

**On Wilkinson's Method of treating the Nine-Points Circle,
with Generalizations.**

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Ever since I first became acquainted with Wilkinson's method of establishing the existence of the nine-points circle of a triangle (see Mackay's "Euclid," Appendix to Bk. IV., Prop. 2, the lettering of which I have followed in the first three sections of this paper), its simple and fundamental character has pleased me. I propose to point out first that this method yields probably the most elementary proof of the concurrence of the perpendiculars from the vertices, and then, after restating the investigation of the nine-points circle, to sketch some generalizations.

§1.

The method depends on the following propositions, which are either contained in Euclid's first book or are easy and well-known deductions from it. I shall use the symbol \equiv to denote "is equal and parallel to, and has the same sense as."

- Prop. (i) If $AB \equiv CD$ then $AC \equiv BD$.
- „ (ii) If $AB \parallel CD$ and $AC \parallel BD$ then $AB \equiv CD$ and $AC \equiv BD$.
- „ (iii) If $AB \equiv CD$ then AD and BC bisect one another.
- „ (iv) If AD and CD bisect one another, then $AB \equiv CD$ and $AC \equiv BD$.
- „ (v) If D is the mid point of AB and E that of AC , then $BC \equiv 2DE$.
- „ (vi) If D is the mid point of AB and $DE \parallel BC$, E being in AC , then E is the mid point of AC .
- „ (vii) If $ABCD$ is a rectangle, AC and BD are equal and bisect one another.
- „ (viii) If AC and BD are equal and bisect one another, $ABCD$ is a rectangle.

Let ABC be a triangle (Fig. 1). Let the perpendiculars BY, CZ let fall from B and C on AC and AB intersect in O .

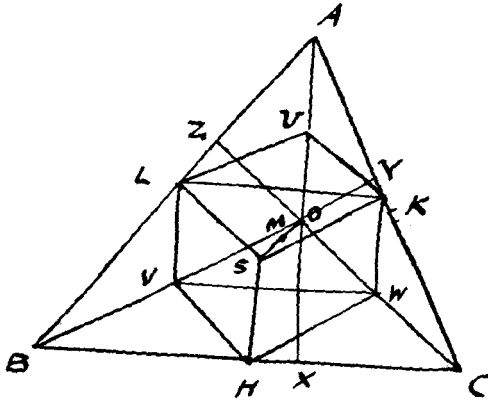


Fig. 1.

Let AO meet BC in X : we have to prove that AX is perpendicular to BC .

Let H, K, L , be the mid points of BC, CA, AB respectively.

„ U, V, W , „ „ „ OA, OB, OC „

By (v) $UV \equiv \frac{1}{2}AB \equiv KH$.

$VH \equiv \frac{1}{2}OC \equiv UK$.

Hence $UVHK$ is a parallelogram whose sides are parallel to AB and CO , which are at right angles. Hence $UVHK$ is a rectangle.

Hence by (vii) HU and KV are equal, and bisect one another.

Similarly $WULH$ is a rectangle, and HU and LW are equal, and bisect one another.

Hence KV and LW are equal, and bisect one another.

Hence $KLWV$ is a rectangle by (viii).

Hence BC and AX , which by (v) are parallel to its sides, are mutually perpendicular.

Thus the three perpendiculars from A, B, C on the opposite sides are concurrent. Q. E. D.

Cor. 1. By (vii) HU, KV, LW are all equal and bisect one another in the same point M , which is therefore the centre of a circle passing through $UVWHKL$.

By (vii) this circle also passes through XYZ , since HXU form two sides of a rectangle whose mid point is M .

Cor. 2. By (v) $OVHW, OWKU, OULV$ are parallelograms, and if we complete the parallelogram $LVHS$, then

$$SH \equiv LV \equiv UO \equiv KW \text{ by (ii).}$$

Hence by (iii) M is the mid point of SO .

$$\begin{aligned} \text{By (i)} \quad SK &\equiv HW \equiv VO \equiv LU \\ SL &\equiv KU \equiv WO \equiv HV. \end{aligned}$$

Hence S is the circumcentre.

Also $AO \equiv 2UO \equiv SH$, etc.

Cor. 3. The triangles UVW and ABC are homothetic, O being the homothetic centre, and M and S corresponding points, the ratio of linear dimensions being $1:2$.

Hence OX produced meets the circumcircle of ABC at a distance from $O = 2OX$.

Cor. 4. If AH meets SO in G , since $AO = 2SH, OG = 2GS$. Similarly BK and CL cut OS in the same point G , which is therefore the point of concurrence of the medians.

§2.

The generalization from the nine-points circle to the nine-points conic worked out by Dr P. Pinkerton in his paper in Vol. XXIV. can be established with great ease by Wilkinson's method.

Let S be any point in the plane of ABC , and H, K, L , as before, the mid points of BC, CA, AB .

Let $CZ \parallel$ to SL and $BY \parallel$ to SK meet AB in Z, CA in Y , and each other in O .

