Two General Theorems in the Differential Calculus.

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1.

Theorem I. Let ${}_na_p$ denote

$$\frac{1}{p\,!}\left\{\delta^n(z^p) - pz\delta^n(z^{p-1}) + \frac{p(p-1)}{1\,.\,2}z^2\delta^n(z^{p-2}) - \ldots \pm pz^{p-1}\delta^n(z)\right\},$$

where $\delta \equiv \frac{d}{dx}$. *Then for any change in the independent variable x, say $z = f(x)$, the coefficient of $\frac{d^p}{dz^p}$ in $\frac{d^n}{dx^n}$ is ${}_na_p$.*

This theorem is true for all positive integral values of n and p. The expression under the brackets becomes symmetrical on adding the term $z^p\delta^n(z^\circ)$, which is zero, and is left out except in the particular case of $n = 0$, when it is z^p.

Example : $x = e^z$, or $z = \log x$.

By the above theorem

$$\frac{d^3}{dx^3} = \left\{\frac{d^3}{dx^3}(\log x)\right\}\frac{d}{dz} + \frac{1}{2\,!}\left\{\frac{d^3}{dx^3}(\log x)^2 - 2z\frac{d^3}{dx^3}(\log x)\right\}\frac{d^2}{dz^2} + \frac{1}{x^3}\,\frac{d^3}{dz^3},$$

$$= e^{-3z}\left\{2\frac{d}{dz} - 3\frac{d^2}{dz^2} + \frac{d^3}{dz^3}\right\},$$

$$= e^{-3z}\left(\frac{d}{dz} - 2\right)\left(\frac{d}{dz} - 1\right)\frac{d}{dz},$$

agreeing with a result well known in the theory of differential equations.

Proof of Theorem I. by Induction.

We have

$$\delta({}_na_p) = \frac{1}{p\,!}\big[\{\delta^{n+1}(z^p) - pz\delta^{n+1}(z^{p-1}) + \ldots \pm pz^{p-1}\delta^{n+1}(z)\}$$
$$- p\delta(z)\{\delta^n(z^{p-1}) - (p-1)z\delta^n(z^{p-2}) + \ldots \mp (p-1)z^{p-1}\delta^n(z)\}\big]$$
$$= {}_{n+1}a_p - \delta(z)\,.\,{}_na_{p-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

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Now if

$$\frac{d^s u}{dx^s} = {}_s a_1 \frac{du}{dz} + \dots + {}_s a_{p-1} \frac{d^{p-1}u}{dz^{p-1}} + {}_s a_p \frac{d^p u}{dz^p} + \dots,$$

then

$$\frac{d^{s+1}u}{dx^{s+1}} = \delta({}_s a_1) \cdot \frac{du}{dz} + \dots + \{{}_s a_{p-1} . \delta(z) + \delta({}_s a_p)\} \frac{d^p u}{dz^p} + \dots,$$

$$= {}_{s+1}a_1 \frac{du}{dz} + \dots + {}_{s+1}a_p \frac{d^p u}{dz^p} + \dots, \text{ by (A).}$$

Thus if the theorem holds for all positive integral values of p when $n = s$, it also holds when $n = s + 1$.

Again, when $n = 1$, ${}_1a_1 = \delta(z)$

and $$(p > 1),\ {}_1a_p = \frac{1}{p!}\{\delta(z^p) - pz\delta(z^{p-1}) + \dots \pm pz^{p-1}\delta(z)\},$$

$$= \frac{z^{p-1}\delta(z)}{(p-1)!}\left\{1 - (p-1) + \frac{p(p-1)}{1 . 2} - \dots \pm 1\right\}$$

$$= 0,$$

whence $\dfrac{du}{dx} = \delta(z)\dfrac{du}{dz}$.

So when $n = 2$, ${}_2a_1 = \delta^2(z)$,

${}_2a_2 = \{\delta(z)\}^2$,

$(p > 2)$, ${}_2a_p = 0$,

whence $$\frac{d^2u}{dx^2} = \delta^2(z) \cdot \frac{du}{dz} + \{\delta(z)\}^2 \cdot \frac{d^2u}{dz^2}.$$

Hence the theorem holds for all values of p when $n = 1, 2$. It follows in the usual way that the theorem holds for all values of p and n.

Cor. When $p > n$, ${}_na_p = 0$.

