$, power of any commutator belongs to Φ_N'', since Φ_N' contains $\bar{\Gamma}(2)_2$. Similarly, the N''th power, and therefore the $\binom{N}{3}$ and $\binom{N+1}{3}$ powers, of an element of $\bar{\Gamma}(2)_3$ belongs to Φ_N''. We can combine these remarks with the formula 2.7 for $n = N$, to establish that, if the Nth powers of α and β both belong to Φ_N'', one has $(\alpha\beta)^N \in \Phi_N''$. The proposition follows, since $\{A, B\}$ generates $\bar{\Gamma}(2)$. ∎

Remark 2.8 Let p be prime number. Stallings introduced the lower p-central series $(S_k)_{k\geq 1}$ as a particular case for $N = p$ of the following construction. One has $S_1 = G$, and, for $k \geq 2$, $S_{k+1} = [G, S_k](S_k)^N$, where the latter expression is the subgroup of G generated by $[G, S_k]$ and $(S_k)^N$. Note that, when $G = \bar{\Gamma}(2)$, one has $S_2 = [\bar{\Gamma}(2), \bar{\Gamma}(2)]\bar{\Gamma}(2)^N = \Phi_N$ and $S_3 = [\bar{\Gamma}(2), \Phi_N]\Phi_N^N$. Note that $S_3 \subset \Phi_N' \subset S_2$. Since A^N and B^N do not belong to S_3, the groups S_3 and Φ_N' do not coincide.

2.4 Odd adelic completions

Recall that, for $k \geq 1$, $\bar{\Gamma}(2)_k$ is the kth term in the lower central series of $\bar{\Gamma}(2)$. Let $D_3 = \{\pm\mathrm{Id}, \pm\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \pm\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}, \pm\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\}$ in $\mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$ be the index 3, 2-Sylow subgroup of $\mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$. It is isomorphic to the Klein group.

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Recall that the derived subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ is the projective congruence subgroup of level 6 whose image in $\mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$ is D_3, and whose image in $\mathrm{PSL}_2(\mathbb{Z}/2\mathbb{Z})$ is cyclic of order 3.

Let

$$\hat{\mathbb{Z}}_{\mathrm{odd}} = \varprojlim_{n\,\mathrm{odd}} \mathbb{Z}/n\mathbb{Z} \simeq \prod_{p\neq 2} \mathbb{Z}_p$$

be the profinite completion of $\mathbb{Z}$ away from the prime 2. Let $\hat{D}_{\mathrm{odd}}$ be the inverse image of D_3 in $\mathrm{PSL}_2(\hat{\mathbb{Z}}_{\mathrm{odd}})$.

Proposition 2.9 *The image of $\bar{\Gamma}(2)_2$ in $\mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$ is equal to D_3. For p prime, $p > 3$, its image modulo p is $\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$.*

Proof Indeed, the images of $\bar{\Gamma}(2)$ and $\mathrm{PSL}_2(\mathbb{Z}))$ in $\mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$ coincide by weak approximation. Thus, $\bar{\Gamma}(2)_2$ modulo 3 coincides with the derived subgroup of $\mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$, which in turn is the reduction modulo 3 of the derived subgroup of $\mathrm{PSL}_2(\mathbb{Z}))$. The second assertion is proved similarly. ∎

Proposition 2.10 *Let k be an integer ≥ 2. The closure of $\bar{\Gamma}(2)_k$ in $\mathrm{PSL}_2(\hat{\mathbb{Z}}_{\mathrm{odd}})$ is equal to $\hat{D}_{\mathrm{odd}}$.*

Proof We prove this first for $k = 2$. Let n be an odd integer divisible by 3. Let D_n be the inverse image of D_3 in $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$. The image of $\bar{\Gamma}(2)_2$ modulo n coincide with the image of the derived subgroup of $\mathrm{PSL}_2(\mathbb{Z}))$ modulo n, which is a subgroup of index 3 of $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$. Such a subgroup can only be D_n. Thus, we obtain the proposition for $k = 2$.

We prove the proposition for $k = 3$. Let I_n be the image of $\bar{\Gamma}(2)_3$ in $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$. Note that we have an exact sequence

$$1 \to K_n \to I_n \to D_3 \to 1.$$

Since we have an exact sequence

$$1 \to \pm 1 + 3\mathrm{M}_2(\mathbb{Z}/\frac{n}{3}\mathbb{Z})_0 \to \mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z}) \to 1,$$

where $\mathrm{M}_2(\mathbb{Z}/\frac{n}{3}\mathbb{Z})_0$ is the subgroup of $\mathrm{M}_2(\mathbb{Z}/\frac{n}{3}\mathbb{Z})$ made of matrices of trace 0, the equality $I_n = D_n$ would follow from the inclusion $1 + 3\mathrm{M}_2(\mathbb{Z}/\frac{n}{3}\mathbb{Z})_0 \subset K_n$. It remains to establish the latter inclusion. Since $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ is equal to its derived subgroup when n is prime to 6, by the Chinese remainder theorem, it is enough to prove it when n is a power of 3.

It follows from the equality $[(1 + p\mathrm{M}_2(\mathbb{Z}_p))_0, \mathrm{SL}_2(\mathbb{Z}_p)] = (1 + p\mathrm{M}_2(\mathbb{Z}_p))_0$, valid for any prime p, and the fact that the closure of $\bar{\Gamma}(2)_2$ in $\mathrm{PSL}_2(\mathbb{Z}_3)$ contains $[(1 + 3\mathrm{M}_2(\mathbb{Z}_3))_0$.

The general case $k \geq 3$ of the proposition is now immediate. Indeed, since $\bar{\Gamma}(2)_3$ and $\bar{\Gamma}(2)_2$ have the same image modulo n, those images are the second and third, respectively, derived subgroups of $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$. Thus, the lower central series of $\mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ stabilizes to the image modulo n of $\bar{\Gamma}(2)_k$ for any $k \geq 3$. ∎

Proposition 2.11 *The closure of Φ_N in $\mathrm{PSL}_2(\hat{\mathbb{Z}}_{\mathrm{odd}})$ is $\mathrm{PSL}_2(\hat{\mathbb{Z}}_{\mathrm{odd}})$ if 3 does not divide N. It is $\hat{D}_{\mathrm{odd}}$ if 3 divides N.*

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Proof If 3 does not divide N, Φ_N contains a nonzero upper triangular matrix (for instance, A'^N) which is not the identity modulo 3. Its closure in $\mathrm{PSL}_2(\hat{\mathbb{Z}}_{\mathrm{odd}})$ contains a maximal proper subgroup of index 3, namely $\hat{D}_{\mathrm{odd}}$, and an element which is not in that subgroup. Therefore, the closure is $\mathrm{PSL}_2(\hat{\mathbb{Z}}_{\mathrm{odd}})$. If 3 divides N, since both A^{3N} and B^{3N} vanish modulo 3, the images of Φ_N and $\bar{\Gamma}(2)_2$ coincide in $\mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$. Hence the result. ∎

Proposition 2.12 *The closure of Φ'_N in $\mathrm{PSL}_2(\hat{\mathbb{Z}}_{\mathrm{odd}})$ is $\mathrm{PSL}_2(\hat{\mathbb{Z}}_{\mathrm{odd}})$ if 3 does not divide N. It is $\hat{D}_{\mathrm{odd}}$ if 3 divides N.*

Proof The closure of $\bar{\Gamma}(2)_2$ and $\bar{\Gamma}(2)_3$ coincide modulo n for every n divisible by 3, as we have just established. Thus, the closure of Φ'_N in $\mathrm{PSL}_2(\hat{\mathbb{Z}}_{\mathrm{odd}})$ contains $\bar{\Gamma}(2)_2$. Since the group Φ'_N contains the matrices A^N and B^N, its closure contains Φ_N. Thus, the closure of Φ'_N in $\mathrm{PSL}_2(\hat{\mathbb{Z}}_{\mathrm{odd}})$ is equal to the closure of Φ_N in $\mathrm{PSL}_2(\hat{\mathbb{Z}}_{\mathrm{odd}})$. ∎

Let $\bar{\Gamma}'(2) = \bar{\Gamma}(2) \cap \hat{D}_{\mathrm{odd}}$. It is a subgroup of index 3 of $\bar{\Gamma}(2)$.

Corollary 2.13 *Let Γ be a congruence subgroup of $\bar{\Gamma}(2)$. One has $\Gamma\Phi'_N = \bar{\Gamma}'(2)$ if $\Gamma \subset \bar{\Gamma}'(2)$ and 3 divides N. One has $\Gamma\Phi'_N = \bar{\Gamma}(2)$ otherwise.*

Let n be an even integer. In particular, one has $\Gamma(n)\Phi'_N = \bar{\Gamma}(2)$ if 3 does not divide either n or N. One has $\Gamma(n)\Phi'_N = \bar{\Gamma}'(2)$ if 3 divides both n and N.