Let AO meet BC in X .

As before, let U, V, W be the mid points of OA, OB, OC .

Obviously $OVHW, OWKU, OULV$ are parallelograms, and $SK \parallel$ to $OB \parallel$ to UL } \therefore $SLUK$ is a parallelogram.

„ $SL \parallel$ to $OC \parallel$ to UK }

$\therefore SK \equiv LU \equiv VO \equiv HW$.

$\therefore SHWK$ and similarly $SHVL$ are parallelograms.

Thus $SH \equiv KW \equiv UO$, *i.e.* the lines through A, B, C parallel to SH, SK, SL are concurrent.

We have also SO, UH, VK, LW bisecting one another in the same point M.

Now UVWMO and ABCSO are homothetic, the ratio of their linear dimensions being 1 : 2.

Hence if any figure be described about ABC, a homothetic figure of half of its linear dimensions can be described about UVW.

If the circumscribing figure be a *central* curve of any sort, with S as centre, then the homothetic curve round UVW will have M as centre, and will therefore pass through HKL.

If the central curve be a *conic*, so that supplementary chords are parallel to conjugate diameters and conversely, then XYZ will also lie on the conic UVWHKL. For SH and BC being conjugate directions as to the conic through ABC, they are so as to the conic UVWHKL. Thus HX, UX, drawn through the extremities of a diameter HU, are parallel to a pair of conjugate diameters, and therefore X lies on the conic. Similarly Y and Z.

Every line from O to a point of the smaller conic meets the larger at double the distance. We note the special cases of the points X, Y, Z, H, K, L.

Thus the properties of the nine-points conic investigated by Dr Pinkerton in his paper in Vol. XXIV. have been arrived at by Wilkinson's method, with the additional aid of the elements of the theory of homothesis, and the proposition about supplementary chords and conjugate diameters of a conic.

§3.

The whole of the preceding theory can be generalized further by first generalizing the fundamental propositions (i)...(viii) on which it is based.

We define certain phrases.

Given a fixed line α , if H is the harmonic conjugate as to BC of its intersection with α , we shall say that H "bisects BC as to α ."

If AB and CD are concurrent in α and also AC and BD, we shall say $AB \equiv CD$ and $AC \equiv BD$.

Hence the symbol \equiv refers only to segments of lines which are concurrent in α .

When α is the line at infinity, this meaning reduces to that previously assigned to the symbol \equiv , and the fundamental

property. "If $AB \equiv CD$ and $CD \equiv EF$, then $AB \equiv EF$," required to make the preceding proofs apply, is obviously a special case of Desargues' Theorem applied to the perspective triangles ACE, BDF.

In place of the word *parallel* we shall in the generalized theory substitute *concurrent in a*.

For our fundamental propositions (i)-(viii) in the generalized theory we have :

- (i) and (ii) are immediate results of the extended definitions.
- (iii) If $AB \equiv CD$, then AD and BC "bisect one another as to a ."
This is the harmonic property of the complete quadrilateral.
- (iv) If AD and BC "bisect one another as to a ," then $AB \equiv CD$ and $AC \equiv BD$.
- (v) If D "bisects AB as to a " and E "bisects AC as to a ," then $BC \equiv 2DE$.
(iv) and (v) are cases of the proposition that if two harmonic ranges have one pair of corresponding points coincident, the joins of the other corresponding pairs are concurrent.
- (vi) If D "bisects AB as to a " and DE and BC are concurrent in a , E being in AC, then E "bisects AC as to a ." This is an application of the theorem that if one transversal of a pencil gives a harmonic range, so does any other.
- (vii) and (viii) do not occur in the generalized theory.

The generalised theory begins with H, K, L "bisecting as to a " the sides BC, CA, AB, and BY, CZ concurrent in a with SK, SL respectively, S being *any* point. O is the intersection of BY, CZ. U, V, W "bisect" OA, OB, OC "as to a ." (See Fig. 2.)

LU and BO are concurrent in a by (v).

\therefore LU and SK are concurrent in a .

Similarly SL and UK " " " "

\therefore $SK \equiv LU \equiv VO \equiv HW$.

\therefore $SH \equiv KW \equiv OU$.

\therefore AX and SH are concurrent in a .

It follows that :

If lines be drawn through A, B, C respectively, concurrent in a with SH, SK, SL respectively, where H, K, L "bisect" BC, CA, AB "as to a ," these lines will be concurrent in a point O.