Note.—A more rigorous proof of this theorem can be based on a 'p' induction for all values of n. The above proof, however, has the advantage of much greater simplicity.

2.

In (A) let $n = p - 1$, then ${}_na_p = 0$.

$$\therefore\ {}_pa_p = \delta(z) \cdot {}_{p-1}a_{p-1}$$

$$= (\delta z)^2 {}_{p-2}a_{p-2}$$

etc.

$$\therefore\ {}_pa_p = \{\delta z\}^p \qquad \text{(B).}$$

Again, from (A) we have, putting $n=p+1$ and using (B),

$${}_{p+1}a_p=\delta\{(\delta z)^p\}+\delta(z)\,.\,{}_pa_{p-1},$$

$$=p(\delta z)^{p-1}\delta^2(z)+\delta(z)\{(p-1)(\delta z)^{p-2}\delta^2 z\}+(dz)^2\{(p-2)(dz)^{p-3}.\,\delta^2 z\}$$
$$+\ldots+(\delta z)^{p-1}\{\delta^2 z\},$$

$$\therefore\ {}_{p+1}a_p=(\delta z)^{p-1}\delta^2 z\{p+(p-1)+(p-2)+\ldots+1\}.$$

$$=\frac{(p+1)p}{2}\,.\,(\delta z)^{p-1}\delta^2 z.$$

Or $${}_pa_{p-1}=\frac{p(p-1)}{2}(\delta z)^{p-2}\delta^2 z\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{C}).$$

Similarly

$${}_pa_{p-2}=\frac{p(p-1)(p-2)(p-3)}{2\,.\,4}(\delta^2 z)^2(\delta z)^{p-4}+\frac{p(p-1)(p-2)}{2\,.\,3}\,.\,(\delta z)^{p-3}\delta^3(z)\ldots\ldots(\mathrm{D})$$

${}_pa_{p-3}$ is of the form $\alpha(\delta^2 z)^3(\delta z)^{p-6}+\beta(\delta^2 z)(\delta^3 z)(\delta z)^{p-5}+\gamma(\delta z)^{p-4}\delta^4 z$, where α, β, γ involve p and not z.

And ${}_pa_r$ is a sum of terms of the form

$$\mathrm{A}(\delta z)^\alpha(\delta^2 z)^\beta(\delta^3 z)^\gamma\ldots\ldots,\qquad(\mathrm{E})$$

where $$\alpha+2\beta+3\gamma+\ldots=p,$$

and A involves p and r, but not z.

Now in the transformation of $\dfrac{d^p}{dx^p}$ by means of the substitution $z=f(x)$, we are dealing with the coefficients ${}_pa_r$, $[r=p,\,p-1,\,\ldots 1]$.

To find when all these coefficients will be constant multiples of one another. We must have from (B) and (C) the relation

$$(\delta z)^p=k(\delta z)^{p-2}\delta^2 z,$$

$$\text{or}\quad k\delta^2 z=(\delta z)^2\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{F}).$$

$$\therefore\quad \delta z=\frac{1}{cx+d}.$$

$$\therefore\quad z=\log(cx+d).$$

By (F) the terms in ${}_pa_r$ all reduce to the form

$$\mathrm{A}(\delta z)^\alpha(\delta z)^{2\beta}(\delta z)^{3\gamma}\ldots,$$

$$i.e.\ \text{ to the form }\ \mathrm{A}(\delta z)^p,$$

and (F) therefore denotes a necessary and sufficient condition.

Hence $z = \log(cx+d)$ alone transforms $\frac{d^p}{dx^p}$ into

$$\phi(x)\left\{a\frac{d^p}{dz^p} + b\frac{d^{p-1}}{dz^{p-1}} + \ldots + k\frac{d}{dz}\right\},$$

where $a, b, \ldots k$ are constants. $\phi(x)$ has then the value $\frac{k}{(cx+d)^p}$ and we see that $z = \log(cx+d)$ transforms the differential equation $\Sigma(cx+d)^n\frac{d^n y}{dx^n} = 0$ into a linear equation with constant coefficients.