3 The associated Riemann surfaces

3.1 The Riemann surface $X_{M,N,L}$

Denote by $X_{M,N,L}$ the compactified modular curve defined by $\Gamma_{M,N,L}$.

Proposition 3.1 *The genus $g_{M,N,L}$ of the curve $X_{M,N,L}$ is given by the following formulas. Denote by T the lowest common multiple of M and N. Suppose T is even and T/L is odd, then one has*

$$g_{M,N,L} := g(X_{M,N,L}) = (NML - NL - ML - NML/2T)/2 + 1.$$

Suppose T is odd or T/L is even, then one has

$$g_{M,N,L} := g(X_{M,N,L}) = (NML - NL - ML - NML/T)/2 + 1.$$

Proof We use Riemann–Hurwitz formula for the morphism $X_{N,M,L} \to X(2)$. Since $\Gamma(2)$ has no elliptic elements, the ramification points of this morphism reside entirely among the cusps.

Concerning the cusps above 0 (resp. ∞), note that the stabilizer of the rational number 0 (resp. ∞) in $P\Gamma(2)$ is generated by B (resp. A). As the morphism $X_{N,M,L} \to X(2)$ is Galois, the ramification index is independent of the chosen cusp, and is the order of the orbit of B (resp. of A) acting on $\Gamma_{M,N,L}\backslash\Gamma(2)$. This is N (resp. M) by definition of $\Gamma_{M,N,L}$. Concerning the cusps above 1, the stabilizer in $P\Gamma(2)$ of the rational number -1 is generated by $A^{-1}B$. It remains to determine the order of $A^{-1}B$ in $\Gamma_{M,N,L}\backslash\Gamma(2) = H_{M,N,L}$.

The abelianization provides a map: $H_{M,N,L} \to \mathbf{Z}/M\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$ which sends $A^{-1}B$ to $(1, -1)$. The latter element is of order T, which leads us to examine the order of $(A^{-1}B)^T$ in $H_{M,N,L}$. Denote by $D = [A^{-1}, B]$. It belongs to and generates the center of $H_{M,N,L}$.

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It is of order L. Note the formula $A^{-1}B = DBA^{-1}$. Thus, one has $(A^{-1}B)^T = D^k A^{-T} B^T$ in $H_{M,N,L}$, where k is the number of factors A^{-1} to the right of a factor B in the $(A^{-1}B)^T$ written as a product of $2T$ factors. One has $k = 1 + 2 + \cdots + (T-1) = T(T-1)/2$. This is why the order of $(A^{-1}B)^T$ is 1 if L is odd or if T/L is even. This order is 2 otherwise.

Thus, the ramification index e of any cusp above 1 is equal to T if L is odd or if T/L is even. It is $2T$ otherwise. We can now apply the Riemann–Hurwitz formula for the dominant morphism $X_{M,N,L} \to X(2)$:

$$2g_{M,N,L} - 2 = -2d + \sum_{x \in X_{M,N,L}} (e_x - 1).$$

We have $d = |H_{M,N,L}| = MNL$. One gets

$$2g_{M,N,L} - 2 = -2MNL + NL(M-1) + ML(N-1) + NML(1-1/e)$$

and

$$g_{M,N,L} = NML - NL - ML - NML/e + 1.$$

The lemma follows from the calculation of e. ∎

3.2 The Riemann surfaces X_N, X'_N, and X''_N

All three groups Φ_N, Φ'_N, and Φ''_N act on the upper half-plane $\mathbb{H}$. We denote, respectively, by X_N, X'_N, and X''_N the corresponding completed modular curves.

Proposition 3.2 *Both morphisms $X'_N \to X_N$ and $X''_N \to X'_N$ (and therefore $X''_N \to X_N$ as well) are unramified. The covering $X'_N \to X_N$ is cyclic of degree N'. The covering $X''_N \to X'_N$ is Galois with group $(\mathbb{Z}/N''\mathbb{Z})^2$. The covering $X''_N \to X_N$ is abelian with Galois group $(\mathbb{Z}/N''\mathbb{Z})^2 \times (\mathbb{Z}/N'\mathbb{Z})$.*

Proof The first statement needs only to be established for the morphism $X''_N \to X_N$. The ramification points can only reside at the cusps. To show that those cusps are unramified, we need only to show that their width in X''_N is equal to their width, equal to N, in X_N. Since the covering $X''_N \to X_1$ is Galois, the width of a cusp of X''_N depends only on its image in the set of cusps $\{0, 1, \infty\}$ of X_1. We just need to look at the order of A, B and $A^{-1}B$ in $\bar{\Gamma}(2)/\Phi''_N$. All three are of order N in the latter quotient.

The other assertions follow immediately from the properties of the groups Φ_N, Φ'_N, and Φ''_N. ∎

The Riemann surface X_N is isomorphic to the Riemann surface obtained from the complex points of the Fermat curve. The covering $X'_N \to X_N$ is obtained from what we call *Heisenberg covering of the Fermat curve* by passing to the complex numbers.

We obtain the genus g_N and g'_N of the curves X_N and X'_N by our formulas for the genus of $X_{N,M,L}$. One has $g_N = g_{N,N,1} = (N-1)(N-2)/2$. Furthermore, if N is odd, one has $g'_N = g_{N,N,N} = (N^3 - 3N^2 + 2)/2 = (N-1)(N^2 - 2N - 2)/2$. If N is even, one gets $g'_N = g_{N,N,N'} = (2N^3 - 5N^2 + 4)/4 = (N-2)(2N^2 - N - 2)/4$.

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The genus of g''_N of the curve X''_N can be deduced from the Riemann–Hurwitz formula. Since we have a covering $X''_N \to X'_N$ of degree N''^2, one has

$$g''_N = N''^2 g'_N - N^2 + 1.$$

The curve X'_N possesses NN' cusps above each of the cusps 0, 1, and ∞ of $X(2)$.

3.3 Some cases of small genus

Proposition 3.3 *The genus g of the curve $X_{M,N,L}$ is equal to 0 for the following values of (N, M, L), and only for those values:* $(N,1,1)$, $(1,M,1)$, $(2,2,1)$, $(2,2,2)$.

Proof The formula just established gives: $2-2g = -L(MN - N - M - MN/e)$. Thus, $g = 0$ implies that $L = 1$ or 2.

If $g = 0$ and $L = 1$, then one has $MN - N - M - MN/e = -2$ and $e = \mathrm{lcm}(M, N)$. Thus, $\gcd(M, N)$ divides 2. If $\gcd(M, N) = 1$, then one has $MN - N - M - 1 = -2$, and thus $(M-1)(N-1) = 0$ that is $M = 1$ or $N = 1$. If $\gcd(M, N) = 2$, one has $MN - N - M - 2 = -2$ and thus $(M-1)(N-1) = 1$; therefore one has $(M, N) = (2, 2)$.

If $g = 0$ and $L = 2$, then one has $MN - N - M - MN/e = -1$. If 4 divides neither M nor N, one has $e = 2\mathrm{lcm}(M, N)$. Then $\gcd(M, N)$ divides 2, and is equal to 2. Then one has $MN - N - M - 1 = -2$, as above. As $L = 2$, the cases $M = 1$ and $N = 1$ are excluded; thus, one has $(M, N) = (2, 2)$. ∎

Proposition 3.4 *The genus g of the curve $X_{M,N,L}$ is equal to 1 for the following values of (N, M, L), and only for those values (up to permutation of N and M):* $(3,2,1)$, $(4,2,1)$, $(4,2,2)$, $(3,3,1)$, $(3,3,3)$.

The Jacobian varieties of those curves are elliptic curves endowed with automorphisms of order 3, 4, 4, 3, and 3, respectively. Consequently, the j-invariants of those curves are 0, 1728, 1728, 0, and 0, respectively.

Proof Consider again the formula: $2-2g = -L(MN - N - M - MN/e)$. Thus, $g = 1$ amounts to $MN - N - M - MN/e = 0$. One has $MN - N - M - MN/e = (N-2)(M-1) - 2 + M(1 - N/e)$. We can suppose that $N > 1$ and $M > 1$ (otherwise $g = 0$).

If $N = 2$, one gets $M(1 - 2/e) = 2$. If $L = 1$, then $e = \mathrm{lcm}(M, 2)$. Thus, $M - 1 = 2$ or $M - 2 = 2$. One has $(N, M, L) = (2, 3, 1)$ or $(N, M, L) = (2, 4, 1)$.

If $N = 2$ and $L = 2$, then $e = 2M$ or $e = M$. If $e = M$, then $M - 2 = 2$. If $e = 2M$, then $M - 1 = 2$ (absurd since $L|M$). One has $(N, M, L) = (2, 4, 2)$ or $(N, M, L) = (2, 4, 1)$.