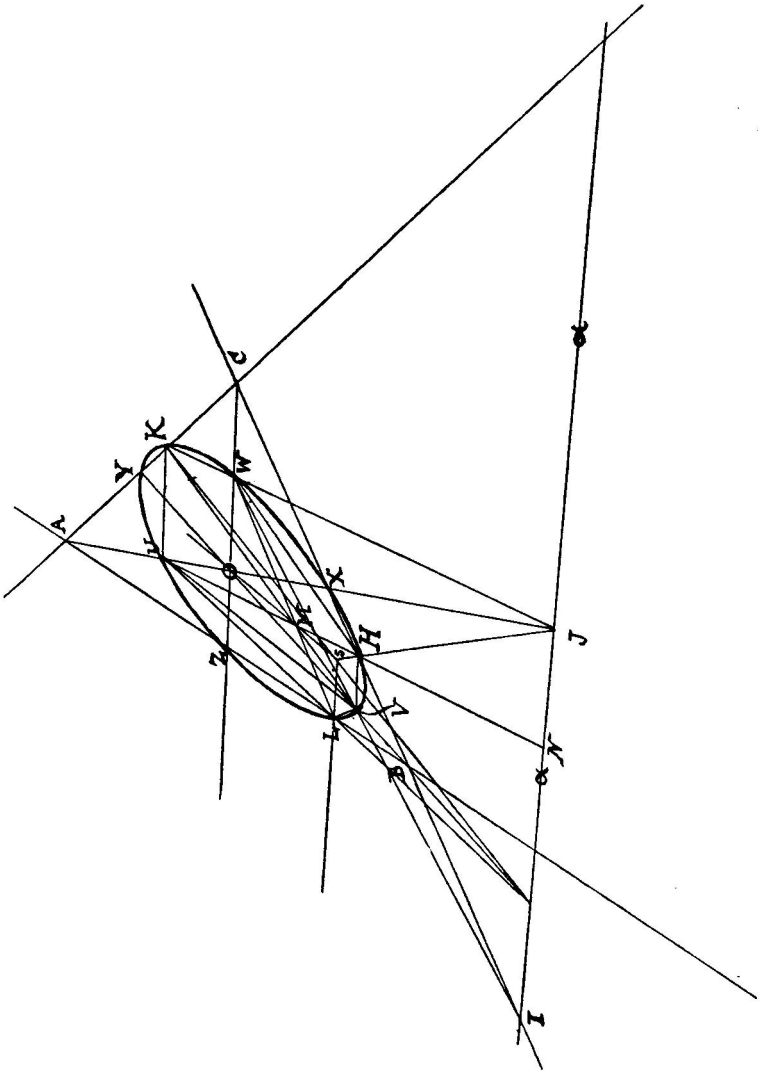


Fig. 2.

The eight points O, S, U, V, W, H, K, L , are such that OS, HU, KV, LW all "bisect one another as to a " in the same point M .

To get the nine-points conic in its generalised form, we must give the appropriate generalization of the theory of homothesis, and of the property that supplementary chords have conjugate directions.

If every point P of a figure ϕ be joined to a fixed point M , and PM be divided by P' and the fixed line a , in a given anharmonic ratio λ , then ϕ' , the assemblage of points P' may be said to be "homothetic as to a " with ϕ , having M as homothetic centre, and λ as homothetic ratio, for lines joining corresponding points of ϕ and ϕ' will obviously be concurrent in a .

(Here ϕ and ϕ' are in perspective, the perspective centre being M and the perspective axis a).

It follows that if PQR in ϕ be collinear, then $P'Q'R'$ in ϕ' will also be collinear. Hence by Pascal's Theorem, if six points in ϕ are on a conic, the corresponding six points in ϕ' are also on a conic, so that every figure "homothetic as to a " with a conic is itself a conic.

If then Γ be a conic round ABC , and S the pole of a as to Γ , and if γ be a conic homothetic with Γ as to a , with O as homothetic centre and homothetic ratio $= -1$, then γ will pass through UVW , and M will be the pole of a as to γ .

Hence if HU meets a in N , HU is divided harmonically by M and N . Hence, U being on γ , so is H . Similarly K and L are on γ .

The generalised form of the proposition that supplementary chords of a conic are in conjugate directions is this :

Two points on a are said to be conjugate as to the conic γ if each lies on the polar of the other.

If PQ is a chord passing through M the pole of a (see Fig. 3), and R is any other point on a , then PR and QR may be said to be chords of γ which are "supplementary as to a ."

Such chords intersect a in points q and p , which are conjugate.

For if a and PQ intersect in r and qM and QR meet in T , then $\{QRTp\} = \{QPMr\} = -1$.

$\therefore T$ is on the polar of p . So is M , $\therefore qM$ is the polar of p .

Thus if PR and QR are "supplementary" chords of γ "as to a ," then PR and QR meet a in conjugate points.