The conditions that $z = f(x)$ transforms

$$X_p\frac{d^p y}{dx^p} + X_{p-1}\frac{d^{p-1}y}{dx^{p-1}} + \ldots + X_0 y = 0,$$

where $X_p \ldots X_0$ are functions of x, into an equation with constant coefficients are

$$\begin{aligned} &X_p\,{}_p a_p = a, \\ &X_p\,{}_p a_{p-1} + X_{p-1}\,{}_{p-1}a_{p-1} = \beta, \\ &X_p\,{}_p a_{p-2} + X_{p-1}\,{}_{p-1}a_{p-2} + X_{p-2}\,{}_{p-2}a_{p-2} = \gamma, \end{aligned}$$

etc.

Thus if $X_p = x^p$, these conditions give $X_{p-1} = ax^{p-1}$, $X_{p-2} = bx^{p-2}$, etc. This is the case above discussed.

If $X_p = \frac{1}{\cos^2 x}$ and $a = 1$, $\therefore\ z = \sin x$,

$$\therefore\ X_{p-1} = \frac{a}{\cos x} + \frac{\sin x}{\cos^3 x},$$

Thus the equations

$$y'' + (a\cos x + \tan x)y' + \cos^2 x \,.\, y = 0,$$
$$y'' + \qquad \tan x \,.\, y' + \cot^2 x \,.\, y = 0,$$

are both reducible by the substitution $z = \sin x$.

3.

Theorem II. For a general transformation of the form

$$z = f(x), \quad y = u \,.\, \phi(x),$$

the coefficient of $\frac{d^r u}{dz^r}$ *in* $\frac{d^n y}{dx^n}$ *is*

$$\frac{1}{r!}\Big[\delta^n\{\phi(x).z^r\} - rz.\delta^n\{\phi(x).z^{r-1}\} + \frac{r(r-1)}{1\,.\,2}z^2\delta^n\{\phi(x).z^{r-2}\} - \ldots z^r\delta^n\{\phi(x)\}$$

In particular the coefficient of u in $\dfrac{d^ny}{dx^n}$ is $\delta^n\{\phi(x)\}$,

and the coefficient of $\dfrac{d^nu}{dz^n}$ in $\dfrac{d^ny}{dx^n}$ is $\phi(x)\,(\delta z)^n$.

Example: $x=e^z$, $y=ue^{2z}=ux^2$.

By the theorem, the coefficient of

$$\frac{d^2u}{dz^2} \text{ in } \frac{d^3y}{dx^3}=\tfrac{1}{2}[\delta^3\{x^2(\log x)^2\}-2\log x\,\delta^3\{x^2\log x\}],$$

$$=\tfrac{1}{2}\left[\frac{6}{x}+4\frac{\log x}{x}-\frac{4\log x}{x}\right],$$

$$=3/x=3e^{-z},$$

the coefficient of

$$\frac{d^3u}{dz^3} \text{ in } \frac{d^3y}{dx^3}=x^2\left(\frac{dz}{dx}\right)^3=x^2e^{-3z}=e^{-z},$$

and the coefficient of u in $\dfrac{d^3y}{dx^3}=0$.

These results agree with the formula used in differential equations, viz.,

$$\frac{d^3y}{dx^3}=e^{-z}\frac{d}{dz}\left(\frac{d}{dz}+1\right)\left(\frac{d}{dz}+2\right)u,$$

$$=e^{-z}\left\{\frac{d^3u}{dz^3}+3\frac{d^2u}{dz^2}+2\frac{du}{dz}\right\}.$$

Theorem II. follows at once from Theorem I., and the Theorem of Leibniz, for if $y=u\,.\,\phi(x)$,

$$\frac{d^ny}{dx^n}=\phi\,.\,\frac{d^nu}{dx^n}+{}_nc_1\delta\phi\,.\,\frac{d^{n-1}u}{dx^{n-1}}+\ldots+u\delta^n\phi,$$

and the coefficient of $\dfrac{d^ru}{dz^r}$ on the right

$$=\phi_na_r+{}_nc_1\delta\phi\,.\,{}_{n-1}a_r+\ldots+\delta^n\phi\,.\,{}_0a_r,$$

$$=\frac{1}{r!}[\{\phi\delta^n(z^r)+{}_nc_1\delta\phi\,.\,\delta^{n-1}(z^r)+\ldots+\delta^n\phi\,.\,z^r\}-\text{etc.}],$$

$$=\frac{1}{r!}[\delta^n\{\phi\,.\,z^r\}-rz\delta^n\{\phi\,.\,z^{r-1}\}+\frac{r(r-1)}{1\,.\,2}z^2\delta^n\{\phi\,.\,z^{r-2}\}$$

$$-\ldots\mp rz^{r-1}\delta^n\{\phi\,.\,z\}\pm z^r\delta^n\phi].$$

Also, since ${}_pa_p=(\delta z)^p$, it follows that the coefficient of

$$\frac{d^nu}{dz^n} \text{ in } \frac{d^ny}{dx^n} \text{ is } \phi(x)\left(\frac{dz}{dx}\right)^n,$$