If $N = 3$, then one has $M - 3 + M(1 - 3/e) = 0$ and $e = \mathrm{lcm}(M, 3)$. One can suppose that $M > 2$. Thus, one has $M = 3$ and $L = 1$ or $L = 3$. One has $(N, M, L) = (3, 3, 1)$ or $(N, M, L) = (3, 3, 3)$.

The case where $M = 2$ or $M = 3$ are treated similarly. If $N > 3$ and $M > 3$, one has $(N-2)(M-1) - 2 + M(1 - N/e) > 0$, which precludes $g = 1$.

The automorphisms come from the action of the image of A in $H_{N,M,L}$ which stabilizes a cusp and therefore is an automorphism of an elliptic curve. ∎

We can derive some information on the Manin–Drinfeld principle in the genus 1 cases.

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Proposition 3.5 *Divisors of the form* $(CP) - (P)$*, where P is any point (in particular, a cusp) of* $X_{3,3,3}$ *(resp.* $X_{4,2,2}$*) and C acts via the map* $\bar{\Gamma}(2) \to H_{N,M,L}$*, are of order dividing* 3 *(resp.* 2*).*

Proof The canonical morphism $X_{3,3,3} \to X_{3,3,1}$ is of degree 3. It gives rise to an isogeny of degree 3 on the Jacobians by Albanese functionality. Thus, the kernel of this isogeny is of order 3. Moreover, any divisor of the form $(CP) - (P)$ is in the kernel of the isogeny. ∎

3.4 The curve $X'(2)$

Recall that the group $\bar{\Gamma}'(2)$ is the subgroup of index 3 of $\bar{\Gamma}(2)$ that is the inverse image of the 2-Sylow subgroup of $\mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$. Denote by $X'(2) = X_{\Gamma'(2)}$ the corresponding modular curve.

Proposition 3.6 *The curve* $X'(2)$ *is of genus* 1 *and its j-invariant is* 0*.*

Proof Consider the morphism of degree $d = 3$: $X'(2) \to X(2)$. Since none of the matrices A (generator of the stabilizer of the cusp ∞), B (generator of the stabilizer of the cusp 0), and AB^{-1} (generator of the stabilizer of the cusp 1) are not in $\bar{\Gamma}(2)$, the morphism is totally ramified at each of the three cusps of $X(2)$, and ramified only over those points. The Riemann–Hurwitz formula expresses the genus g of $X'(2)$ as $(2g-2) = -2d + \sum_P (e_P - 1) = 6 - 6 = 0$ (where P runs through points of ramification and e_P designates the ramification index at that point), hence $g = 1$.

The curve $X'(2)$ has an automorphism (the class of A in $\bar{\Gamma}(2)/\bar{\Gamma}'(2)$) of order 3 which leaves fixed the cusp ∞. Since it is of genus 1, it is an elliptic curve of with an automorphism of order 3. It has necessarily j-invariant 0. ∎

Remark 3.7 Since $\Gamma_{3,3,3} = \Phi'_3$ is a subgroup of index 3 of $\Gamma'(2)$. One has a morphism $X_{3,3,3} \to X'(2)$ of degree 3. Both curves involved are of genus 1, so we have an isogeny of degree 3. Note that $\bar{\Gamma}'(2)$ is a congruence subgroup of level 12 and a subgroup of index 3 of the derived subgroup $\tilde{\Gamma}$ of $\mathrm{PSL}_2(\mathbb{Z})$. The latter subgroup defined a modular curve $X_{\tilde{\Gamma}}$ of genus 1, which happens to have j-invariant 0. Thus, we get an isogeny $X'(2) \to X_{\tilde{\Gamma}}$ of degree 3.

We now prove Theorem 1.3.

Proof The correspondence is obtained by composing pushing to $X_{\Gamma.\Phi'_d}$ and pulling back to X_Γ. By Proposition 2.12, one has $\Gamma.\Phi'_d = \bar{\Gamma}(2)$ except if 3 divides N and Γ is contained in $\bar{\Gamma}'(2)$. In the latter case, $\theta_{N,\Gamma}$ factorizes through the Jacobian of $X(2)$, which is 0. Otherwise, namely if 3 divides N and Γ is contained in $\bar{\Gamma}'(2)$, $\theta_{N,\Gamma}$ factorizes through the surjective map $J'_N \to J'(2)$. Since $J'(2)$ is the Jacobian of a curve of genus 1, and j-invariant 0, it is an elliptic curve of j-invariant 0. Moreover, the map $J'(2) \to J_\Gamma$ has finite kernel. The result follows. ∎

It is well-known that the modular curve $X_0(27)$ has j-invariant 0, and that the Fermat curve is a model for $X_0(27)$. We might expect a connection between $X_0(27)$ and X'_3. Let $\Gamma = \bar{\Gamma}(2) \cap \Gamma_0(27)$. It is a congruence subgroup. But it is not contained in $\Gamma'(2)$. Thus, if we apply Theorem 1.3 to the group Γ, we obtain, counterintuitively, the 0-morphism $J'_3 \to J_\Gamma$. *A fortiori*, if we push forward $J_\Gamma \to J_0(27)$ we obtain 0.

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3.5 Mixed homology groups

In [5], the homology group of X_N relative to the whole set of cusps is thoroughly studied, by the method of Manin [9]. We found fruitful in [2] to consider the following slightly different point of view: for Γ a subgroup of finite index of $\bar{\Gamma}(2)$, the corresponding modular curve X_Γ covers $X(2)$, which admits three cusps: $\Gamma(2)0$, $\Gamma(2)1$, and $\Gamma(2)\infty$. Let ∂_Γ^+ (resp. ∂_Γ^-) be the set of cusps above $\Gamma(2)0 \cup \Gamma(2)\infty$ (resp. $\Gamma(2)1$)). It is thus possible to consider the mixed homology group $\mathrm{H}_1(X_\Gamma - \partial_\Gamma^-, \partial_\Gamma^+; \mathbb{Z})$ (and its dual $\mathrm{H}_1(X_\Gamma - \partial_\Gamma^+, \partial_\Gamma^-; \mathbb{Z})$). One gets a group isomorphism

$$\xi_\Gamma^+ : \mathbb{Z}[\Gamma\backslash\bar{\Gamma}(2)] \to \mathrm{H}_1(X_\Gamma - \partial_\Gamma^-, \partial_\Gamma^+; \mathbb{Z})$$

which, for $g \in \bar{\Gamma}(2)$, associates to Γg the class $\xi_\Gamma(g)$ in $\mathrm{H}_1(X_\Gamma - \partial_\Gamma^-, \partial_\Gamma^+; \mathbb{Z})$ of a path from $g0$ to $g\infty$ in the upper half-plane.

Consider now the case where $\Gamma = \Phi'_N$. To simplify notations, set $\partial^+ = \partial^+$ and $\partial^- = \partial^-$. Recall that we have a group isomorphism $\Phi'_N\backslash\Gamma(2) \to H_{N,N,N'}$ which, for $(a, b, c) \in (\mathbb{Z})^3$, to $\Phi'_N A^a C^c B^b$ associates $x^a z^c y^b$. We get thus a group isomorphism

$$\mathbb{Z}[H_{N,N,N'}] \simeq \mathrm{H}_1(X'_N - \partial^-, \partial^+; \mathbb{Z}).$$

For $(a, b, c) \in (\mathbb{Z}/N\mathbb{Z})^2 \times (\mathbb{Z}/N'\mathbb{Z})$, set $i(a, b, c) = \xi^+_{\Phi'_N}(\Phi'_N A^a C^c B^b)$.

Thus, $H_{N,N,N'}$ acts on the curve X'_N, and transitively on the sets of cusps of X'_N above 0, 1, and ∞, respectively. Thus, the stabilizer of a cusp is cyclic of order N.

The long exact sequence of relative homology yields:

$$0 \to \mathrm{H}_1(X'_N - \partial^-; \mathbb{Z}) \to \mathrm{H}_1(X'_N - \partial^-, \partial^+; \mathbb{Z}) \xrightarrow{\delta_N} \mathbb{Z}[\partial^+]^0 \to 0,$$

where δ_N is the boundary map. Similarly we have a dual exact sequence:

$$0 \to \mathbb{Z}[\partial^-]^0 \xrightarrow{\delta_N^*} \mathrm{H}_1(X'_N - \partial^-, \partial^+; \mathbb{Z}) \to \mathrm{H}_1(X'_N, \partial^+; \mathbb{Z}) \to 0,$$

where δ_N^* is the dual boundary map. It induces

$$0 \to \mathbb{Z}[\partial^-]^0 \xrightarrow{\delta_N^*} \mathrm{H}_1(X'_N - \partial^-; \mathbb{Z}) \to \mathrm{H}_1(X'_N; \mathbb{Z}) \to 0.$$

Hence, $\mathrm{H}_1(X'_N; \mathbb{Z})$ can be described as a subquotient of the group $\mathrm{H}_1(X'_N - \partial^-, \partial^+; \mathbb{Z})$. We will make this explicit by spelling out the maps δ_N and δ_N^*.