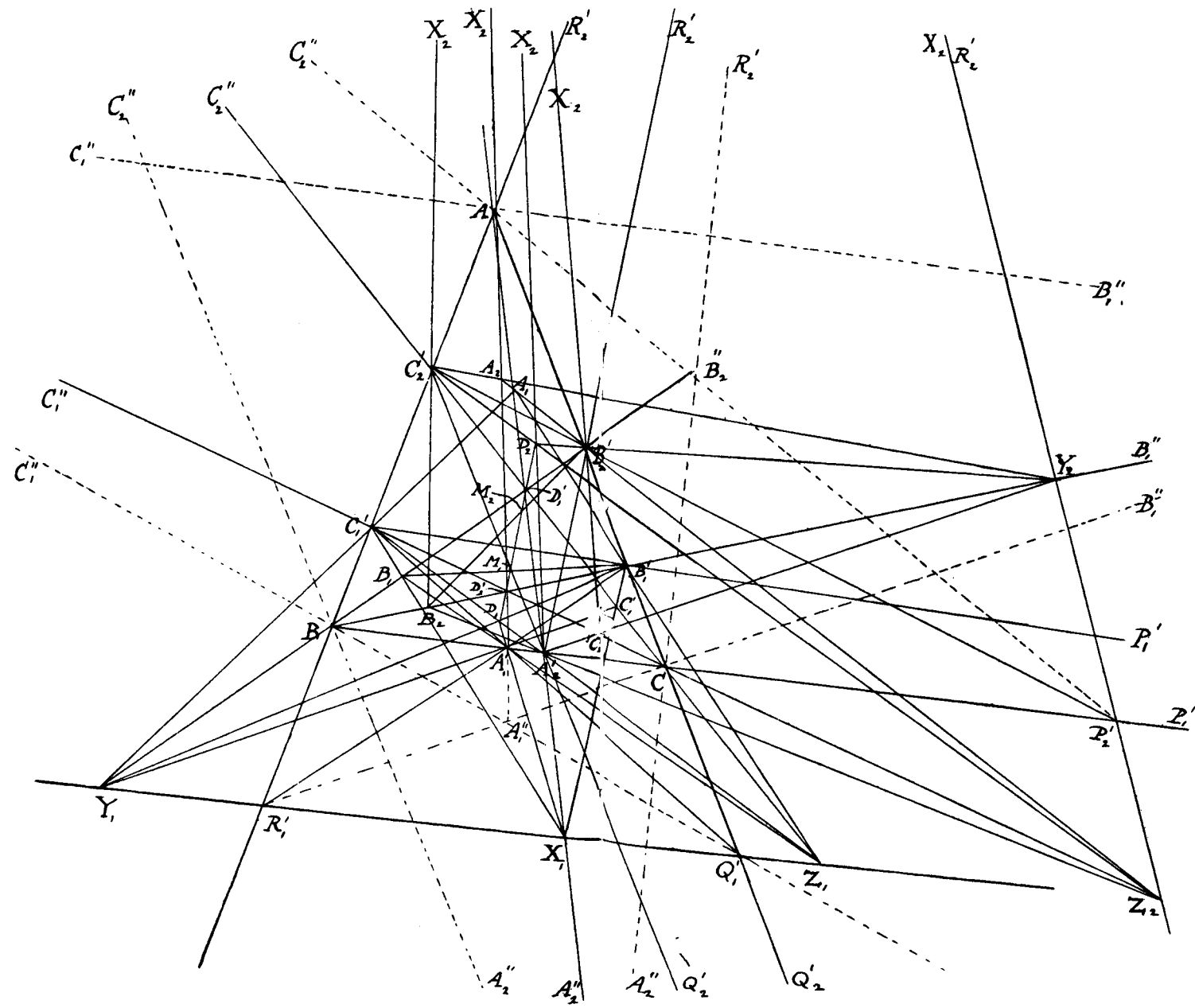


Fig. 1.

Conversely, if through the extremities of a chord of γ through M lines be drawn to conjugate points on a , these will meet in a point R on the conic.

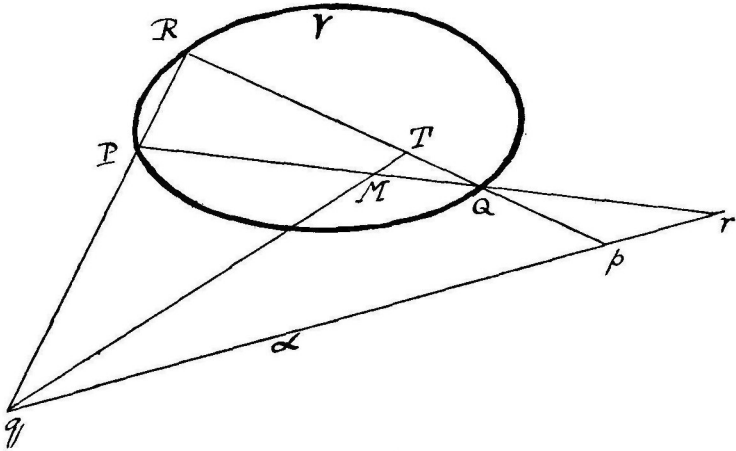


Fig. 3.

Returning to Fig. 2, HX and VW are concurrent in a point I of a , and UX and KW in a point J of a .

But KW and VW are "supplementary" chords of γ "as to a ." Hence I and J are "conjugate points as to γ ."

Hence HX and UX are "supplementary" chords of γ as to a , i.e. X lies on the conic γ . So do Y and Z .