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and the coefficient of u is clearly $\delta^n\{\phi(x)\}$. Hence Theorem II. is proved.

4.

Theorem: By the substitution $z=(ax+b)/(cx+d)$, $y=u(cx+d)^{n-1}$ $\frac{d^n y}{dx^n}$ is transformed into $\frac{p^n}{(cx+d)^{n+1}}\frac{d^n u}{dz^n}$, where $p=ad-bc$.

This Theorem follows from Theorem II. Thus, using the substitution $z=f(x)$, $y=u\,.\,\phi(x)$,

$$\frac{d^n y}{dx^n}\text{ becomes }\sum_{r=0}^{r=n}\frac{1}{r\,!}\Big[\delta^n\{\phi(x)\,.\,z^r\}-rz\delta^n\{\phi(x)\,.\,z^{r-1}\}+\ldots\pm z^r\delta^n\{\phi(x)\}\Big]\frac{d^r u}{dz^r},$$

The sufficient conditions that all the differential coefficients up to the $(n-1)^{\text{th}}$, as well as the term in u, may vanish, are

$$\delta^n\{\phi(x)\}=0\ldots\ldots(\text{i}),$$
$$\delta^n\{\phi(x)\,.\,z\}=0\ldots\ldots(\text{ii}),$$
$$\delta^n\{\phi(x)\,.\,z^2\}=0\ldots\ldots(\text{iii}),$$
$$\delta^n\{\phi(x)\,.\,z^{n-1}\}=0\ldots\ldots(n).$$

(i) gives $\phi(x)=a_0+a_1x+\ldots+a_{n-1}x^{n-1}$.

(n) then gives $z^{n-1}=\dfrac{A_0+A_1x+\ldots+A_{n-1}x^{n-1}}{a_0+a_1x+\ldots+a_{n-1}x^{n-1}}$,

and from (i) it follows that $z=(ax+b)/(cx+d)$.

Hence also $\phi(x)=(cx+d)^{n-1}$, and the theorem follows at once.

Thus if we take the equation

$(ax^2+\beta x+\gamma)^n y^{(n)}=ky$ where $ax^2+\beta x+\gamma=(ax+b)(cx+d)$ and apply the substitution $z=(ax+b)/(cx+d)$, $y=u(cx+d)^{n-1}$, it becomes $(ax+b)^n(cx+d)^n\dfrac{p^n}{(cx+d)^{n+1}}\dfrac{d^n u}{dz^n}=ku(cx+d)^{n-1}$,

i.e.
$$\left(\frac{ax+b}{cx+d}\right)^n\frac{d^n u}{dz^n}=k'u,$$

or
$$z^n\frac{d^n u}{dz^n}=k'u,$$

which has the solution
$$u=\sum_{m=m_1}^{m=m_n}A_m z^m$$

where $m_1,\ldots,m_n$ are the roots of the equation
$$m(m-1)\ldots(m-n+1)-k'=0.$$

5.

Generally speaking, it is only when we are dealing with linear equations that the discovery of a particular integral helps us to the complete solution. Thus for the equation

$$9xy^2y'' + 2 = 0,$$

it is easy to find the particular integral $y = x^{1/3}$, but since the equation is not linear, this does not lead to a complete solution. If we apply the transformation $z = 1/x$, $y = ux$, which is a particular case of the transformation of §4, $p = 1$, and the equation reduces to

$$u^2u'' + 2/9 = 0.$$

The complete solution

$$\int \frac{du}{(1/u + c_1)^{1/2}} = 2/3(z + c_2) = 2/3(1/x + c_2)$$

is now easily obtained.