The sets of cusps of X'_N lying above ∞, 0 and 1 are, respectively, $\Phi'_N\backslash\Gamma(2)/A^{\mathbb{Z}}$, $\Phi'_N\backslash\Gamma(2)/B^{\mathbb{Z}}$ and $\Phi'_N\backslash\Gamma(2)/(AB^{-1})^{\mathbb{Z}}$. All three sets can be understood as follows.

Proposition 3.8 *We have three bijective maps as follows:*

$$\Phi'_N\backslash\Gamma(2)/A^{\mathbb{Z}} \to (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N'\mathbb{Z})$$

given by $x^a z^c y^b \mapsto (b, c + ab)$,

$$\Phi'_N\backslash\Gamma(2)/B^{\mathbb{Z}} \to (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N'\mathbb{Z})$$

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given by $x^a z^c y^b \mapsto (a, c)$, *and*

$$\Phi'_N \backslash \Gamma(2)/(AB^{-1})^{\mathbb{Z}} \to (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N'\mathbb{Z})$$

given by $x^a z^c y^b \mapsto (a + b, c - b(b+1)/2)$.

Proof The first two identifications are straightforward. We establish the third one. Let k be an integer. One finds by induction on k, $x^a z^c y^b (xy^{-1})^k = x^{a+k} z^{c-kb+k(k-1)/2} y^{b-k}$. Since $(a + k) + (b - k) = a + b$ and

$$c - kb + k(k-1)/2 - (b-k)(b-k+1)/2 = c - b(b+1)/2,$$

the map passes indeed to the quotient $\Phi'_N \backslash \Gamma(2)/(AB^{-1})^{\mathbb{Z}}$. It is surjective (take $b = 0$). Since there are NN' cusps above 1, it is bijective. ∎

Denote by $j_\infty(b, c)$ the cusp $\Phi'_N A^a C^{c-ab} B^b A^{N\mathbb{Z}}$, for any $a \in \mathbb{Z}$, by $j_0(a, c)$ the cusp $\Phi'_N A^a C^c B^b B^{N\mathbb{Z}}$, for any $b \in \mathbb{Z}$, and $j_1(d, c)$ the cusp $\Phi'_N A^a C^{c-b(b+1)/2} B^b (AB^{-1})^{N\mathbb{Z}}$, for any $a, b \in \mathbb{Z}$ such that $a + b = d$. With these conventions we can express δ_N.

Proposition 3.9 *Let $a, b, c \in \mathbb{Z}$. One has $\delta_N(i(a, b, c)) = j_\infty(b, c - ab) - j_0(a, c)$.*

Proof The boundary of the modular symbol $\{A^a C^c B^b 0, A^a C^c B^b \infty\}$ is

$$[\Phi'_N A^a C^c B^b A^{\mathbb{Z}}] - [\Phi'_N A^a C^c B^b B^{\mathbb{Z}}],$$

which translates immediately into the claimed formula. ∎

We use [2, Proposition 5] to determine δ_N^*.

Proposition 3.10 *One has, for $d \in \mathbb{Z}/N\mathbb{Z}$ and $c \in \mathbb{Z}/N'\mathbb{Z}$,*

$$\delta_N^*(j_1(d, c)) = \sum_{\substack{a,b \in \mathbb{Z}/N\mathbb{Z}, \\ a+b=d}} i(a, b+1, c - b(b+1)/2) - i(a, b, c - b(b+1)/2).$$

Proof We just need to translate the third statement [2, Proposition 5]. With the notations of that proposition, we have $w_1 = N$. It remains to use the third bijection of Proposition 3.8 and the definition of i. ∎

Let S_N be the subgroup of $\mathbb{Z}[(\mathbb{Z}/N\mathbb{Z})^2 \times \mathbb{Z}/N'\mathbb{Z}]$ formed by the elements of the form $\sum_{a,b,c} \lambda_{a,b,c}[a, b, c]$, satisfying, for every $(a, c) \in (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N'\mathbb{Z})$, the relation $\sum_{b \in \mathbb{Z}/N\mathbb{Z}} \lambda_{a,b,c} = 0$ and, for every $(b, c) \in (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N'\mathbb{Z})$, the relation $\sum_{a \in \mathbb{Z}/N\mathbb{Z}} \lambda_{a,b,c+ab} = 0$. By Proposition 3.9, its image by i has boundary 0.

Let R_N be the subgroup of $\mathbb{Z}[(\mathbb{Z}/N\mathbb{Z})^2 \times \mathbb{Z}/N'\mathbb{Z}]$ spanned by elements of the form

$$e_{c,d} = \sum_{\substack{a,b \in \mathbb{Z}/N\mathbb{Z}, \\ a+b=d}} [a, b+1, c - b(b+1)/2)] - [a, b, c - b(b+1)/2],$$

for $(d, c) \in (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N'\mathbb{Z})$. By Proposition 3.10, it is a subgroup of S_N. Thus, we get a presentation by generators and relations of the homology of X'_N.

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Corollary 3.11 *The map i produces an exact sequence*

$$0 \to R_N \to S_N \to \mathrm{H}_1(X'_N;\mathbb{Z}) \to 0.$$

Remark 3.12 By exchanging the roles of ∂^+ and ∂^-, it is possible to give a dual presentation of $\mathrm{H}_1(X'_N;\mathbb{Z})$. We leave this to the reader.

4 The Heisenberg covering and its models

In this section, we assume N to be odd. Therefore $N' = N$.

4.1 Modular functions

Let $z \in \mathbb{H}$. For $q_2 = \exp(\pi i z)$, consider the classical λ-function [13]:

$$\lambda(z) = 16q_2 \prod_{n\geq 1}\left(\frac{1+q_2^{2n}}{1+q_2^{2n-1}}\right)^8, 1-\lambda(z) = \prod_{n\geq 1}\left(\frac{1-q_2^{2n-1}}{1+q_2^{2n-1}}\right)^8.$$

From the above expression, it is clear that $\lambda(1) = 1$ and $(1-\lambda(1)) = 0$. The Nth roots

$$x := \sqrt[N]{\lambda}, y := \sqrt[N]{1-\lambda}$$

define modular units for Φ_N. We recover thus the familiar model of the Fermat curve.

Since the λ function identifies $\mathbb{P}^1 - \{0,1,\infty\}$ to $Y(2)$, a covering of $\mathbb{P}^1 - \{0,1,\infty\}$ can be understood as a covering of $Y(2)$, i.e., a modular curve.

4.2 Reminder on Fermat curves

The Nth Fermat curve F_N is given by the projective model:

$$X^N + Y^N = Z^N.$$

Fermat curves and their points at infinity (cusps) are studied extensively by Rohrlich [16, 17], Vélu [18], and Posingies [15]. In particular, these authors consider the map

$$\beta_N : F_N \to \mathbb{P}^1$$

given by $(X:Y:Z) \to (X^N:Z^N)$. The map β_N is of degree N^2. It is ramified only above the points $0, 1, \infty$. The corresponding ramification points are given by $a_j = (0:\zeta^j:1)$, $b_j = (\zeta^j:0:1)$, $c_j = (\varepsilon\zeta^j:1:0)$, for $j \in \mathbb{Z}/N\mathbb{Z}$.

Recall that ζ is a primitive Nth root of unity and ε is a square root of ζ. Each of the above points has ramification index N over $\mathbb{P}^1$. For all $j \in \mathbb{Z}/N\mathbb{Z}$, the cusps a_j, b_j, c_j are all defined over the cyclotomic field $\mathbb{Q}(\mu_N)$. Among them, only a_0, b_0, and c_0 are defined over $\mathbb{Q}$.

4.3 The cuspidal subgroup of the Fermat curve

The divisors of following modular functions are given by:

$$\mathrm{div}(x-\zeta^j) = Nb_j - \sum_j c_j, \mathrm{div}(y-\zeta^j) = Na_j - \sum_j c_j, \mathrm{div}(x-\varepsilon\xi^j y) = Nc_j - \sum_j c_j.$$

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Rohrlich [17] has determined the structure of the cuspidal group of the Jacobian of F_N (see also Vélu's alternative proof and description [18]). Since every cuspidal divisor on F_N is annihilated by N in the Jacobian, the cuspidal group is a quotient of $\mathbb{Z}/N\mathbb{Z}[\partial_{\Phi_N}]^0$. The additional relations are given as follows. Recall that N is odd.