Thus for every point S and line a associated with the triangle ABC there exists a generalized nine-points conic γ which passes through $U, V, W, H, K, L, X, Y, Z$, when H, K, L "bisect" BC, CA, AB , "as to a ," and AX, BY, CZ are concurrent in a with SH, SK, SL respectively, and U, V, W "bisect as to a " the segments AO, BO, CO , O being the point of concurrence of AX, BY, CZ .

And the pole M of a as to this conic "bisects" SO "as to a ," and S is the pole of a of a conic round ABC which is "homothetic as to a " with γ , O being the homothetic centre.

By sending a to infinity this reduces to the ordinary theory of the nine-points conic.

§4.

We may pursue this line of investigation a little further, as follows. For the sake of symmetry we will change the lettering to some extent, as may be seen by comparing Fig. 4 with Fig. 2. $P_1'Q_1'R_1'$ is a transversal of the triangle ABC , forming with it a complete quadrilateral. (See Fig. 4).

$A_1'' B_1'' C_1''$ is its diagonal triangle.

AA_1'', BB_1'', CC_1'' , concur in D_2' , and cut BC , CA , AB in A_1', B_1', C_1' .

Again $P_2'Q_2'R_2'$ is another transversal of ABC , giving another complete quadrilateral, which, treated like the preceding one, gives rise to points $A_2'' B_2'' C_2'' A_2' B_2' C_2' D_1'$.

We then combine the quadrilaterals.

$P_1'Q_1'R_1'$ is cut by AA_2' in X_1 , by BB_2' in Y_1 , in CC_2' in Z_1

Y_1C_1', Z_1B_1' and AD_1' concur in A_1

Z_1A_1', X_1C_1' and BD_1' ,, ,, B_1

X_1B_1', Y_1A_1' and CD_1' ,, ,, C_1 .

Similarly $Y_2C_2', Z_2B_2',$ and AD_3' concur in A_2 , etc.

$X_1A_1', Y_1B_1', Z_1C_1'$ concur in D_1

$X_2A_2', Y_2B_2', Z_2C_2'$,, ,, D_2

$D_1D_2' D_1' D_2$ are collinear.

$A_1A_1', B_1B_1', C_1C_1'$ concur in M_1

$A_2A_2', B_2B_2', C_2C_2'$,, ,, M_2 .

A twelve-points conic passes through

$$A_1B_1C_1 A_1' B_1' C_1' A_2B_2C_2 A_2' B_2' C_2'.$$

With respect to this conic, M_1 is the pole of $P_1'Q_1'R_1'$, and M_2 that of $P_2'Q_2'R_2'$.

D_1 is pole of $P_1'Q_1'R_1'$ as to the circumconic, with D_1' as centre, and $P_1'Q_1'R_1'$ as axis of perspective.

D_2 is pole of $P_2'Q_2'R_2'$ as to a circumconic of ABC in perspective with the twelve-points conic, and having D_2' as centre of perspective and $P_2'Q_2'R_2'$ as axis.

If we send $P_1'Q_1'R_1'$ off to infinity, $A_1' B_1' C_1'$ become the mid points of the sides of the triangle ABC , and we get the special twelve-points conic, which again becomes a circle, when for the transversal $P_2'Q_2'R_2'$ we take the "orthic axis" of the triangle ABC .

Similarly, if $P_2' Q_2' R_2'$ be sent to infinity, and the "orthic axis" of ABC is taken for $P_1' Q_1' R_1'$, then AA_2', BB_2', CC_2' become the medians and AA_1', BB_1', CC_1' the perpendiculars of the triangle ABC , the twelve-point conic again reducing to the nine-points circle (or twelve-points circle, as it might be called) of ABC .

Going back to the general case, we might note other sets of collinear points and concurrent lines. This will appear in the next section, where the figure is considered from another point of view.

Meanwhile, we may note the set of six collinear points $D_1 D_2' M_1 M_2 D_1' D_2$, where we have

$$\begin{aligned} D_2' M_1 &= \frac{1}{4} D_2' D_1' \text{ as to the line } P_1' Q_1' R_1' \\ D_1' M_2 &= \frac{1}{4} D_1' D_2' \text{ ,, ,, ,, } P_2' Q_2' R_2' \\ D_1 D_2' &= \frac{1}{2} D_2' D_1' \text{ ,, ,, ,, } P_1' Q_1' R_1' \\ D_2 D_1' &= \frac{1}{2} D_1' D_2' \text{ ,, ,, ,, } P_2' Q_2' R_2'. \end{aligned}$$

The first statement of the four means, by analogy with the expression previously defined, "bisection of a line segment as to a line" that M_1 and the point of intersection of $D_1 D_2$ with $P_1' Q_1' R_1'$ divide $D_2' D_1'$ in the anharmonic ratio $-\frac{1}{4}$. This implies, of course, that when $P_1' Q_1' R_1'$ goes off to infinity $D_2' M_1$ becomes actually $= \frac{1}{4} D_2' D_1'$.