In this example we reduced the equation to a known form. We shall consider from this point of view the general equation

$$\frac{d^2y}{dx^2} + \psi(xy) = 0 \quad \text{.........................(1)}$$

Putting $z = f(x)$, and $y = u \, . \, \phi(x)$, (1) becomes

$$\frac{d^2u}{dz^2} + P\frac{du}{dz} + Q = 0,$$

where $P = \frac{\delta^2\{z \, . \, \phi\} - z\delta^2\phi}{\phi \, . \, (\delta z)^2} = \frac{2\delta z \, . \, \delta\phi + \phi \, . \, \delta^2 z}{\phi \, . \, (\delta z)^2}$

and $Q = \frac{u\delta^2\phi + \psi(xy)}{\phi \, . \, (\delta z)^2}.$

$$\frac{dQ}{dz} = \frac{\phi \, . \, \delta z \, . \, u \, . \, \delta^3\phi + \phi \, . \, \delta z \, . \, \psi_x + \phi \, . \, \delta z \, . \, \psi_y \, . \, u\delta\phi - (u\delta^2\phi + \psi)(\delta\phi \, . \, \delta z + 2\phi \, . \, \delta^2 z)}{\phi^2(\delta z)^4}$$

$$\frac{dP}{dz} = \frac{2\phi \, . \, (\delta z)^2 \, . \, \delta^2\phi - 2\phi \, . \, \delta z \, . \, \delta^2 z \, . \, \delta\phi + \phi^2\delta z \, . \, \delta^3 z - 2(\delta z)^2(\delta\phi)^2 - 2\phi^2(\delta^2 z)^2}{\phi^2(\delta z)^4}.$$

Hence P and Q are independent of z if

$$\phi \, . \, \delta z \, . \, u\delta^3\phi + \phi\delta z\psi_x + \phi\delta z\psi_y u\delta\phi - (u\delta^2\phi + \psi)(\delta\phi \, . \, \delta z + 2\phi \, . \, \delta^2 z) = 0, \quad \text{(A)}$$

and $2\phi(\delta z)^2\delta^2\phi - 2\phi\delta z\delta^2 z\delta\phi + \phi^2\delta z\delta^3 z - 2(\delta z)^2(\delta\phi)^2 - 2\phi^2(\delta^2 z)^2 = 0.$ (B)

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We have a particular solution of (B) when $P=0$,

i.e. when $$\phi\delta^2 z + 2\delta\phi\,.\,\delta z = 0 \quad (\alpha)$$

or $$\delta z = a/\phi^2 \quad (\beta).$$

Using (α) in (A) it reduces to

$$\delta z\,.\{\phi\,.\,\psi_x + u\phi\,.\,\delta\phi\,.\,\psi_y + 3\delta\phi\,.\,\psi + u(\phi\delta^3\phi + 3\delta^2\phi\,.\,\delta\phi)\} = 0.$$

or $$\phi\psi_x + y\delta\phi\,.\,\psi_y + 3\delta\phi\,.\,\psi + y\delta^3\phi + \frac{3y\delta^2\phi\,.\,\delta\phi}{\phi} = 0 \quad [\delta z \neq 0], \quad \text{(C)}.$$

The Lagrangian Subsidiary System is

$$\frac{dx}{\phi} = \frac{dy}{y\,.\,\phi'} = \frac{d\psi}{-3\phi'\psi - y\phi''' - \dfrac{3y\phi''\phi'}{\phi}}$$

$$\frac{dx}{\phi} = \frac{dy}{y\phi'} \text{ gives } \frac{d\phi}{\phi} = \frac{dy}{y}.$$

$$\therefore\quad y/\phi = a(\text{const.}).$$

Using this in the last equation, we have

$$\frac{d\psi}{d\phi} + \frac{3}{\phi}\,.\,\psi + a\frac{\phi'''}{\phi'} + 3a\frac{\phi''}{\phi} = 0,$$

$$\textit{i.e.}\quad \frac{d}{d\phi}\{\phi^3(\psi + a\phi'')\} = 0.$$

Hence $$\phi^3(\psi + a\phi'') = b$$

or $$\phi^3\left(\psi + y\frac{\phi''}{\phi}\right) = b \text{ (const.)}.$$

Hence the general solution of (C) may be written

$$\psi = \frac{1}{\phi^3}\chi(y/\phi) - y/\phi\,.\,\phi'' \quad [\chi \text{ arbitrary}],$$

or $$= \frac{1}{y^3}\chi(y/\phi) - y/\phi\,.\,\phi''.$$

Hence $\dfrac{d^2y}{dx^2} + \dfrac{1}{y^3}\chi(y/\phi) - y/\phi\,.\,\phi'' = 0$ can be reduced to $\dfrac{d^2u}{dz^2} + Q = 0$, where Q does not contain z, by means of the substitution

$$y = u\phi,\quad z = \int\frac{1}{\phi^2}dx.$$

Special cases :