Theorem 4.1 (Rohrlich [17]) *The group $\mathcal{P}$ of principal divisors is spanned by the following set:*

$$\left\{ \sum_{i=0}^{N-1}[a_i]-[P], \sum_{i=0}^{N-1}[b_i]-[P], \sum_{i=0}^{N-1}[c_i]-[P], \right.$$
$$\left. \sum_{i=0}^{N-1} i([a_i]-[b_i]), \sum_{i=0}^{N-1} i([a_i]-[c_i]), \sum_{i=0}^{N-1} i^2([a_i]+[b_i]+[c_i]-3[P]) \right\},$$

where P is any cusp of F_N. Thus, the cuspidal subgroup of the Jacobian of F_N is a free $\mathbb{Z}/N\mathbb{Z}$-module of rank $3N-7$.

We set

$$D_A = \sum_{i\in\mathbb{Z}/N\mathbb{Z}} \{i\}[a_i]$$

(resp. $D_B = \sum_{i\in\mathbb{Z}/N\mathbb{Z}}\{i\}[b_i]$, resp. $D_C = \sum_{i\in\mathbb{Z}/N\mathbb{Z}}\{i\}[c_i]$) and

$$f_A = \prod_{i\in\mathbb{Z}/N\mathbb{Z}} (-y+\zeta^i)^{\{i\}}.$$

Corollary 4.2 *The class of D_A is of order N in the cuspidal subgroup of the Jacobian of F_N. Moreover, it is congruent to D_B and D_C modulo $\mathcal{P}$.*

Proof Indeed, the divisor of f_A is ND_A. By Rohrlich's theorem, D_A is of order N. The congruence of D_A, D_B, and D_C modulo $\mathcal{P}$ follows from Rohrlich's relations. ∎

4.4 The covering as a function field extension

By Rohrlich's theorem, the order of D_A in the cuspidal group is N, therefore an Nth root of f_A defines a cyclic covering $G \to F_N$ (a provisional notation, since G will be shown to be equal to F'_N) of degree N. Such a morphism is indeed unramified, since the divisor of f_A belongs to $N\mathbb{Z}[\partial_{\Phi_N}]^0$. Since $D_A - D_B$ is a principal divisor, exchanging the roles of X and Y would give the same covering. A similar reasoning with respect to D_C holds. The cyclic covering $G \to F_N$ translates into a covering of Riemann surfaces $W \to X_N$.

Consider now the third term $[\Gamma(2),[\Gamma(2),\Gamma(2)]]$ in the lower central series of $\Gamma(2)$. It is not a subgroup of finite index of $\Gamma(2)$, but one can still consider the Riemann surface $X_{[\Gamma(2),[\Gamma(2),\Gamma(2)]]}$.

Proposition 4.3 *The covering $W \to X_N$ factorizes through $X_{[\Gamma(2),[\Gamma(2),\Gamma(2)]]}$.*

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Proof Let $U \in \Gamma(2)$ and $V \in [\Gamma(2), \Gamma(2)]]$. Let g be an Nth root of f_A. One has to prove that g is invariant under $[U, V]$, that is, $g_{|UV} = g_{|VU}$. Note first that $y_{|U} = \zeta^r y$, with $r \in \mathbb{Z}$. One has

$$g_{|U}^N = \prod_{i \in \mathbb{Z}/N\mathbb{Z}} (-\zeta^r y + \zeta^i)^{\{i\}} = \prod_{i \in \mathbb{Z}/N\mathbb{Z}} (-y + \zeta^{i-r})^{\{i\}} = \prod_{i \in \mathbb{Z}/N\mathbb{Z}} (-y + \zeta^i)^{\{i+r\}}.$$

Write $\{i + r\} = \{i\} + r + t$, with $t \in N\mathbb{Z}$. Thus, we get

$$g_{|U}^N = g^N \prod_{i \in \mathbb{Z}/N\mathbb{Z}} (-y + \zeta^i)^r \prod_{i \in \mathbb{Z}/N\mathbb{Z}} (-y + \zeta^i)^t.$$

The last factor is obviously an Nth power. By the description of Rohrlich of the cuspidal subgroup of F_N, the factor $\prod_{i \in \mathbb{Z}/N\mathbb{Z}} (-y + \zeta^i)^r$ is an Nth power in the function field of F_N. So there exists a function h on F_N, and an integer s such that $g_{|U} = g\zeta^s h$. Note that $h_{|V} = h$ and $g_{|V} = \zeta^q g$, with $q \in \mathbb{Z}$. Therefore, one has

$$g_{|UV} = g_{|V} \zeta^s h_{|V} = \zeta^{s+q} gh = \zeta^q g_{|U} = g_{|VU}. \quad \blacksquare$$

Proposition 4.4 *Any covering of degree N of the Fermat modular curve X_N that factors through $X_{[\Gamma(2),[\Gamma(2),\Gamma(2)]]}$, factors through the Heisenberg covering.*

Proof Such a covering corresponds to a cyclic quotient of order N of Φ_N. Denote by Γ the corresponding cocyclic subgroup of Φ_N. Since the covering is unramified at the cusps, Γ contains the matrices A^N and B^N. Recall that Φ_N is generated by A^N, B^N and the commutator subgroup of $\Gamma(2)$. Since the covering factors through $X_{[\Gamma(2),[\Gamma(2),\Gamma(2)]]}$, the group Γ contains $[\Gamma(2), [\Gamma(2), \Gamma(2)]]$. In particular, Γ contains $[A, C]$ and $[B, C]$. Thus, the image of C in Φ_N/Γ is of order N. Consequently, Γ is the group generated by $[\Gamma(2), [\Gamma(2), \Gamma(2)]]$, A^N, B^N, C^N. $\blacksquare$

Corollary 4.5 *Let ζ be a primitive Nth root of unity in $\mathbb{Q}(\mu_N)$. The function*

$$f'_A = \prod_{i \in \mathbb{Z}/N\mathbb{Z}} (-y + \zeta^i)^{\{i\}}$$

admits an Nth root in the function field of F'_N.

Proof Indeed, the covering of F_N defined by an Nth root of f'_A is cyclic of order N and factorizes through $X_{[\Gamma(2),[\Gamma(2),\Gamma(2)]]}$. Consequently, it is F'_N. $\blacksquare$

Let K be the field of fractions of the curve given in the introduction over the complex numbers. In inhomogeneous form, it is generated by the variable X, Y, T_ζ for every primitive Nth root of unity ζ, with the relations

$$X^N + Y^N = 1$$

and, for every primitive Nth root of unity ζ (in $\mathbf{Q}(\mu_N)$)

$$\prod_{j=1}^{(N-1)/2} (Y - \zeta^{-j})^j T_\zeta^N = \prod_{j=1}^{(N-1)/2} (Y - \zeta^j)^j.$$

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Corollary 4.6 *The function field of F'_N over $\mathbb{C}$ is isomorphic to K.*

Proof Indeed, the function field of F'_N is the subfield of K generated by an Nth root of f_A over the function field of F_N. Let ζ' be a primitive Nth root of unity. By the preceding corollary, $T_{\zeta'}$ belongs to K. Thus, all of K is contained in the function field of F'_N. ∎

Thus, we have shown that the curve F'_N extends the Riemann surface X'_N.

Corollary 4.7 *The Riemann surfaces X'_N and $F'_N(\mathbb{C})$ are isomorphic.*

4.5 The Heisenberg covering

Denote by G the Galois group of the field extension $K|\mathbb{Q}(X^N)$. It sits in an exact sequence

$$1 \to H_N \to G \to (\mathbb{Z}/N\mathbb{Z})^\times \to 1$$

by the transitive action of G on Nth roots of unity, and the fact that the Heisenberg covering is defined over $\mathbb{Q}(\zeta)$. Recall that for $i \in (\mathbb{Z}/N\mathbb{Z})$, $\{i\}$ denotes the representative of i in $\{-(N-1)/2, \ldots, (N-1)/2\}$.

Proposition 4.8 *For $\sigma \in G$, there exists $u, v, s \in (\mathbb{Z}/N\mathbb{Z})$ and $r \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $\sigma(X) = \zeta^u X$, $\sigma(Y) = \zeta^v(Y)$, $\sigma(\zeta) = \zeta^r$. For ρ a representative of r in $\mathbb{Z}$,*

$$\sigma(T^\rho) = \zeta^s T X^v \prod_{i \in (\mathbb{Z}/N\mathbb{Z})} (-Y + \zeta^i)^{(\rho\{i/\rho\} - \{i\})/N}.$$

Furthermore, $(u, v, s, r) \in (\mathbb{Z}/N\mathbb{Z})^3 \times (\mathbb{Z}/N\mathbb{Z})^\times$ characterizes σ.