We remark that five lines, consisting of a triad (the sides of the triangle ABC) and a pair (the two transversals) determine uniquely a certain conic (the twelve-points conic).

Hence if any five straight lines be given, since they can be divided into a triad and a pair in ${}^5C_3 = 10$ different ways they determine a set of ten twelve-points conics. Into the relations existing between these ten conics I do not enter.

In place of determining the conic by a triangle and two transversals, we might of course let it be determined by the triangle and two points, viz., the "poles" of the transversals with respect to the triangle.

§5.

The figure $A_1 B_1 C_1 D_1 A_1' B_1' C_1' D_1'$ is formed by three four-ray pencils with collinear apices $X_1 Y_1 Z_1$, intersecting so as to form a figure similar to a plane projection of a parallelepiped. We shall study it separately, generalize it, and apply it to a further generalization of the original figure. In this section, and the next following, we will for simplicity drop the suffixes of the letters.

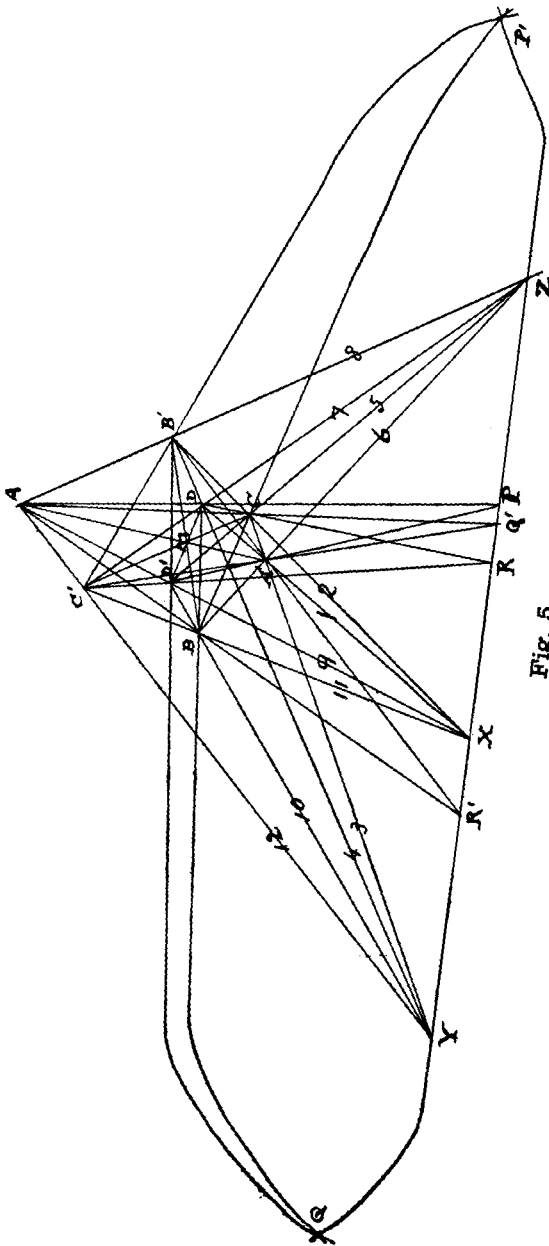


Fig. 5.

The figure we are now to consider may be constructed by following the order of drawing the lines indicated by the numerals on Fig. 5.

Thus, choosing X and Y and drawing any two pairs of lines 1, 2; 3, 4 through X and Y respectively, intersecting one another in $DB'CA'$, we then choose a point Z in XY and join it to C, A', D, B' by the lines 5, 6, 7, 8.

Then choose D' in ZC .

The line XD' meets ZB' in A , while YD' meets ZA' in B . Then YA and XB will concur with ZD in C' . This is easily proved, for *as to* XYZ we have, taking C' as the intersection of XB and ZD , $C'B \equiv DA' \equiv B'C \equiv AD'$. Hence by Prop. (i) $BD' \equiv C'A$ *as to* XYZ , *i.e.* YA produced passes through C' .

Since $C'A \equiv A'C$ it follows that CA' and AC concur with XYZ in a point Q' .

Similarly $B'C'$ and BC concur with XYZ in a point P' .

„	$A'B'$	„	AB	„	„	„	„	„	R'
„	DA	„	$D'A'$	„	„	„	„	„	P
„	DB	„	$D'B'$	„	„	„	„	„	Q
„	DC	„	$D'C'$	„	„	„	„	„	R

From Prop. (iii) it follows that AA', BB', CC', DD' all bisect one another *as to* XYZ in the same point M .

If we leave out two opposite points, say DD' , the remaining six, $AC'BA'CB'$ will lie on a conic in virtue of the converse of Pascal's Theorem, and this conic has obviously M as the pole of XYZ .

The number of independent variables (freedoms) in the figure may be reckoned thus: X and Y have two freedoms each, 1, 2, 3, 4 have one freedom each.

Z, D' have one each. The other points and lines are then fixed. Thus we have a total of 10 freedoms.