1° $\phi''=0, \therefore \phi = ax+b$ and $\psi = \frac{1}{y^3}\chi\left(\frac{y}{ax+b}\right)$, and from ($\beta$), $z=\frac{cx+d}{ax+b}$.

Hence substitution $y=u(ax+b)$, $z=\frac{cx+d}{ax+b}$ will reduce the equation

$$\frac{d^2y}{dx^2}+\frac{1}{y^3}\chi\left(\frac{y}{ax+b}\right)=0$$

2° $$\phi''=c/\phi^3.$$

$$\therefore \quad \phi'^2=\frac{a\phi^2+\beta}{\phi^2},$$

$$i.e. \quad \frac{\phi d\phi}{\sqrt{a\phi^2+\beta}}=dx.$$

$$\therefore \quad x+\gamma = 1/a\sqrt{a\phi^2+\beta}$$

$$\text{or} \quad \phi = \sqrt{ax^2+bx+c} \quad \therefore \psi = \frac{1}{y^3}\chi\left(\frac{y}{\sqrt{ax^2+bx+c}}\right).$$

$\therefore$ the substitution $y=u\,.\,\sqrt{ax^2+bx+c}$, $z=\int\frac{dx}{ax^2+bx+c}$

reduces the equation $\frac{d^2y}{dx^2}+\frac{1}{y^3}\chi\left(\frac{y}{\sqrt{ax^2+bx+c}}\right)=0.$

6.

The substitution $z=(ax+b)/(cx+d)$, $y=u(cx+d)^{n-1}$ will reduce the more general equation $\frac{d^ny}{dx^n}+\psi(xy)=0$ to a known form if

$$\psi=\frac{1}{(cx+d)^{n-1}}\chi\left(\frac{1}{(cx+d)^{n-1}}\right), \quad \chi \text{ arbitrary.}$$

7.

The equation $\frac{d^2y}{dx^2}+\frac{1}{y^3}\chi(y/x)=0$ (1) is homogeneous in the sense that all the terms are of the same order when y and x are considered of order 1, and y'' of order -3. In certain cases it is also homogeneous when y is considered of order n, y' of order $n-1$, etc., and x of order 1, when it will be reducible to an equation of the 1st order by the substitution $x=e^z$, $y=ux^n$ (2).

Therefore, corresponding to the cases where (1) is homogeneous in both senses, we have a soluble class of equations of the 1st order.

(1) is homogeneous in the 2nd sense when and only when $4n-2=a(n-1)$ (3), and the equation is $y^3y''=A(y/x)^a$.

Using (2) and putting $p=\frac{du}{dz}$, this equation becomes

$$p\frac{dp}{du}+(2n-1)p+n(n-1)u-\frac{A}{u^{3-a}}=0.$$

Hence $$p\frac{dp}{du}+(2n-1)p+n(n-1)u+Au^{\frac{n+1}{n-1}}=0$$

is a soluble class of equations.

Examples :

1° $$y\frac{d^2y}{dx^2}-y^2=\sec^2x \text{ can be put in the form}$$

$$\frac{d^2y}{dx^2}=\frac{1}{\cos^3x}\cdot\left(\frac{\cos x}{y}\right)+y,$$

which is of the form of § 5 when $\phi''=-\phi$, *i.e.* $\phi=\cos x$.

Therefore substitution $z=\tan x$, $y=u\cos x$ reduces this equation.

So for $yy''+y^2=\text{sech}^2x$.

2° $$z=\frac{1}{x^3},\quad y=ux^2 \text{ reduces}$$

$$y''=2y\left(\frac{1}{x^8}-\frac{1}{x^2}\right).$$

3° $$p\frac{dp}{du}+3p+2u+u^3=0.$$

$$p\frac{dp}{du}-3p+2u+1=0$$

$$p\frac{dp}{du}-p+\frac{1}{u}=0$$

are of soluble type of § 7.

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