Proof The first three identities are evident. A simple calculation establishes the last one. Indeed,

$$\sigma(T^\rho)^N = \sigma(T^N)^\rho = \sigma\Big(\prod_{i \in (\mathbb{Z}/N\mathbb{Z})} (-Y + \zeta^i)^{\{i\}}\Big)^\rho = \prod_{i \in (\mathbb{Z}/N\mathbb{Z})} (-\zeta^v Y + \zeta^{\rho i})^{\{i\}\rho}.$$

By factoring T^N and $\prod_i \zeta^{v\{i\}}$, and replacing the variable i by $j = \rho i - v$, one gets

$$\sigma(T^\rho)^N = T^N \prod_i \zeta^{v\{i\}} \prod_{j \in (\mathbb{Z}/N\mathbb{Z})} (-Y + \zeta^j)^{\rho\{(j+v)/\rho\}} \prod_{j \in (\mathbb{Z}/N\mathbb{Z})} (-Y + \zeta^j)^{-\{j\}}.$$

We use $\prod_i \zeta^{v\{i\}} = 1$ and we get

$$\sigma(T^\rho)^N = T^N \prod_{j \in (\mathbb{Z}/N\mathbb{Z})} (-Y + \zeta^j)^{\rho\{(j+v)/\rho\} - \{j\} - v} \prod_{j \in (\mathbb{Z}/N\mathbb{Z})} (-Y + \zeta^j)^v.$$

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Using the identity $X^N = (1 - Y^N) = \prod_{j\in(\mathbb{Z}/N\mathbb{Z})}(-Y + \zeta^j)$, we get

$$\sigma(T^\rho)^N = T^N X^{N\nu} \prod_{j\in(\mathbb{Z}/N\mathbb{Z})} (-Y + \zeta^j)^{\rho\{(j+\nu)/\rho\}-\{j\}-\nu}.$$

The desired formula follows by taking Nth roots. ∎

With the notations of the proposition, we note $\sigma = \sigma_{u,v,r,s}$. Murty and Ramakrishnan note that Deligne showed that F'_N can be defined over $\mathbb{Q}$, without giving a reference. We spell out this assertion.

Proposition 4.9 *The curve F'_N can be defined over $\mathbb{Q}$. More precisely, the surjective map $G \to (\mathbb{Z}/N\mathbb{Z})^\times$ admits the map $r \mapsto \sigma_{0,0,r,0}$ as a section. Denote by S the corresponding subgroup of G. It acts trivially on Nth roots of unity. Consequently, the field of invariants by S in K is the function field of a curve over $\mathbb{Q}$, and defines F'_N over $\mathbb{Q}$. The Heisenberg covering $F'_N \to F_N$ extends also over $\mathbb{Q}$.*

Proof Group theoretic arguments about the structure of G as an extension imply easily the existence of the section. We will show that our explicit map is indeed a section. Let $r, r' \in (\mathbb{Z}/N\mathbb{Z})^\times$ and ρ and ρ' be representatives of r and r', respectively, in $\mathbb{Z}$. We need to check that $\sigma_{0,0,r',0}\sigma_{0,0,r,0} = \sigma_{0,0,rr',0}$, which need to be verified only by application on T, or equivalently on $T^{\rho\rho'}$, and on ζ. This is trivial for ζ. Here is the computation on $T^{\rho\rho'}$. We first simplify the formula of the previous proposition

$$\sigma_{0,0,r,0}(T^{\rho\rho'}) = T^{\rho'} \prod_{j\in(\mathbb{Z}/N\mathbb{Z})} (-Y + \zeta^j)^{(\rho\{j/\rho\}-\{j\})\rho'/N}.$$

Thus, one has

$$\sigma_{0,0,r',0}\sigma_{0,0,r,0}(T^{\rho\rho'}) = T \prod_{j\in(\mathbb{Z}/N\mathbb{Z})} (-Y + \zeta^j)^{(\rho'\{j/\rho'\}-\{j\})/N} \prod_{j\in(\mathbb{Z}/N\mathbb{Z})} (-Y + \zeta^{\rho' j})^{(\rho\{j/\rho\}-\{j\})\rho'/N}).$$

A change of variable in the second product of the right-hand side gives

$$\sigma_{0,0,r',0}\sigma_{0,0,r,0}(T^{\rho\rho'}) = T \prod_{j\in(\mathbb{Z}/N\mathbb{Z})} (-Y + \zeta^j)^{(\rho\rho'\{j/\rho\rho'\}-\{j\})/N} = \sigma_{0,0,rr',0}(T^{\rho\rho'}).$$

∎

4.6 Automorphisms

The group $(\mathbb{Z}/N\mathbb{Z})^2 \simeq \Gamma(2)/\Phi_N$ acts on F_N: $(i, j) \in (\mathbb{Z}/N\mathbb{Z})^2$ acts by the rule $(x, y) \mapsto (\zeta^i x, \zeta^j y)$. Such an action is defined on $\mathbb{Q}(\mu_N)$. It lifts to an action of $H_{\mathbb{Z}/N\mathbb{Z}} \simeq \Gamma(2)/\Phi'_N$ on F'_N.

Lemma 4.10 *The action of $H_{\mathbb{Z}/N\mathbb{Z}}$ on F'_N is defined over $\mathbb{Q}(\mu_N)$.*

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Proof For $U \in H_{\mathbb{Z}/N\mathbb{Z}}$, one needs to check that the action of U on a $\mathbb{Q}(\zeta)$-rational function is still $\mathbb{Q}(\zeta)$-rational. It suffices to verify this for g, an Nth root of f_A. We repeat a previous calculation and get

$$g_{|U}^N = g^N \prod_{i\in\mathbb{Z}/N\mathbb{Z}} (-y+\zeta^i)^r \prod_{i\in\mathbb{Z}/N\mathbb{Z}} (-y+\zeta^i)^t,$$

with $t \in N\mathbb{Z}$. Both factors $\prod_{i\in\mathbb{Z}/N\mathbb{Z}}(-y+\zeta^i)^r$ and $\prod_{i\in\mathbb{Z}/N\mathbb{Z}}(-y+\zeta^i)^t$ are Nth power, (of, say, g_1 and g_2) in the function field of F_N over $\mathbb{Q}(\zeta)$. We thus find that $g_{|U}$ is equal to $\zeta^p g g_1 g_2$, which belongs to the function field of X_N over $\mathbb{Q}(\zeta)$. ∎

Remark 4.11 We do not give an explicit algebraic model of the curve X_N''. But it can be obtained by taking N''th roots of the functions whose divisors are $\sum_{i=0}^{N-1} i^2([a_i]-[P])\}$, $\sum_{i=0}^{N-1} i^2([b_i]-[P])\}$ and $\sum_{i=0}^{N-1} i^2([c_i]-[P])\}$.

4.7 Regular integral models of Heisenberg curves

We relate the Heisenberg covering to the model given in the introduction. Note that the inhomogeneous version of the projective model is given by

$$X^N + Y^N = 1$$

and, for every primitive Nth root of unity ζ (in a fixed cyclotomic extension of $\mathbf{Q}$)

$$\prod_{j=1}^{(N-1)/2} (Y-\zeta^{-j})^j T_\zeta^N = \prod_{j=1}^{(N-1)/2} (Y-\zeta^j)^j.$$

We now prove Theorem 1.1 (in the spirit of, e.g., [3, Proposition 1.1.13]).

Proof We have indeed a scheme of relative dimension 1 over $\mathrm{Spec}(\mathbb{Z}[\mu_N, 1/N])$. We just have to establish the smoothness. We use the Jacobian criterion (e.g., [8, p. 130, Theorem 2.19]).

Since the Heisenberg covering is obtained by taking the Nth root of a function which does not vanish outside the zero locus of XY, all points away from $X = 0$ and $Y = 0$ are regular in all characteristics prime to N. It remains to establish the regularity of a point P_ζ of the form $(X, Y) = (0, \zeta)$ or $(X, Y) = (\zeta, 0)$ with ζ a primitive Nth root of unity. Suppose $(X, Y) = (0, \zeta)$. For points of this form, one has $T_\zeta = 0$. We set

$$F_\zeta = \prod_{j=1}^{(N-1)/2} (Y-\zeta^{-j})^j T_\zeta^N - \prod_{j=1}^{(N-1)/2} (Y-\zeta^j)^j$$

and compute the partial derivative at P_ζ. One obtains

$$\frac{\partial F_\zeta}{\partial Y}(P_\zeta) = -\prod_{j=2}^{(N-1)/2} (\zeta-\zeta^j)^j,$$

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which is a cyclotomic unit that belongs to $\mathbf{Z}[\mu_N, 1/N]^\times$. Thus, the Jacobian matrix is nonzero over any fiber in characteristic prime to N. This is sufficient to establish the regularity of P_ζ over $\mathrm{Spec}(\mathbf{Z}[\mu_N, 1/N])$ since the structural morphism is of relative dimension 1. This reasoning applies to the zero locus of Y, by exchanging the roles of X and Y. ∎

4.8 Cusps

The term cusp refers to the cusps of the modular curve X_Γ attached to a subgroup Γ of $\Gamma(2)$. These points are not intrinsic to the corresponding algebraic curves. In the cases of interest to us, we have a morphism $X_N \to X(2)$, given by the function x^N. Thus, the cusps of F_N are the $3N$ points above the points 0, 1, and ∞ described above.