We arrive at the same result by noting that a conic has 5 freedoms, while the choice of 5 points on it gives 5 more freedoms, the 6th point being thereby fixed.

It is clear, too, that we may begin by choosing any 5 points (10 freedoms) out of the six $ACB A'C'B'$, after which the 6th point and $DD'XYZ PQR P'Q'R'$ are determined.

The number of different ways they can be determined for a given set of 5 points is 5×3 , for this is the number of ways in which we can single out one of the 5 points and arrange the other 4 in pairs.

§6.

If we remove the restriction that XYZ should be collinear, then one freedom is added to the figure, bringing the total up to 11. In this case Pascal's Theorem does not apply, and the six points do not necessarily lie on a conic.

The generalized figure (which is to be supplied by the reader) may be studied by means of a three-dimensional construction as follows :

Begin with the complete quadrilateral consisting of the triangle ZB'D and its transversal YC'A, having U'PP' as its diagonal triangle, U' being the intersection of DA and B'C', P that of DA and YZ, P' that of YZ and B'C'.

Let X be an arbitrary point outside the plane of AB'DC'YZ. Take a plane through YZ and let it cut the lines XA, XB', XD, XC' in D'CA'B respectively. We thus get a pair of quadrilaterals triply in perspective. To prove this it is only necessary to show that BD' passes through Y, etc.

Now BD'Y is the intersection of the planes YZA' and XYC', so that B, D', Y are collinear. Similarly A'C passes through Y, and of course DB' and C'A do so.

In like manner we can show that DC', D'C, BA', B'A are concurrent in Z.

The solid figure AD'BC'DA'CB' is topologically the same as a cube or parallelepiped, but having 3 sets of 4 edges *concurrent* instead of *parallel*.

Let DA, B'C' meet in U' ; D'A', BC in U.

„ DB, C'A „ „ V' ; D'B', CA „ V.

„ DC, A'B' „ „ W' ; D'C', AB „ W.

The planes XA'DAD', XCB'C'B both contain U, U', X, hence U, U', X are collinear. So are V, V', Y and W, W', Z.

Let DB meet XZ in Q and DC meet XY in R. The three planes XPADA'D', YQBDB'D', ZRCDC'D' all contain DD', hence they have DD' as common section, and DD' will meet the plane

XYZ in a point δ , which will be the point of concurrence of the 3 lines XP, YQ, ZR, in which these three planes intersect the plane XYZ.

Thus AA', BB', CC' all meet DD', since each of these lies in one of the planes of which DD' is the common intersection. Again BB' and CC' lie in the plane XBC'CB', and therefore intersect one another, hence their point of intersection, M, must lie on DD'.

In other words, M is the common point of the three planes YQBDB'D', ZRCDC'D', XBC'CB', so that their mutual intersections, BB', CC', DD' are concurrent in M. Similarly AA' cuts DD' where it meets CC', *i.e.* at M.

Again UU' lies in the planes XPADA'D'M, XBCB'C'M, so that U, U', M, X are collinear. So are V, V', M, Y; and W, W', M, Z.

UU' and WW' being in one plane, UW' and U'W meet one another; but U'W lies in the plane PADBQ and UW' in the plane RCDBQ, therefore they meet in BDQ. But UU' and WW' lie in the plane XMZ \therefore UW' and U'W lie in that plane and meet BDQ where it meets that plane, *i.e.* in Q.

Thus	UW'	and	U'W	meet	in	Q.
Similarly	UV'	„	U'V	„	„	R
„	VW'	„	V'W	„	„	P.
Also	VW	„	V'W'	„	„	P'
„	WU	„	W'U'	„	„	Q'
„	UV	„	U'V'	„	„	R'

where P' is the point of intersection of CB and C'B' with YZ, etc.

P, Q, R' are collinear since they are on the intersections of two planes C'U'B'W'A'V' and AWBUCV. Hence also these planes and the plane XYZ have P'Q'R' as common intersection.

Also B'V' and BV which lie in the plane YB'DV'BQD'V (and respectively in the planes C'U'B'W'A'V' and AWBUCV) must intersect where P'Q'R' meets this plane, *i.e.* where YQ meets P'Q'R', say in Y'.

Similarly	C'W'	and	CW	concur	with	P'Q'R'	in	Z'.
„	A'U'	„	AU	„	„	„	„	X'.

Again A'U', B'V', C'W' concur in d and AU, BV, CW' in d' , which both lie on DMD' δ .

Note that δ is the point in which DD' meets the plane XYZ . Similarly we have points α , β , γ which are the intersections of AA' , BB' , CC' with the same plane.

α lies in XPX' in YQ' and in ZR'

β „ YQY' „ ZR' „ XP'

γ „ ZRZ' „ XP' „ YQ' .

Our figure has been developed with special regard to DD' , but it might equally have been developed with special regard to AA' , in which case $MXYZ P'Q'R PQR'$ would take the places of $MXYZ PQR P'Q'R'$, etc. It might also be developed with special reference to BB' or CC' .