Since the morphism $F'_N \to F_N$ is unramified, the modular curve F'_N possesses $3N^2$ cusps.

Proposition 4.12 *If N is prime to 3, the cusps of F'_N are defined over $\mathbb{Q}(\zeta)$. If 3 divides N, the cusps of F'_N are defined over the cyclotomic field generated by the $3N$th roots of unity.*

Proof Let a be a cusp of F'_N above a_0. It is defined over the field generated by $g(a)$ and $\mathbb{Q}(\zeta)$, where g is an Nth root of f_A. One has

$$f_A(a_0) = \prod_{i\in\mathbb{Z}/N\mathbb{Z}} (-1+\zeta^i)^{\{i\}} = \prod_{i=1}^{(N-1)/2} \frac{(-1+\zeta^i)^i}{(-1+\zeta^{-i})^i} = \prod_{i=1}^{(N-1)/2} (-\zeta^i)^i.$$

Thus, we get

$$f_A(a_0) = (-1)^{\sum_{i=1}^{(N-1)/2} i} \zeta^{\sum_{i=1}^{(N-1)/2} i^2} = (-1)^{(N^2-1)/8} \zeta^{(N-1)(N+1)N/24},$$

which is a sixth root of unity.

Suppose N is prime to 3. Then $g(a)$ is an Nth root of unity, up to sign. Since the group H_N acts transitively on the cups above ∞, and its action is $\mathbb{Q}(\zeta)$-rational, we deduce that all the cusps above ∞ are defined over $\mathbb{Q}(\zeta)$. A similar reasoning apply to the cusps above 0, and above 1.

A similar reasoning applies when 3 divides N. Indeed, $g(a)$ is a $3N$th root of unity, up to sign. ∎

The cusps of F'_N above the cusp ∞ (resp. 0, resp. 1) of $X(2)$ coincide with the classes $\Gamma\backslash\Gamma(2)\infty$ (resp. $\Gamma\backslash\Gamma(2)0$, resp. $\Gamma\backslash\Gamma(2)1$), which in turn can be identified with the double classes $\Gamma\backslash\Gamma(2)/A^{\mathbb{Z}}$ (resp. $\Gamma\backslash\Gamma(2)/B^{\mathbb{Z}}$, resp. $\Gamma\backslash\Gamma(2)/(AB^{-1})^{\mathbb{Z}}$).

4.9 About the Manin–Drinfeld principle for F'_3

Recall that $g_3 = 1$ and observe that F'_3 has 27 cusps. Fix one cusp P_0 of F'_3, which becomes thus an elliptic curve (F'_3, P_0). Since a cyclic group of order 3 acts on F'_3 and stabilizes P_0, the elliptic curve (F'_3, P_0) admits an automorphism of order 3. Thus, the j-invariant of (F'_3, P_0) is 0.

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Proposition 4.13 *Divisors supported on the cusps of F_3' above ∞ (resp. 0, resp. 1) are of order dividing 3 in the Jacobian of F_3'. Furthermore, cuspidal divisors of degree 0 are torsion in the Jacobian of F_3', and of order dividing 9.*

Proof Let X be a compact connected Riemann surface of genus 1. Let J be the Jacobian of X. Recall the exact sequence

$$0 \to J(\mathbf{C}) \to \mathrm{Aut}(X) \to \mathrm{Aut}(J) \to 0,$$

where Aut denotes the automorphisms over $\mathbf{C}$. The first map associates to the class of a divisor D the translation by D in X.

For $X = F_3'$, the group $\mathrm{Aut}(J)$ is cyclic of order 6. One gets a group homomorphism $H_{\mathbf{Z}/3\mathbf{Z}} \to \mathrm{Aut}(J)$, whose kernel is of order 9, and therefore isomorphic to $(\mathbf{Z}/3\mathbf{Z})^2$. Since this kernel identifies to a subgroup of the one dimensional complex torus $J(\mathbf{C})$, the latter subgroup is $J(\mathbf{C})[3]$. We have proved that the orbit of any cups Q by $H_{\mathbf{Z}/3\mathbf{Z}}$ contains $Q + J(\mathbf{C})[3]$. But these sets are both of cardinality 9, and are therefore equal. Since $H_{\mathbf{Z}/3\mathbf{Z}}$ acts transitively on the cusps above ∞ (resp. 0, resp. 1), the first statement of the proposition is proved.

About the second statement, it is sufficient to prove this for a divisor of the form $(\alpha) - (\beta)$, where α and β are cusps of F_3' not above the same point of $\{0, 1, \infty\}$. Without loss of generality, say they are above 0 and ∞, respectively. Let a and b be the cusps of F_3 below α and β, respectively. We have

$$3((\alpha) - (\beta)) = (3(\alpha) - \sum_{\alpha'}(\alpha')) + (\sum_{\alpha'}(\alpha') - \sum_{\beta'}(\beta')) + (\sum_{\beta'}(\beta') - 3(\beta)),$$

where α' (resp. β') runs through the cusps of F_3' above a (resp. b). By the first statement of the proposition and the torsion properties of the cuspidal subgroup of the F_3, each of the three terms of the right-hand side is of order dividing 3. ∎

Recall that the dessin for X_N' is a graph with the following additional structure: the vertices are bicolored (white and black) and the of set edges attached to any given vertex are endowed with a cyclic ordering (a transitive action of $\mathbb{Z}$). The vertices are the cusps of X_N' above 0 and ∞. The edges form the coset $\Phi_N' \backslash \Gamma(2) \simeq H_{\mathbb{Z}/N\mathbb{Z}}$, which is in bijection with $(\mathbb{Z}/N\mathbb{Z})^3$. the edge associated with $\Phi_N' g$ has extremities $\Phi_N' g 0$ and $\Phi_N' g \infty$. The cyclic ordering of the edges attached to the vertices $\Phi_N' g 0$ (resp. $\Phi_N' g \infty$) is given by the action of B (resp. A^{-1}). To sum up, the dessin can be drawn on X_N'.

To be more concrete, each edge is in bijection with $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N'\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, via the map $\Phi_N' A^a C^c B^b \mapsto (a, c, b)$. The edge thus labeled (a, c, b) is connected to the edge labeled $(a, c, b + 1)$ via a black vertex (cusp above 0) and the edge labeled (a, c, b) is connected to the edge labeled $(a - 1, c - ab, b)$ via a white vertex (cusp above ∞). The line segments represent the arcs on X_3' above the geodesic arc from 0 to ∞ in the upper half-plane.

We illustrate all this for $N = 3$. In that case, the genus of X_3' is equal to 1. According to these rules, the drawing (dessin) for X_3' is given as follows.