We observe that a doubly infinite family of quadric surfaces exists each member of which passes through $ABCD A'B'C'D'$ (since these eight points are common to three pairs of planes), and has M as the pole of the plane XYZ .

Note the plane figure consisting of the triangle XYZ and the transversal $P'Q'R'$. These determine $PQR X'Y'Z' \alpha\beta\gamma\delta$ in the same plane; or if PQR and its transversal $P'Q'R'$ be taken, they determine the diagonal triangle XYZ and also the points $X'Y'Z' \alpha\beta\gamma\delta$.

Now let us consider a plane passing through $A'B'C'$. It passes also through $P'Q'R'$. The lines $P'A'$, $Q'B'$, $R'C'$ will form a triangle which we may name $A_0B_0C_0$, and it is clear that A_0 lies in AD' , B_0 in BD' , and C_0 in CD' .

The solid figure studied in this section can be projected on any plane (the plane of the paper), and the properties obtained above by solid geometry might be worked out by the aid of the theorems of Ceva and Menelaus and the theory of anharmonic ratio.

§7.

Let us now suppose the figure of §6 to be projected on the plane of the paper, the projections of all the points being denoted by corresponding letters with suffix 1, with the exception of $A_0B_0C_0$, whose projections will be denoted by ABC .

Let $B_1'C_1'$ meet BC in P_2'
 „ $C_1'A_1'$ „ CA „ Q_2'
 „ $A_1'B_1'$ „ AB „ R_2'
 „ AA_1D_1' „ BC „ A_2'
 „ BB_1D_1' „ CA „ B_2'
 „ CC_1D_1' „ AB „ C_2' .

Let D_2' be the point of concurrence of AA_1' , BB_1' , and CC_1' .

Through P_2' draw an arbitrary line meeting BD_2' in B_2 and CD_2' in C_2 .

Then $Q_2'C_2$ and $R_2'B_2$ will meet in AD_2' in a point A_2
 B_2C_2' , $B_2'C_2$, and A_2D_2' will concur in a point X_2
 C_2A_2' , $C_2'A_2$, „ B_2D_2' „ „ „ „ Y_2
 A_2B_2' , $A_2'B_2$ „ C_2D_2' „ „ „ „ Z_2 .

X_2A_2' , Y_2B_2' and Z_2C_2' will concur in a point D_2 , and the figure may be further developed until there is a point with suffix 2 corresponding to every point with suffix 1.

We have now arrived at a generalization of the figure of §4, since the points $X_1Y_1Z_1$ are no longer restricted to be collinear, nor are $X_2Y_2Z_2$ so restricted. The twelve points

$$A_1B_1C_1 A_1' B_1' C_1' A_2B_2C_2 A_2' B_2' C_2'$$

no longer necessarily lie on a conic.

To get the figure of §4 as a special case of that now considered we must suppose the projection of the solid figure to be from a centre in the intersection of the planes $X_1Y_1Z_1$ and $X_2Y_2Z_2$, so that each of these planes will project into a straight line. The points $X_1Y_1Z_1 P_1Q_1R_1 P_1' Q_1' R_1' X_1' Y_1' Z_1' \alpha_1\beta_1\gamma_1\delta_1$ etc., in fact all in the plane $X_1Y_1Z_1$ become collinear, as also $X_2Y_2Z_2 P_2Q_2R_2$, etc., and the twelve points $A_1B_1C_1 A_1' B_1' C_1' A_2B_2C_2 A_2' B_2' C_2'$ would lie on a conic, as in §4.

If we wish to get the generalized figure of the present section by a plane construction we can begin as in §4 with the triangle ABC , and proceed as in §4 to get the points

$$P_1' Q_1' R_1' A_1'' B_1'' C_1'' A_1' B_1' C_1' D_2' P_2' Q_2' R_2' A_2'' B_2'' C_2'' A_2' B_2' C_2' D_1',$$

but instead of taking X_1 as intersection of AA_2' with $P_1'Q_1'R_1'$ let X_1 be an *arbitrary* point in AA_2' . Then let X_1B_1' and X_1C_1' cut C_0C_2' and B_0B_2' in C_1 and B_1 which will be collinear with P_1' . Then Z_1 and Y_1 are the intersections of B_1A_1' and C_1A_1' with X_1Q_1' and X_1R_1' . Z_1B_1' and Y_1C_1' concur with AA_2' in A_1 . X_1 will then be the point of concurrence of B_1C_1' , C_1B_1' , A_1D_1' and $A_1'D_1$; and similarly for Y_1 and Z_1 . P will be the point of concurrence of A_1D_1 , $A_1'D_1'$ and Y_1Z_1 , and similarly for Q and R .

The other points with suffix 2 can then be constructed as in the first part of the present section, in which it is to be noted there is *one* arbitrary element which may be taken as the point B_2 in BD_2' .

Thus on the whole the figure of the present section has two additional freedoms as compared with that of §4.

Note.—The last section (§7) has been re-written and made more complete since the paper was read.