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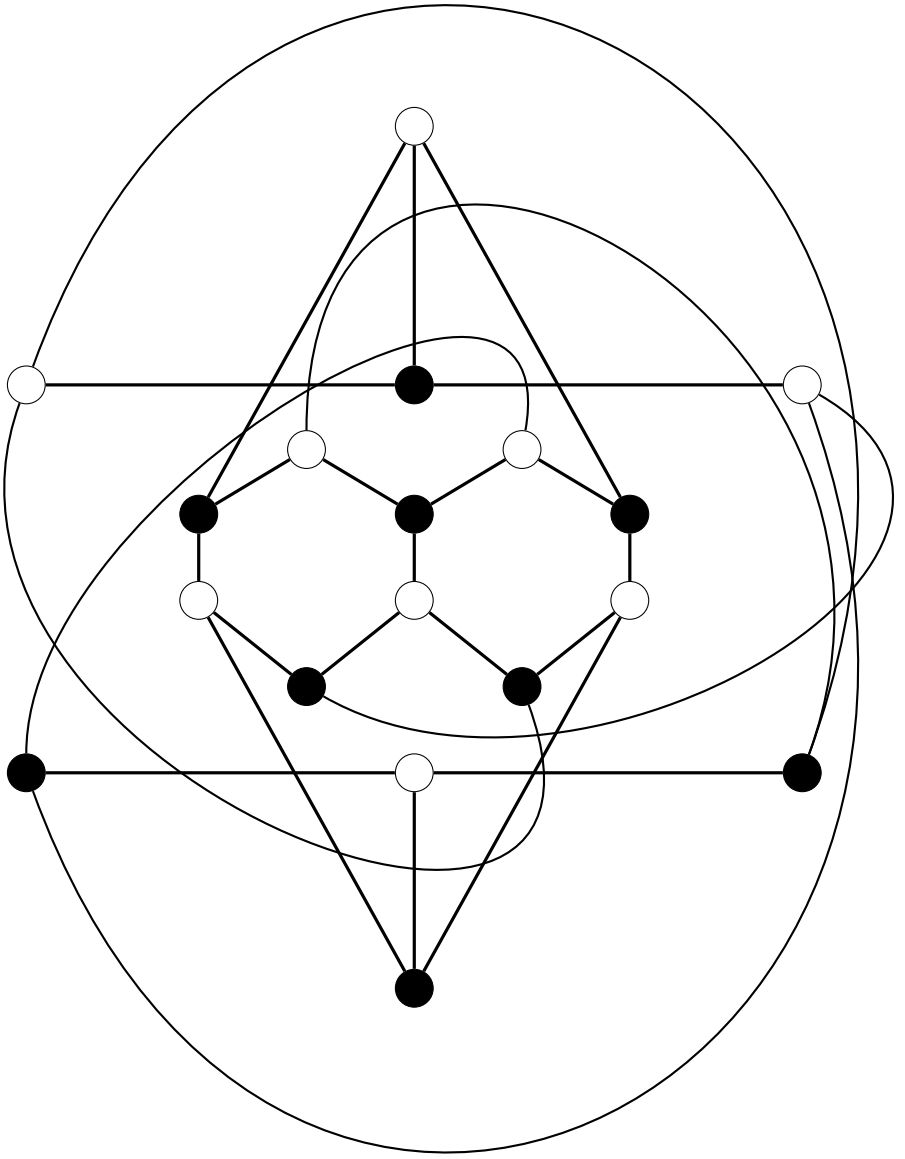

As the group Φ_3 is not a congruence subgroup [14], Φ_3' is *a fortiori* not a congruence subgroup. The latter fact can be derived alternately from Wolfart's criterion and an examination of the dessin. Indeed, the width of the each cusps is equal to 6 and $[\Gamma(2) : \Phi_3'] = 27$. Wolfart's criterion, to check that Φ_3' is a congruence subgroup or not it is enough to check $\Gamma(6) \subset \Phi_3' \subset \Gamma(2)$. However, $[\Gamma(2) : \Gamma(6)] = 144$ is not divisible by the index 27.

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4.10 About the failure of the Manin–Drinfeld principle for F_5'

Suppose $N = 5$. We show that the Manin–Drinfeld theorem fails for the Heisenberg covering $\mathcal{F}_5'$ of $\mathcal{F}_5$. Consider the scheme $\mathcal{C}_5$ over $\mathrm{Spec}(\mathbf{Z}[\mu_5, 1/5])$ given by the system of equations

$$T_\zeta^5 = \frac{(-Y+\zeta)(-Y+\zeta^2)^2}{(-Y+\zeta^{-1})(-Y+\zeta^{-2})^2},$$

where ζ runs through the primitive fifth roots of unity in $\mathbb{Q}(\mu_5)$. It is smooth, as can be shown by applying the Jacobian criterion. One has an obvious morphism of schemes $\mathcal{F}_5' \to \mathcal{C}_5$.

Denote by C_5 the generic fiber of $\mathcal{C}_5$. Over $\mathbb{C}$, C_5 identifies to a modular curve as follows. Consider the morphism $\bar{\Gamma}(2) \to H_{5,5,5}$ and the inverse image Γ' of the subgroup of $H_{5,5,5}$ generated by B. Then Γ' defines a corresponding modular curve isomorphic to the Riemann surface $C_5(\mathbb{C})$.

The curve C_5 possesses 19 cusps, given by the following planar coordinates (Y, T_ζ): $(0, \varepsilon)$, (∞, ε), $(1, -\delta)$, $(\zeta, 0)$, $(\zeta^2, 0)$, (ζ^{-1}, ∞), (ζ^{-2}, ∞), where ε and δ run through the fifth roots of unity. Set $T = T_\zeta$. The function fields of F_5' and C_5 are $\mathbf{Q}(\zeta, X, Y, T)$ and $\mathbf{Q}(\zeta, Y, T)$, respectively.

The obvious morphism $\pi\colon F_5' \to C_5$ sends the cusps of F_5' to the cusps of C_5. We show that C_5 does not satisfy the Manin–Drinfeld principle. Since $\mathcal{C}_5$ is smooth, the Jacobian of C_5 extends to an abelian scheme $\mathcal{J}_5$ over $\mathrm{Spec}(\mathbf{Z}[\mu_5, 1/5])$.

Proposition 4.14 *There exists a divisor of infinite order supported on the cusps of C_5.*

Proof We suppose that all cuspidal divisors are torsion in C_5. Our proof is organized around the following calculation. One has

$$T^5 + 1 = \frac{(1-Y)(2Y^2 + (2 - 2(\zeta^2 + \zeta^{-2}) - \zeta - \zeta^{-1})Y + 2)}{(-Y+\zeta^{-1})(-Y+\zeta^{-2})^2}. \tag{4.1}$$

Let y_1 and y_2 be the roots of the polynomial $2Y^2 + (2 - 2(\zeta^2 + \zeta^{-2}) - \zeta - \zeta^{-1})Y + 2$. Let ζ_1 be a fifth root of unity in $\mathbf{Z}[\mu_5]$. Consider the function $T + \zeta_1$, which divides $T^5 + 1$. The divisor of the function $T + \zeta_1$ is (in terms of planar coordinates for (Y, T)): $(1, -\zeta_1) + (y_1, -\zeta_1) + (y_2, -\zeta_1) - D$, where $D = (\zeta^{-1}, \infty) + 2(\zeta^{-2}, \infty)$. Apparently fortuitously, this divisor is cuspidal in the fibers at 11 and at 2 of C_5. ∎

Lemma 4.15 *Let ζ_1 and ζ_2 be distinct primitive fifth roots of unity in $\mathbb{Z}[\mu_5]$. The divisors $3(1, \zeta_1) - 3(1, \zeta_2)$ and $3(1, \zeta_1) - D$ are principal in any fiber above 11 of C_5.*

Proof In characteristic 11, the polynomial $2Y^2 + (2 - 2(\zeta^2 + \zeta^{-2}) - \zeta - \zeta^{-1})Y + 2$ has the providential property of having a double root equal to 1. Therefore, the divisor of the function $T + \zeta_1$ over $\mathbb{F}_{11}$ is cuspidal and equal to $3(1, \zeta_1) - D$. It follows that the function $(T + \zeta_1)/(T + \zeta_2)$ has divisor $3(1, \zeta_1) - 3(1, \zeta_2)$. ∎

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We return to the proof of the proposition. By our Manin–Drinfeld assumption in C_5, the divisor $3(1,\zeta_1) - D$ is torsion in the Jacobian of C_5. It extends to a torsion point of $\mathcal{J}_5$, whose order is determined in any special fiber at a prime π of residual characteristic p, provided $p-1 > e$, where e is the ramification index at π of the extension $\mathbb{Q}(\mu_5)|\mathbb{Q}$ [11]. This applies for any prime except perhaps $p = 2$ and $p = 5$. The calculation for $p = 11$ ensures that the divisors $3(1,\zeta_1) - 3(1,\zeta_2)$ and $3(1,\zeta_1) - D$ are principal in C_5.

Therefore, those divisors are principal in any special fiber of $\mathcal{C}_5$. Consider any fiber $\bar{C}$ above 2 of C_5. In that fiber, the function $T - \zeta = T + \zeta$ has divisor, in view of Equation (4.1), $(0,\zeta) + (1,\zeta) + (\infty,\zeta) - D_\zeta$, which is principal. Thus, by taking $\zeta_1 = \zeta$, the divisor

$$(3(1,\zeta) - D_\zeta) - ((0,\zeta) + (1,\zeta) + (\infty,\zeta) - D_\zeta) = 2(1,\zeta) - (0,\zeta) - (1,\zeta)$$

is principal in $\bar{C}$. Thus, there exists $f\colon \bar{C} \to \mathbb{P}^1$ of degree 2. So $\bar{C}$ is hyperelliptic. The principality of the divisor $3(1,\zeta_1) - 3(1,\zeta_2)$ ensures that there is a degree 3 morphism $\bar{C} \to \mathbb{P}^1$. By Castelnuovo–Severi-type inequalities, this imposes that the genus of $\bar{C}$ is ≤ 2. But the genus of $\bar{C}$ equals the genus of C_5; it is equal to 6 and we have reached a contradiction.

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